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PAPERS

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PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY

NEW SOLUTION OF AN INTEGRAL EQUATION

By G. H. HARDY *and* E. C. TITCHMARSH.

[Received 15 June, 1935.—Read 20 June, 1935.]

1. *Statement of the theorem.*

1.1. The equation

$$(1.1.1:\mathrm{M}) \qquad f'(x) = \tfrac{1}{2}\int_0^\infty \frac{e^{-u}}{u}\{f(x+u)-f(x-u)\}\,du$$

originates with Milne†. It is satisfied by any quadratic $f(x)$, and has no other obvious solutions. The problem is to prove that all solutions are quadratic‡, or, failing this, that every solution belonging to some specified class of functions is quadratic.

We have already given several solutions of the problem§, but that which we give here is a good deal more general than any obtained before. In our previous solutions we assumed that

$$f(x) = O(e^{A|x|}),$$

where $A < 1$, for all real x, and this is a great deal more than is needed for the convergence of the integral. A natural and much more general

† Milne (7).

‡ That is to say, that every $f(x)$ such that (i) the integral exists, in some specified sense, for every x, (ii) the equation is true for every x, is quadratic. The precise meaning of the assertion would depend on the particular sense chosen in (i).

§ Hardy and Titchmarsh (3, 4, 5).

assumption would be that $f(x) = e^{|x|}\, g(x)$, or $f(x) = \cosh x\, g(x)$, where $g(x)$ is L^2 in $(-\infty, \infty)$†. We are still unable to prove that this assumption is sufficient, but the assumption which we prove sufficient is "all but" the same.

THEOREM. *Any solution of* (M) *of the form*

$$f(x) = \cosh x\, g(x), \tag{1.1.2}$$

where

$$\int_{-\infty}^{\infty} g^2\{1+(\log|g|)^2\}\,dx < \infty, \tag{1.1.3}$$

is a quadratic.

Our proof of this theorem is quite elaborate, and depends upon a long series of lemmas, many of which are more interesting than the final result. Some of our arguments are capable of considerable generalization, while others depend much more on the particular idiosyncrasies of the equation. We have not attempted to generalise the result, since our analysis must be justified by its intrinsic interest rather than by any importance in its conclusion.

Our new formal method consists in putting $f(x) = \cosh x\, g(x)$, and deducing from the given equation one satisfied by $G(x)$, the Fourier transform of $g(x)$. This is found to be

$$G(x-i)+G(x+i) = 0.$$

The analysis also shows that $G(x)$ is an analytic function, periodic and regular except possibly for a certain sequence of triple poles. These conditions determine $G(x)$, apart from the values of certain constants, and $G(x)$ in turn determines $g(x)$ and $f(x)$. The main argument begins in § 3.1, and the reader can begin there also, and refer to the lemmas as they are required. There is nothing particularly surprising in the results of the lemmas, however intricate the proofs may be.

2. *Preliminary lemmas.*

2.1. In this section we collect a number of lemmas (some of which are very well known theorems) required for the proof.

† The two forms of the hypothesis are plainly equivalent.

LEMMA α. *If $g(x)$ is L^2 in $(-\infty, \infty)$, then*

$$h(x) = \frac{1}{\pi}\int_0^\infty \frac{g(x+u)-g(x-u)}{u}\,du \tag{2.1.1}$$

exists for almost all x, and is L^2.

This is a well-known theorem in the theory of conjugate functions†. The integral is a Lebesgue integral at ∞, a Cauchy integral at 0.

Lemmas concerning inequalities.

2.2. LEMMA β. *If $p(x)$ belongs to L, and $q(x)$ to L^2, in $(-\infty, \infty)$, then*

$$r(x) = \int_{-\infty}^{\infty} p(u)\,q(x+u)\,du$$

belongs to L^2.

For, if

$$\int_{-\infty}^{\infty} |p(u)|\,du = P, \qquad \int_{-\infty}^{\infty} q^2(u)\,du = Q^2,$$

we have

$$\{r(x)\}^2 \leqslant P\int_{-\infty}^{\infty} |p(u)|\,q^2(x+u)\,du,$$

$$\int_{-\infty}^{\infty} \{r(x)\}^2\,dx \leqslant P\int_{-\infty}^{\infty} |p(u)|\,du\int_{-\infty}^{\infty} q^2(x+u)\,dx = P^2Q^2.$$

The lemma is a special case of "Young's inequality"‡.

2.3. LEMMA γ. *If*

$$\int_0^\infty \{g(x)\}^2\{1+(\log x)^2\}\,dx < \infty \tag{2.3.1}$$

and

$$g_1(x) = \int_0^x g(t)\,dt,$$

then

$$\int_0^\infty \left\{\frac{g_1(x)}{x}\right\}^2\{1+(\log x)^2\}\,dx < \infty \tag{2.3.2}$$

and

$$\int_0^\infty \frac{|g(x)\,g_1(x)|}{x}\{1+(\log x)^2\}\,dx < \infty. \tag{2.3.3}$$

† See, for example, Titchmarsh (10).

‡ See Hardy, Littlewood, and Pólya (2), Theorem 280. The theorem stated there for $\lambda > 0$, $\mu > 0$ is still true when $\lambda = 0$ or $\mu = 0$, and it is one of these cases ($\lambda = 0$, $\mu = \frac{1}{2}$) which we need here. We therefore insert a proof.

Since $g(x)\log x$ is L^2,

$$\int_0^x g(t)\log t\,dt = o(x^{\frac{1}{2}})\dagger,$$

and

$$g_1(x) = o\left(\frac{x^{\frac{1}{2}}}{|\log x|}\right)$$

both at 0 and ∞. Also, if

$$1+(\log x)^2 = \varpi(x),$$

then

$$\chi(x) = \int_x^\infty \frac{\varpi(t)}{t^2}\,dt < A\,\frac{\varpi(x)}{x},$$

where A is a constant, and $g_1{}^2\chi = o(1)$ at 0 and ∞. Hence

$$\int_0^\infty \frac{g_1{}^2\varpi}{x^2}\,dx = -\int_0^\infty g_1{}^2\chi'\,dx = 2\int_0^\infty g_1 g\chi\,dx$$

$$\leqslant 2A\int_0^\infty \left|\frac{g_1}{x}\right||g|\,\varpi\,dx \leqslant 2A\left(\int_0^\infty \frac{g_1{}^2\varpi}{x^2}\,dx\int_0^\infty g^2\varpi\,dx\right)^{\frac{1}{2}},$$

$$\int_0^\infty \frac{g_1{}^2\varpi}{x^2}\,dx \leqslant 4A^2\int_0^\infty g^2\varpi\,dx,$$

and

$$\int_0^\infty \frac{|gg_1|\,\varpi}{x}\,dx \leqslant 2A\int_0^\infty g^2\varpi\,dx.$$

2.4. In the lemmas which follow we use g^* in the senses defined (with a slightly different notation) by Hardy, Littlewood, and Pólya. There are two such senses. In the first place, when the interval considered is $(0, \infty)$, g^* is the decreasing rearrangement of $|g|$. Secondly, when the interval is $(-\infty, \infty)$, g^* is the "symmetrically decreasing" rearrangement of $|g|$, the even function equimeasurable with $|g|$ and decreasing both ways from the origin‡.

LEMMA 8. *If g satisfies* (1.1.3), *then g^* satisfies* (2.3.1).

The hypothesis (1.1.3) is unaltered by the substitution of g^* for g. It is therefore sufficient to prove that (1.1.3) implies (2.3.1) when g is a positive symmetrically decreasing function.

† Hardy, Littlewood, and Pólya (2), 164, Theorems 222, 223.

‡ Hardy, Littlewood, and Pólya (2), ch. X. See in particular 276–278, where, however, the fundamental intervals are $(0, 1)$ and $(-\infty, \infty)$, and what we should call g^* in the first case is denoted by $\bar{g}$.

We have then

$$\tfrac{1}{2}xg^2(x) \leqslant \int_{\frac{1}{2}x}^{x} g^2(t)\,dt < A$$

and so $$g < Ax^{-\frac{1}{2}}, \quad (\log x)^2 < 4(\log g - \log A)^2$$

for $x > 1$, with A independent of x. Hence, first,

$$(2.4.1) \qquad \int_1^\infty g^2(\log x)^2\,dx \leqslant 8\int_1^\infty g^2\{(\log g)^2+(\log A)^2\}\,dx < \infty.$$

On the other hand,

$$uv \leqslant u\log u + e^{v-1}$$

for $u > 0$ and all real v†. Taking $u = g$, $v = \frac{1}{4}\log(1/x)$, $x < 1$, we obtain

$$g\log\frac{1}{x} = 4\left(g\,.\,\tfrac{1}{4}\log\frac{1}{x}\right) \leqslant 4(g\log g + e^{-1}x^{-\frac{1}{4}}),$$

$$(2.4.2) \qquad \int_0^1 g^2(\log x)^2\,dx \leqslant A\int_0^1 g^2(\log g)^2\,dx + A\int_0^1 x^{-\frac{1}{2}}\,dx < \infty.$$

The conclusion follows from (2.4.1) and (2.4.2).

LEMMA ϵ‡. *If $h(x)$ is symmetrically decreasing, then*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)\,g(y)\,h(x-y)\,dx\,dy \leqslant \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g^*(x)\,g^*(y)\,h(x-y)\,dx\,dy.$$

2.5. LEMMA ζ. *If $g(x)$ satisfies* (1.1.3), *then*

$$h(x) = \int_1^\infty \frac{g(x+u)}{u}\,du$$

is L^2 in $(-\infty, \infty)$.

We may suppose g positive. The integral exists for all x, by Schwarz's inequality, and it is enough to show that

$$\int_{-\infty}^{\infty} h^2(x)\,dx = \int_{-\infty}^{\infty} dx\left(\int_{x+1}^{\infty}\frac{g(u)}{u-x}\,du\right)^2 < \infty.$$

This is

$$\int_{-\infty}^{\infty} dx\int_{x+1}^{\infty}\frac{g(u)}{u-x}\,du\int_{x+1}^{\infty}\frac{g(v)}{v-x}\,dv = \int_{-\infty}^{\infty} g(u)\,du\int_{-\infty}^{\infty} g(v)\,dv\int_{-\infty}^{w}\frac{dx}{(u-x)(v-x)},$$

† "Young's inequality": see Hardy, Littlewood, and Pólya (2), Theorem 63.

‡ See Hardy, Littlewood, and Pólya (2), Theorem 380.

where $w = \min(u-1, v-1)$. Hence

$$\int_{-\infty}^{\infty} h^2(x)\,dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(u)\,g(v)\,\frac{\log(1+|u-v|)}{|u-v|}\,du\,dv$$

$$\leqslant \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g^*(u)\,g^*(v)\,\frac{\log(1+|u-v|)}{|u-v|}\,du\,dv,$$

since

$$\frac{\log(1+|x|)}{|x|}$$

is symmetrically decreasing, so that we may use Lemma ϵ.

It is therefore sufficient to prove the last integral finite, and we may confine ourselves to the octant $0 \leqslant v \leqslant u$. But

$$\int_0^u g^*(v)\,\frac{\log(1+u-v)}{u-v}\,dv \leqslant \frac{1}{u}\int_0^u g^*(v)\,dv \int_0^u \frac{\log(1+u-v)}{u-v}\,dv,$$

by Tchebychef's inequality†; and the second factor is

$$\int_0^u \frac{\log(1+v)}{v}\,dv < A\{1+(\log u)^2\}.$$

It is therefore sufficient that

$$\int_0^\infty \frac{g^*(u)\,g_1{}^*(u)}{u}\,\{1+(\log u)^2\}\,du < \infty,$$

and this follows from Lemmas δ and γ.

2.6. LEMMA θ. *If $g(x)$ satisfies* (1.1.3), *then*

(2.6.1)
$$\phi(x) = \int_0^\infty \frac{g(x+u)-e^{-2u}\,g(x-u)}{u}\,du,$$

$$\psi(x) = \int_0^\infty \frac{e^{-2u}\,g(x+u)-g(x-u)}{u}\,du$$

are L^2.

It is sufficient to consider $\psi(x)$, which we write in the form

$$\psi(x) = \int_0^\infty \frac{g(x+u)-g(x-u)}{u}\,du - \int_0^1 \frac{1-e^{-2u}}{u}\,g(x+u)\,du$$

$$+\int_1^\infty \frac{e^{-2u}}{u}\,g(x+u)\,du - \int_1^\infty \frac{g(x+u)}{u}\,du.$$

The four integrals here are L^2, the first by Lemma α, the second and third by Lemma β, and the last by Lemma ζ.

† Hardy, Littlewood, and Pólya (2), Theorems 43 and 236.

Lemmas concerning Fourier transforms.

2.7. We use the formulae

$$(2.7.1)\quad F(x)=\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty}f(t)\,e^{ixt}\,dt,\quad f(x)=\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty}F(t)\,e^{-ixt}\,dt,$$

the functions f and F being L^2 and the integrals "mean square integrals" in the sense of Plancherel†. The possible ambiguity in the meaning of "transform", in respect to the factors e^{ixt} and e^{-ixt}, may be avoided by standardising the distinction between small and capital letters as in the formulae (2.7.1).

LEMMA η. *If $f(x)$ and $f'(x)$ are L^2, and $F(x)$ and $H(x)$ are their transforms, then $xF(x)$ is L^2 and*

$$H(x)=-ixF(x).$$

Since ff' is L, we have

$$|\{f(X)\}^2-\{f(x)\}^2|=2\left|\int_x^X f(t)\,f'(t)\,dt\right|<\epsilon$$

for large x and X. Hence $f(x)$ tends to a limit when $x\to\infty$, and the limit is 0 since f is L^2. Also

$$\int_{-a}^{a}f'(t)\,e^{ixt}\,dt+ix\int_{-a}^{a}f(t)\,e^{ixt}\,dt=f(a)\,e^{iax}-f(-a)\,e^{-iax}.$$

The left-hand side has the mean square limit

$$\sqrt{(2\pi)}\,\{H(x)+ix\,F(x)\},$$

when $a\to\infty$, in any finite interval (x_1, x_2), and the right-hand side tends to 0, uniformly in this interval. Hence $H+ixF=0$.

LEMMA κ. *If f is L^2 and F is its transform; and*

$$f(x)=\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty}\chi(t)\,e^{-ixt}\,dt,$$

where χ is L; then $\chi\equiv F$ (so that χ is L^2).

For
$$\lim_{T\to\infty}\frac{1}{\sqrt{(2\pi)}}\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)f(t)\,e^{ixt}\,dt$$

is equal, almost everywhere, to both $\chi(x)$ and $F(x)$‡.

† See Hobson (6), Paley and Wiener (8), or Titchmarsh (9).

‡ See Hobson (6), 740 and 748 respectively.

LEMMA λ. *If the transform of $g(x)$ is $G(x)$, then the transform of $h(x)$, defined by* (2.1.1), *i.e. the transform of the conjugate of $g(x)$, is*

$$-i \operatorname{sgn} x\, G(x).$$

If $g(x) = l(x)+m(x)$, where l is even and m odd, then

$$G(x) = L_c(x)+i\, M_s(x),$$

where L_c is the cosine transform of l and M_s the sine transform of m. Also† $h(x)$ is the sum of the sine transform of $-L_c(x)$ and the cosine transform of $M_s(x)$, and hence the transform of $h(x)$ is

$$-i \operatorname{sgn} x\, L_c(x)+\operatorname{sgn} x\, M_s(x) = -i \operatorname{sgn} x\, G(x).$$

Lemmas concerning analytic functions.

2.8. LEMMA μ. *Suppose that (a, b) is a finite stretch of the real axis, and that $G_1(z)$ and $G_2(z)$ are analytic functions satisfying the following conditions*:

(i) *$G_1(z)$ is regular, except perhaps for a finite number of poles, not on the real axis, in the rectangle $a \leqslant x \leqslant b$, $-2 < y < 0$*;

(ii) *$G_2(z)$ has the same properties in the rectangle $a \leqslant x \leqslant b$, $0 < y < 2$*;

(iii) *$G_1(z)$ and $G_2(z)$ have the same mean square limit $G(x)$, over the interval (a, b), when $y \to -0$ and $y \to +0$ respectively.*

Then $G_1(z)$ and $G_2(z)$ are elements of an analytic function $G(z)$ regular, except for the poles mentioned, in the region

$$a \leqslant x \leqslant b, \quad -2 < y < 2.$$

This is a simple extension of a classical theorem‡.

Suppose, first, that $G_1(z)$ and $G_2(z)$ have no poles. Since

$$\int_a^b |G_2(x+iy)-G(x)|^2\, dx \to 0$$

when $y \to +0$,

$$\int_a^b |G_2(x+iy)|^2\, dx$$

is bounded for $0 < y < 2$, and

$$\int_a^b dx \int_0^1 |G_2(x+iy)|^2\, dy = \int_0^1 dy \int_a^b |G_2(x+iy)|^2\, dx < \infty.$$

† See Titchmarsh (10), § 2.5–3.2. The sign of $g(x)$ on p. 115 should be changed and p. 119, line 5 should read " minus the sine transform . . . ".

‡ For which see, for example, Titchmarsh (9), 155–157.

Hence

$$\int_0^1 |G_2(x+iy)|^2\,dy < \infty \tag{2.8.1}$$

for almost all x. Similarly,

$$\int_{-1}^0 |G_1(x+iy)|^2\,dy < \infty \tag{2.8.2}$$

for almost all x. We suppose that α and β are two numbers of (a, b) for each of which both these conditions are satisfied.

Suppose that δ is small and positive and that z_0 lies in the upper of the two regions in question, and apply Cauchy's theorem to

$$\frac{G_2(z)}{z-z_0}$$

and the rectangle $\alpha+i\delta$, $\beta+i\delta$, $\beta+i$, $\alpha+i$. We obtain

$$2\pi i\,G_2(z_0) = \left(\int_{\alpha+i\delta}^{\beta+i\delta} + \int_{\beta+i\delta}^{\beta+i} + \int_{\beta+i}^{\alpha+i} + \int_{\alpha+i}^{\alpha+i\delta}\right) \frac{G_2(z)}{z-z_0}\,dz = J_1+J_2+J_3+J_4,$$

say. When $\delta \to 0$,

$$J_1 = \int_{\alpha+i\delta}^{\beta+i\delta} \frac{G_2(z)}{z-z_0}\,dz \to \int_\alpha^\beta \frac{G(x)}{x-z_0}\,dx,$$

since $G(x)$ is the mean-square limit of $G(x+i\delta)$;

$$J_2 = \int_{\beta+i\delta}^{\beta+i} \frac{G_2(z)}{z-z_0}\,dz \to \int_\beta^{\beta+i} \frac{G_2(z)}{z-z_0}\,dz$$

on account of (2.8.1); and J_4 tends to a limit similarly. Hence

$$2\pi i\,G_2(z_0) = \int_\alpha^\beta \frac{G(x)}{x-z_0}\,dx + \left(\int_\beta^{\beta+i} + \int_{\beta+i}^{\alpha+i} + \int_{\alpha+i}^{\alpha}\right) \frac{G_2(z)\,dz}{z-z_0}. \tag{2.8.3}$$

If we apply the same argument, with the same z_0, to a rectangle below the real axis, we obtain

$$0 = \left(\int_\alpha^{\alpha-i} + \int_{\alpha-i}^{\beta\ \ i} + \int_{\beta-i}^{\beta}\right) \frac{G_2(z)\,dz}{z-z_0} - \int_\alpha^\beta \frac{G(x)}{x-z_0}\,dx. \tag{2.8.4}$$

Combining (2.8.3) and (2.8.4), we have

$$G_2(z_0) = \frac{1}{2\pi i} \int \frac{G_2(z)\,dz}{z-z_0},$$

the contour being now the rectangle $(\alpha-i, \beta-i, \beta+i, \alpha+i)$. Since the integral is an analytic function $G(z_0)$, regular in this rectangle, $G_2(z_0)$ is an

element of such a function. Similarly, starting with a z_0 below the axis, we show that $G_1(z_0)$ also is an element of $G(z_0)$.

The complication introduced by poles of $G_1(z)$ and $G_2(z)$ is trivial. There is a rational function $R(z)$ such that $G_1^* = G_1 - R$ and $G_2^* = G_2 - R$ are regular in the regions of the enunciation, and G_1^*, G_2^*, and so G_1, G_2, are analytic continuations of one another.

2.9. LEMMA *v*. *If $G(z)$ is regular for*

$$a \leqslant y \leqslant b, \quad x \geqslant 1,$$

where $b-a > 2$, and

$$\int_1^\infty |G(x+iy)|^2 dx$$

is bounded for $a \leqslant y \leqslant b$, then $G(z)$ is bounded for

$$a+1 \leqslant y \leqslant b-1, \quad x \geqslant 2\dagger. \tag{2.9.1}$$

For if x and y satisfy (2.9.1), and $\rho \leqslant 1$, we have

$$G(z) = \frac{1}{2\pi}\int_0^{2\pi} G(z+\rho e^{i\phi})\, d\phi,$$

$$\tfrac{1}{2}G(z) = \frac{1}{2\pi}\int_0^1 \rho d\rho \int_0^{2\pi} G(z+\rho e^{i\phi})\, d\phi,$$

$$\tfrac{1}{4}|G(z)|^2 \leqslant \frac{1}{4\pi^2}\int_0^1\int_0^{2\pi} |G|^2 \rho d\rho d\phi \int_0^1\int_0^{2\pi} \rho d\rho d\phi$$

$$\leqslant A\int_{y-1}^{y+1} dv \int_{x-1}^{x+1} |G(u+iv)|^2 du \leqslant A\int_{y-1}^{y+1} dv \int_1^\infty |G(u+iv)|^2 du.$$

Proof of the theorem.

3.1. When we substitute from (1.1.2) into (1.1.1), we obtain

$$g'(x)\cosh x + g(x)\sinh x \tag{3.1.1}$$

$$= \tfrac{1}{2}\int_0^\infty \frac{e^{-u}}{u}\{g(x+u)\cosh(x+u) - g(x-u)\cosh(x-u)\}\, du$$

$$= \tfrac{1}{4}e^x\phi(x) + \tfrac{1}{4}e^{-x}\psi(x),$$

where $\phi(x)$ and $\psi(x)$ are defined as in (2.6.1). The integrals exist for almost

† For more general theorems of the same character, see Hardy, Ingham, and Pólya (1), especially Theorem 11 (561).

all x, and $\phi(x)$ and $\psi(x)$ are L^2, by Lemma θ. We write (3.1.1) in the form

$$e^x p(x) = e^{-x} q(x), \tag{3.1.2}$$

where

$$p(x) = g'(x)+g(x)-\tfrac{1}{2}\phi(x), \quad q(x) = -g'(x)+g(x)+\tfrac{1}{2}\psi(x). \tag{3.1.3}$$

Since

$$g'(x) = -\frac{e^x-e^{-x}}{e^x+e^{-x}}\, g(x)+\tfrac{1}{2}\frac{e^x}{e^x+e^{-x}}\,\phi(x)+\tfrac{1}{2}\frac{e^{-x}}{e^x+e^{-x}}\,\psi(x),$$

and ϕ and ψ are L^2, g' is L^2; and its transform is $-ixG(x)$, where $G(x)$ is the transform of $g(x)$, by Lemma η. Finally (3.1.3) shows that $p(x)$ and $q(x)$ are L^2; we proceed to calculate their transforms, $P(x)$ and $Q(x)$.

We write $p(x)$ in the form

$$p(x) = g'(x)+g(x)-\tfrac{1}{2}\int_0^\infty \frac{g(x+u)-g(x-u)}{u}\,du-\tfrac{1}{2}\int_0^\infty \frac{1-e^{-2u}}{u}\,g(x-u)\,du$$

$$= g'(x)+g(x)-\tfrac{1}{2}\pi h(x)-\tfrac{1}{2}\phi_1(x).$$

The transforms of the first three terms are $-ix\,G(x)$, $G(x)$, and $\frac{1}{2}\pi i \operatorname{sgn} x\, G(x)$, the last by Lemma λ. To find that of the last term we make use of Parseval's theorem, in the form

$$\int_{-\infty}^{\infty} a(u)\,\bar{b}(u)\,du = \int_{-\infty}^{\infty} A(u)\,B(u)\,du,$$

taking $$a(u) = g(x-u)$$

and $$b(u) = \frac{1-e^{-2u}}{u} \quad (u>0), \quad b(u) = 0 \quad (u \leqslant 0).$$

Then $A(u) = e^{ixu}\,G(-u)$ and

$$B(u) = \frac{1}{\sqrt{(2\pi)}}\int_0^\infty \frac{1-e^{-2t}}{t}\,e^{iut}\,dt = \frac{1}{\sqrt{(2\pi)}}\log\frac{2-iu}{-iu},$$

where the logarithm has an imaginary part between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$. Hence

$$\phi_1(x) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} G(-u)\log\frac{2+iu}{iu}\,e^{ixu}\,du$$

$$= \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} G(u)\log\frac{2-iu}{-iu}\,e^{-ixu}\,du.$$

The integral is absolutely convergent, since

$$G(u), \quad \log\frac{2-iu}{-iu}$$

are both L^2; and the transform of $\phi_1(x)$ is the product of these two functions, by Lemma κ.

Hence the transform of $p(x)$ is

$$P(x) = G(x)\left(1-ix+\tfrac{1}{2}\pi i \operatorname{sgn} x-\tfrac{1}{2}\log\frac{2-ix}{-ix}\right) = G(x)\left(1-ix-\tfrac{1}{2}\log\frac{2-ix}{ix}\right),$$

the imaginary part of the logarithm now lying between $-\pi$ and π. Similarly the transform of $q(x)$ is

$$Q(x) = G(x)\left(1+ix-\tfrac{1}{2}\log\frac{2+ix}{-ix}\right).$$

3.2. If

$$P(z) = P(x+iy) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} p(u)\, e^{izu}\, du,$$

then, by (3.1.2),

$$P(z) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{0} p(u)\, e^{ixu-yu}\, du + \frac{1}{\sqrt{(2\pi)}}\int_{0}^{\infty} q(u)\, e^{ixu-(y+2)u}\, du.$$

Each of these integrals is uniformly convergent in any strip

$$-2 < -2+\delta \leqslant y \leqslant -\delta < 0,$$

so that $P(z)$ is regular in the open strip $-2 < y < 0$. Also

$$P(x+iy)-P(x)$$

is the transform of $\qquad (e^{-yx}-1)\,p(x),$

so that

$$\int_{-\infty}^{\infty} |P(x+iy)-P(x)|^2\, dx = \int_{-\infty}^{\infty} (e^{-yx}-1)^2\, |p(x)|^2\, dx,$$

which tends to zero with y. Hence $P(x)$ is the mean-square limit of $P(x+iy)$.

Similarly there is an analytic function $Q(x+iy)$, regular for $0 < y < 2$, whose mean square limit, when $y \to 0$, is $Q(x)$.

We define $G_1(z)$ and $G_2(z)$ by

$$G_1(z)\left(1-iz-\tfrac{1}{2}\log\frac{2-iz}{iz}\right) = P(z), \quad G_2(z)\left(1+iz-\tfrac{1}{2}\log\frac{2+iz}{-iz}\right) = Q(z).$$

Then $G_1(z)$ is regular for $-2 < y < 0$, except perhaps for a triple pole at $z = -i$, and $G_2(z)$ is regular for $0 < y < 2$, except perhaps for a triple pole at $z = i$. Also

$$\underset{y\to-0}{\text{l.i.m.}}\, P(x+iy) = P(x), \quad \underset{y\to+0}{\text{l.i.m.}}\, Q(x+iy) = Q(x)$$

over the whole interval $(-\infty, \infty)$ of x, and *a fortiori* over any finite interval (a, b); and so

$$\underset{y\to-0}{\text{l.i.m.}}\, G_1(x+iy) = G(x), \quad \underset{y\to+0}{\text{l.i.m.}}\, G_2(x+iy) = G(x)$$

over any such finite interval.

It now follows from Lemma μ that $G_1(z)$ and $G_2(z)$ are elements of an analytic function $G(z)$. This function is regular in the strip

$$-2 < y < 2,$$

except perhaps for triple poles at $z = \pm i$. In particular, it is regular at the origin.

3.3. Next,

$$P(x-i) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} p(u)\, e^{u} \,.\, e^{ixu}\, du,$$

and so

$$p(u)\, e^{u} = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} P(x-i)\, e^{-iux}\, dx;$$

and similarly

$$q(u)\, e^{-u} = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} Q(x+i)\, e^{-iux}\, dx.$$

All these integrals are mean square integrals. Using (3.1.2), we obtain

$$\int_{-\infty}^{\infty} \{P(x-i) - Q(x+i)\}\, e^{-iux}\, du = 0,$$

and therefore

$$P(x-i) = Q(x+i)$$

for almost all real x. This is

$$G(x-i)\left(-ix - \tfrac{1}{2}\log\frac{1-ix}{1+ix}\right) = G(x+i)\left(ix + \tfrac{1}{2}\log\frac{1-ix}{1+ix}\right);$$

and it follows that

$$G(x-i) + G(x+i) = 0 \tag{3.3.1}$$

for almost all, and therefore for all, real x.

We may now use (3.3.1) to effect the continuation of $G(z)$ all over the plane. The function $G(z)$ so defined has the period $4i$, and is regular, except perhaps for triple poles at

$$z = \ldots, \quad -3i, \quad -i, \quad i, \quad 3i, \quad \ldots. \tag{3.3.2}$$

Further, the integrals

$$\int_1^\infty |G(x+iy)|^2\,dx, \quad \int_{-\infty}^{-1} |G(x+iy)|^2\,dx$$

are bounded, and therefore, by Lemma ν, $G(z)$ is bounded for $|x| \geqslant 2$.

Finally, we write

$$z = \frac{2}{\pi}\log\zeta, \quad \zeta = e^{\frac{1}{2}\pi z}, \quad G(z) = G\left(\frac{2}{\pi}\log\zeta\right) = H(\zeta).$$

The points (3.3.2) correspond to $\zeta = -i$ and $\zeta = i$. Thus $H(\zeta)$ is odd, is regular except perhaps for triple poles at these two points, and is bounded for large ζ. Hence, for some A, B, C,

$$H(\zeta) - \left\{\frac{A}{\zeta-i} + \frac{B}{(\zeta-i)^2} + \frac{C}{(\zeta-i)^3} + \frac{A}{\zeta+i} - \frac{B}{(\zeta+i)^2} + \frac{C}{(\zeta+i)^3}\right\}$$

is a constant, which must be 0 because $H(\zeta)$ is odd.

It follows that

$$G(z) = \alpha\frac{1}{\cosh\frac{1}{2}\pi z} + \beta\frac{d}{dz}\left(\frac{1}{\cosh\frac{1}{2}\pi z}\right) + \gamma\frac{d^2}{dz^2}\left(\frac{1}{\cosh\frac{1}{2}\pi z}\right),$$

for some α, β, γ, and therefore that

$$g(x) = \frac{a+bx+cx^2}{\cosh x}$$

for some a, b, c. Hence $f(x)$ is a quadratic.

3.4. We may add in conclusion that, if we assume a little more about $f(x)$, the "inequality" part of the proof may be much simplified without affecting the remainder. If $g(x)$ belongs to L^2 and also to some L^p with $1 < p < 2$, then

$$\int_1^\infty \frac{g(x+u)}{u}\,du$$

belongs to L^2 by Young's inequality†. The proof then proceeds as before.

† Hardy, Littlewood, and Pólya (2), Theorem 280: take

$$\lambda = \frac{1}{p} - \tfrac{1}{2}, \quad \mu = 1 - \frac{1}{p}, \quad f(-t) = \frac{1}{t} \ (t \geqslant 1), \quad f(-t) = 0 \ (t < 1).$$

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THE CLASSIFICATION OF THE LINES OF CUBIC SURFACES PASSING THROUGH A PLANE CUBIC CURVE WHICH HAS SIX GIVEN POINTS ON SIX OF THE LINES

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1. *Introduction.*

The general problem of finding the points of intersection of a plane cubic curve with the twenty-seven lines of certain cubic surfaces through the curve, when six of these points are given, has been solved by T. L. Wren*. Taking six points on the curve, not on a conic and no three of them collinear, he has given some allocations for the lines through these points such that the remaining twenty-one points are determined. His discussion is based on straight line and conic constructions, which, however, can be reduced readily to a construction by straight lines alone by the use of residual points. We propose in this paper to obtain all the modes in which the allocation can be made. We shall extend the solution to cases when the six points are not general in position. From this extension we shall obtain a discussion of the lines of all the singular cubic surfaces and show their inter-relationships to be simple degeneracies of the general system of twenty-seven lines of the non-singular cubic surface. We demonstrate these both in geometrical terms and in terms of the Schläfli notation. By this method the treatment of each singular surface as a separate entity is not necessary. Further, from this we obtain systems of natural development of the singular cubic surfaces from one another.

We shall follow the usual notation, denoting by a_r, b_s, c_{rs} $(r, s = 1, 2, \ldots, 6)$ the lines a_r, b_s of a double-six and their transversals c_{rs}. The point where a line meets the plane cubic curve will be denoted by the corresponding capital letter with the same suffix.

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* *Journal London Math. Soc.*, 6 (1931), 16.

2. *Given six points on the curve in general position, to determine the possible allocations of lines to the points.*

We shall first show that at least three of the six allotted lines must be mutually skew.

(*a*) No three of the lines can be coplanar, for this implies three collinear points on the cubic base curve.

(*b*) If the allocation contains three lines which meet a fourth, these three are necessarily skew.

(*c*) If two lines are skew and no one of the other four is skew to both, then either three or four of these intersect one of the first two lines and are therefore skew by (*b*), or two of them meet the first only and two meet the second. The lines of a pair do not intersect, by (*a*), and each pair is skew with the intersecting line of the other pair.

It follows that at least three lines must be mutually skew, and we may, without loss of generality, consider the lines as being a_1, a_2, a_3 with corresponding points A_1, A_2, A_3. Further, to any six lines on a given cubic surface there is a definite number of possible systems of lettering. Suppose that one of these systems, when allocated to the six points on the plane cubic curve, gives rise to a set of twenty-one other points: if we then apply any of the other systems of lettering to these six points we shall obtain the same twenty-one points on the curve with corresponding changes in their lettering. It follows, therefore, that it is sufficient to obtain the various *types* of allocation. We shall give a typical one based on the three primary lines a_1, a_2, a_3.

I. All six lines mutually skew.

With the allocation a_1, a_2, a_3, a_4, a_5, a_6 to the lines, the six points belong to nine possible sets of twenty-seven points if the curve is elliptic, three if nodal, and one if cuspidal.

II. Five of the lines mutually skew.

(*a*) Five lines which have only one common transversal pass through points of type A_1, A_2, A_3, A_4, A_5. The allotment of either (i) B_6 or (ii) C_{56} to the sixth point determines the remaining twenty-one in four, two, or one ways according to the nature of the curve. The allotment of (iii) B_1 or (iv) C_{12} to the sixth point determines the rest uniquely by a direct straight line construction. There is only one set in this case.

(*b*) Five lines which have two common transversals pass through points of the type A_1, A_2, A_3, A_4, C_{56}. The allotment of either (i) B_6 or (ii) C_{15} to the sixth point determines the rest of the group in four, two, or

one ways according to the type of the base curve. It is not possible to allot either B_1 or C_{12} to the sixth point, since these points are already determined by the five given ones.

III. Four of the lines mutually skew.

Allot A_1, A_2, A_3, A_4 to the points on these four lines. The lines not skew to all of a_1, a_2, a_3, a_4 are given by the types b_5, b_3, c_{34}, c_{35}. The allotment to the fifth point of one of the pair B_5, C_{35} or of the pair B_3, C_{34} forces the other of the pair; such pairs, therefore, cannot be allocated to the fifth and sixth points, and the only possible allocations to these points are of the types

(1) B_5, B_6; (2) B_3, B_6; (3) C_{34}, B_6; (4) C_{35}, B_6;

(5) C_{35}, B_3; (6) C_{35}, B_1; (7) C_{35}, C_{24}; (8) C_{35}, C_{34}.

In each of these cases there is one solution.

IV. Three of the lines mutually skew.

Allot A_1, A_2, A_3 to the points on these three lines. The lines not skew to all of a_1, a_2, a_3 pass through points which may be arranged in three groups: (*a*) three of type B_3, three of type C_{23}; (*b*) three of type B_4; (*c*) nine of type C_{34}. The allotment of any member of group (*a*) to the fourth point determines the rest of the group: thus only one member of group (*a*) can be allocated. The allotment of two members of group (*b*) determines the third member of (*b*) and the whole of (*c*). The allotment of B_4 or C_{34} determines the other three points with suffix 4 from groups (*b*) and (*c*). The only types which give different systems are

(1) B_3, B_4, B_5; (2) B_3, B_4, C_{35}; (3) B_1, B_4, C_{35};

(4) B_1, C_{14}, C_{35}; (5) B_1, C_{24}, C_{35}.

Clearly these groups could have been defined equally well by means of C_{12} instead of B_3 and C_{23} instead of B_1.

3. *The general theory of the sets of twenty-seven points.*

In the discussion of the possible allocations in the preceding paragraph we have limited ourselves to the case of six points of general position which are connected with non-singular surfaces through the given cubic curve. By means of a general theorem we shall now extend the discussion to include cases in which the six given points are not independent. In some cases we shall find that, in order to use a non-singular cubic surface, the latter must bear special relations to the plane of the cubic curve, while in others a singular cubic surface is required.

We begin with six given points on a given cubic curve C which may be general in position or have geometrical inter-relationships. An allocation is given to these six points. Then we can construct a system of cubic surfaces through C, in which the six lines corresponding to the allocation pass through the six points, if we can find a cubic surface with a plane section (the cubic curve C') in which six of the points of intersection of straight lines on the surface bear the same allocation and have the same geometrical inter-relationships as the original six on C.

The parameters of the points corresponding to the intersections of the remaining lines on the surface can be expressed linearly in terms of those of the six. Since the relations between these parameters are independent of the cubic curve and surface, we can find a corresponding set of points on C bearing the same allocations. From the aggregate of points we can select the points A_1, A_2, A_3, A_4, B_5, B_6. If these do not lie on a conic we can construct a cubic surface through C satisfying the required conditions*. If they lie on a conic (or degenerate conic), take $F = 0$ as a cubic cone through C, $L = 0$ as the plane containing C, and $S = 0$ as the quadric cone through the six points and having common vertex with $F = 0$. Then $F + kLS = 0$ represents a singular cubic surface such that the six points A_1, A_2, A_3, A_4, B_5, B_6 lie on the appropriate lines of the surface. In either case the parameters of all the other points are expressible uniquely in terms of these and hence the original six lie on lines of the surface. (It should be noted that the set of remaining points derived from the *original* six is not necessarily unique. There may be a finite number of sets or, if the positions of the six points are not sufficiently general, a definite degree of freedom.)

Further, since the positions of the points can be expressed parametrically in terms of the parameters of the original six, the solution of these relations exposes the type of the surface and the speciality of the plane (by abnormal relations among the points). It follows at once that in any particular case all suitable cubic surfaces are of the same type and bear the same relationship to the plane of the cubic curve.

4. *The six points lying on a general conic.*

In the discussion of the possible allocations in paragraph 2 it was assumed that the six original points were general in position. By virtue of the preceding theorem we are now in a position to discuss the case of any special distribution, since it is sufficient to discover a suitable plane section of some cubic surface on which the six points of intersection of the appro-

* T. L. Wren, *loc. cit.*, 18.

priate lines show the desired relationships. In illustration we tabulate the results which arise from taking the six original points as lying on a non-degenerate conic in general position; the numbers refer to the cases as given in paragraph 2, and it is assumed that the plane curve is non-singular. Cases where the cubic surface required by the plane distribution is non-singular will be denoted by **O** and where the required surface is singular by the appropriate singularity, *e.g.* $\mathbf{C}_2$, for a surface with a conic node*. Where, starting with the surface, special planes are needed to obtain distributions of the stated nature on a plane section, these are also shown, a freely selected point on the surface being denoted by P. In two cases [I and II (a) (iv)] no simple constructions have been obtained, but indirect ones are given in notes 1, 2 at the end of the paragraph.

I $\mathbf{C}_2$ for one set. **O** for eight sets: see note 1.

II (a) (i) $\mathbf{C}_2$ for one set. **O** for three sets: plane of transversals Pa_1a_5, Pb_5b_1.

(ii) **O** for four sets: plane of transversals PPa_5, Pb_5b_6.

(iii) **O**: plane of transversals Pa_1a_5, Pb_5b_6.

(iv) **O**: see note 2.

II (b) (i) **O**: plane of transversals Pa_1c_{56}, Pb_5b_1.

(ii) **O**: plane of transversals PPc_{15}, Pb_5b_6.

III (1) $\mathbf{C}_2$.

(2) **O**: plane through common point of a_3, b_5.

(3) **O**: plane through a transversal of a_1, a_2, a_6.

(4) **O**: plane of transversals Pa_1a_2, Pb_5c_{64}.

(5) **O**: plane of transversals Pa_1a_2, Pb_5c_{34}.

(6) **O**: plane of transversals Pa_1a_2, Pb_5c_{14}.

(7) **O**: plane of transversals Pa_1a_2, Pb_5b_2.

(8) **O**: plane of transversals Pa_1a_2, Pb_5b_3.

IV (1) $\mathbf{C}_2$.

(2) **O**: plane through common point of c_{56}, b_5.

(3) **O**: plane through a transversal of c_{12}, c_{14}, b_5.

(4) **O**: plane through a transversal of c_{12}, b_4, b_5.

(5) **O**: plane of transversals Pa_1a_2, $Pc_{13}c_{16}$.

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* Further detail of this case is given in the next paragraph. The notation used for the singular cubic surface is that of Salmon, *Analytical geometry of three dimensions*, 2 (1915), 170.

Note 1. The essential relation can be reduced to $b_4 - 2a_4 \equiv a_4 - 2b_4$. At a point A_4 taken arbitrarily on a_4 draw a tangent line t to cut the surface at the point $-2A_4$. The plane through $-2A_4$ and b_4 cuts the surface again in a conic through $-2A_4$. Any plane through the line t cuts b_4 in the point B_4. At B_4 draw the tangent line in this plane to cut the surface again at the point $-2B_4$. As the plane through t varies, the join of A_4 and $-2B_4$ traces out a locus on the plane of the conic. (Algebraically, this locus can be shown to be a quintic with a 4-ple point at $-2A_4$ and also passing through the intersection of a_4 with the conic.) If Q is any other point of intersection of this locus with the conic, the plane through Q and t gives a plane section of the cubic surface with the desired distribution.

Note 2. The essential relation can be reduced to $b_4 - 2a_4 \equiv a_3 + a_5$. Construct the tangent line t and the conic section in the plane of $-2A_4$, b_4 as in note 1. Any plane through t cuts a_3, a_5 in points A_3, A_5 whose join describes a quadric surface of which a_3, a_5, and t are generators. This quadric cuts the plane of the conic in a second conic through $-2A_4$. If Q is a point of intersection of these conics other than $-2A_4$, the plane through Q and t gives a plane section of the cubic surface with the desired distribution.

5. *The classification of singular cubic surfaces and their lines.*

We take the case in which A_1, A_2, A_3, A_4, B_5, B_6 lie on a conic and obtain the various forms of singular surface in their orders of geometrical degeneracy by considering successive degeneracies of the conic. We have already shown (in paragraph 3) that, if these six base points lie on a conic, a singular cubic surface can be drawn to satisfy the conditions. We now add a converse theorem.

If, in the complete set of points on the plane cubic curve, any six points of type equivalent to that of A_1, A_2, A_3, A_4, B_5, B_6 lie on a conic, then the cubic surface is certainly singular; the pairs of lines equivalent to a_r, b_r (of a double-six) coincide and pass through the corresponding singularity.

Consider the quadric with a_1, a_2, a_3 as generators of one system and b_5, b_6 as generators of the other. The points A_1, A_2, A_3, B_5, B_6 lie on the quadric, and its section by the plane of the cubic curve is therefore the conic through these points and A_4. It follows that a_4 (meeting b_5 and b_6) is a generator of the first system and that the complete intersection of the quadric and cubic surface is the six lines a_1, a_2, a_3, a_4, b_5, b_6. Now b_1 meets

a_2, a_3, a_4, and is therefore a generator of the quadric and part of the curve of intersection: the lines a_1 and b_1 must coincide and the quadric must be a cone. The pair of lines a_r, b_r coincide and the six lines so obtained (corresponding to a double-six) are concurrent in a singularity of the surface.

[The points A_r, B_r (or equivalents) on the cubic curve coincide, but this does not imply that an initial group of four A's and two B's on a conic can be replaced by an initial group of six A's, which does not necessarily enforce a singular surface.]

For a $\mathbf{C}_2$ the six lines corresponding to a double-six lie on a non-degenerate cone through the node, for a $\mathbf{B}_3$ these become three lines on one plane and three on a second both through the node and so on, so that to identify the types of singularity on the surface it is sufficient to construct the set of points on the plane cubic and discover the cases where appropriate points lie on a conic, which may be degenerate. The distribution of the points on the plane cubic curve indicates the form in which the twenty-seven letters a, b, c of the normal Schläfli system are to be allocated to the lines of the singular cubic surface: all special properties among these points correspond to properties of the lines of the surface since, for example, the point of contact of two double-six conics establishes the line joining two conic nodes. We do not give the details of these as each case can be readily constructed on the cubic curve.

In Table I we show the cubic surfaces which are demanded by particular degeneracies. This table is not entirely exhaustive as some of the surfaces can be obtained by other forms of degeneracy. The cubic curve is assumed non-singular and the plane of section not special. Since, with six points A_1, A_2, A_3, A_4, B_5, B_6 lying on a conic, the points B_1, B_2, B_3, B_4, A_5, A_6 coincide with them one by one, it is unnecessary to define in these degenerate cases the distribution of the four A's and two B's. We make the convention that a line-pair meeting on the cubic C at the point P has 2-point contact with C at P; a tangent to C at P and a general line through P is a line-pair having 3-point contact with C at P; while an inflexional tangent and a general line through the inflexion forms a line-pair having 4-point contact with C.

It is evident from this table that, when the conic is proper and two of the initial six points coincide, an extra conic node is introduced; when a third point of the six coincides with this, the additional $\mathbf{C}_2$ becomes a $\mathbf{B}_3$, and each further coincidence with the same point increases by unity the reduction in class due to the binode. The initial six points, so long as the conic does not degenerate, lie on six lines through the original conic-node, which remains a conic node.

When the conic is a line-pair not meeting on the cubic curve, there is a B_3 on the surface, and the same considerations apply. When the line-pair meet at a point P on the cubic, the lines of the surface through the initial six points meet in a B_4; each contact of the line-pair at a point other than P introduces a new conic node, whereas increase of the order of contact at P transforms the B_4 into a binode of higher effect on the class of the surface. The unodes arise when the two lines of the pair coincide.

When there is both binode and conic node the surface can arise in two ways according as the lines of the surface through the initial six points are the lines through a conic node or through a binode.

TABLE I.

Nature of conic through $A_1A_2A_3A_4B_5B_6$	Proper	Two lines not meeting on C	Two lines meeting on C	Two coincident lines
No contact	C_2	B_3	—	—
1 simple contact	$2C_2$	B_3+C_2	B_4	—
2 simple contacts	$3C_2$	B_3+2C_2	B_4+C_2	—
3 simple contacts	$4C_2$	—	B_4+2C_2	U_6
1 3-point contact	B_3+C_2	$2B_3$	B_5	—
1 3-point and 1 simple contact	B_3+2C_2	$2B_3+C_2$	B_5+C_2	—
2 3-point contacts	$2B_3+C_2$	$3B_3$	—	—
1 4-point contact	B_4+C_2	—	B_6	—
1 4-point and 1 simple contact	B_4+2C_2	—	B_6+C_2	U_7
1 5-point contact	B_5+C_2	—	—	—
1 6-point contact	B_6+C_2	—	—	U_8

[It is a matter of some value to note that a concept of the forms of many of the singular surfaces can readily be obtained from this plane distribution. Since the lettering of the points gives us the behaviour of the lines through them (a coincidence of points implies a coincidence of lines, since the section is not special), we can, in a rough sketch, show these lines and picture the form of the surface by imagining a series of parallel plane cubic sections.]

Table II shows the modes in which the singular cubic surfaces can be evolved, one from another, and summarises the degeneracies given in Table I. The development can be made from any position in the table to any adjacent position in the row below (*e.g.* from B_3+2C_2 we can obtain $2B_3+C_2$, B_5+C_2, U_7, or B_4+2C_2).

TABLE II.

			B_3	C_2 B_3			$2C_2$	C_2 —	$2C_2$		
		B_3+C_2	—	B_3+C_2	B_4	—	B_4	$3C_2$	B_3+C_2	$3C_2$	
	$2B_3$	—	B_3+2C_2	B_4+C_2	B_5	—	U_6	—	B_3+2C_2	B_4+C_2	$4C_2$
B_6	—	$2B_3+C_2$	—	B_5+C_2	B_6	U_7	—	—	U_7	B_4+2C_2	—
—	U_8	$3B_3$	B_6+C_2	B_6+C_2	—	U_8	—	—	—	—	B_6+C_2

6. *The interpretation of the results in a form independent of the initial group.*

The existence of any singularity on the cubic surface implies that a double-six, coinciding in pairs, is given by lines passing through the singularity and lying on a cone (the join of the singularity to the conic of the plane section). If the surface has $r+1$ conic-nodes there are $r+1$ double-sixes lying on non-degenerate cones. For $r = 1, 2, 3$, any two of the double-sixes have four elements in common (paired in different ways) coinciding in a line of the cubic surface, one of the $\frac{1}{2}r(r+1)$ joins of the nodes; each one of the $r+1$ cones touches the cubic along r of these lines, and touches one of the remaining r cones along each of these r lines. Only a single letter can be attached to each of the remaining lines.

A binode is found on the surface when a double-six cone breaks up into two separate planes. The twenty-seven letters of the lines can be written in the form of three Steiner trihedrals (i), (ii), (iii). As an example we write down the system suitable for the initial group $A_1, A_2, A_3, A_4, B_5, B_6$, but the method is quite general. The trihedrals (i) and (ii) are so arranged that the leading diagonal of either represents the three lines through the binode which lie on one of the planes. If we regard these forms as determinants:

$$\begin{vmatrix} a_1 & c_{12} & b_2 \\ b_3 & a_2 & c_{23} \\ c_{13} & b_1 & a_3 \end{vmatrix} \quad \begin{vmatrix} a_4 & c_{45} & b_5 \\ b_6 & a_5 & c_{56} \\ c_{46} & b_4 & a_6 \end{vmatrix} \quad \begin{vmatrix} c_{14} & c_{35} & c_{26} \\ c_{36} & c_{24} & c_{15} \\ c_{25} & c_{16} & c_{34} \end{vmatrix}$$

(i) (ii) (iii)

two positive terms, one from (i) and one from (ii), form one-half the lines of a double-six (they either are six mutually skews or lie on a quadric). The nine lines of either trihedral coalesce in threes with the lines of the leading diagonal. Any three which form a negative term coalesce in one of these points. If there is a second binode, the two binodes have a plane in common which we can take as that of the trihedral (ii); we can arrange trihedral (iii) to give the other plane of the second binode by its leading

diagonal. The lines of the common trihedral (ii) coalesce in the join of the binodes. For three binodes the join of any two is represented by the nine coalescing lines of the common trihedral.

For $\mathbf{B}_4$, $\mathbf{B}_5$, $\mathbf{B}_6$ there are coincidences among the lines of one binode plane with one line of the other, and for $\mathbf{U}_6$, $\mathbf{U}_7$, $\mathbf{U}_8$ the planes coincide; the modifications in the deductions which may be made from the trihedrals are obvious. Further, it can readily be shown that the lines of the reverse diagonal of (ii) and the leading diagonal of (iii), giving half a double-six, will lie on a non-degenerate quadric cone if two of the lines of the plane in (ii) coincide, and the surface with a binode will also possess a conic node. The six lines of (ii) coalescing in the join of the nodes are three pairs of the double-six of lines through the conic node, and its quadric cone has therefore three-line contact.

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ON AN EXTENSION OF SOME FORMULAE OF RAMANUJAN

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1. Among a series of remarkable results left to us by Srinivasa Ramanujan the following formulae* call for attention:

$$\sum_{n=1}^{\infty} \frac{n^{13}}{e^{2\pi n}-1} = \tfrac{1}{24}; \tag{1}$$

$$\frac{1}{\sqrt[4]{\alpha}}\left\{1+4\alpha\int_0^{\infty}\frac{xe^{-\alpha x^2}}{e^{2\pi x}-1}\,dx\right\} = \frac{1}{\sqrt[4]{\beta}}\left\{1+4\beta\int_0^{\infty}\frac{xe^{-\beta x^2}}{e^{2\pi x}-1}\,dx\right\} \quad (\alpha\beta=\pi^2); \tag{2}$$

$$\left.\begin{aligned} \phi(n) &= \frac{1}{n}\sqrt{\left(\frac{2}{n}\right)}\,\psi\left(\frac{1}{n}\right)-\psi(n), \qquad \phi(n)=\int_0^{\infty}\frac{\cos\pi nx}{e^{2\pi\sqrt{x}}-1}\,dx,\\ \psi(n) &= \frac{1}{n}\sqrt{\left(\frac{2}{n}\right)}\,\phi\left(\frac{1}{n}\right)+\phi(n), \qquad \psi(n)=\int_0^{\infty}\frac{\sin\pi nx}{e^{2\pi\sqrt{x}}-1}\,dx+\frac{1}{2\pi n}; \end{aligned}\right\} \tag{3}$$

and so on.

These formulae are connected with the function

$$\sigma_0(x) = \sum_{n=1}^{\infty} e^{-2\pi nx} = \frac{1}{e^{2\pi x}-1}, \tag{4}$$

which, with respect to its poles, has the following expansion

$$\sigma_0(x) = -\tfrac{1}{2}+\frac{1}{2\pi x}+\frac{x}{\pi}\sum_{n=1}^{\infty}\frac{1}{x^2+n^2}. \tag{5}$$

The object of the present note is to extend the above-mentioned formulae to a function $\sigma(x)$, having an infinite number of poles at the points $x=\pm ni$, but with residues $\tau(n)$, where $\tau(n)$ denotes the number of divisors of n.

* *Collected papers*, 326 (Question 387), 75 (14), 60 (10) and (10′).

2. I have proved in my note "On Voronoï's sum-formula"* that the function

$$\sigma(x) = 2 \sum_{n=1}^{\infty} \tau(n) \left[K_0\{4\pi\epsilon \sqrt{(nx)}\} + K_0\{4\pi\bar{\epsilon} \sqrt{(nx)}\}\right], \tag{6}$$

$$\epsilon = e^{\frac{1}{4}i\pi}, \quad \bar{\epsilon} = e^{-\frac{1}{4}i\pi},$$

where $K_0(x)$ denotes Macdonald's function, is represented by the series

$$\sigma(x) = -\gamma - \tfrac{1}{2}\log x - \frac{1}{4\pi x} + \frac{x}{\pi} \sum_{n=1}^{\infty} \frac{\tau(n)}{x^2+n^2}, \tag{7}$$

where γ is Euler's constant.

Since

$$\int_0^{\infty} K_0(ax)\, x^{\mu-1}\, dx = \frac{2^{\frac{1}{2}\mu}\, \Gamma^2(\frac{1}{2}\mu)}{a^{\mu}} \quad [R(\mu) > 0], \tag{8}$$

we have, by Mellin's formula,

$$K_0\{4\pi\epsilon \sqrt{(nx)}\} + K_0\{4\pi\bar{\epsilon} \sqrt{(nx)}\} = \frac{1}{2\pi i} \int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \frac{\Gamma^2(s) \cos \frac{1}{2}\pi s}{(2\pi)^{2s}(nx)^s}\, ds. \tag{9}$$

On account of the absolute convergence of the series

$$\zeta^2(s) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} \quad [R(s) > 1], \tag{10}$$

it is permissible to interchange the signs $\int$ and Σ, and so we get

$$\sigma(x) = \frac{1}{2\pi i} \int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \frac{\zeta^2(1-s)\, ds}{2 \cos \frac{1}{2}\pi s\, x^s}. \tag{11}$$

By using this formula we may establish the results analogous to (1), (2), and (3).

3. Since

$$\int_0^{\infty} J_0(ax)\, x^{1-s}\, dx = \frac{\pi}{a^{2-s}\, 2^{s-1} \sin \frac{1}{2}\pi s\, \Gamma^2(\frac{1}{2}s)} \quad [\tfrac{1}{2} < R(s) < 2], \tag{12}$$

we get, from (11),

$$\int_0^{\infty} x J_0(ax)\, \sigma(x)\, dx = \frac{1}{2\pi i} \int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \frac{\pi \zeta^2(1-s)\, ds}{\sin \pi s\, 2^{s-1}\, \Gamma^2(\frac{1}{2}s)\, a^{2-s}}. \tag{13}$$

* *Messenger of Math.* (2), 58 (1928), 30–32.

Let $ABCD$ be the contour of the rectangle

$$A(\tfrac{3}{2}+it), \quad B(-\tfrac{1}{2}+it), \quad C(-\tfrac{1}{2}-it), \quad D(\tfrac{3}{2}-it).$$

Consider the integral

$$\frac{1}{2\pi i}\int_{ABCD}\frac{\pi\zeta^2(1-s)\,ds}{\sin\pi s\,2^{s-1}\Gamma^2(\frac{1}{2}s)\,a^{s-2}} \quad (a>0).$$

The sum of the residues of the function

$$\omega(s)=\frac{\pi\zeta^2(1-s)}{\sin\pi s\,2^{s-1}\Gamma^2(\frac{1}{2}s)\,a^{s-2}}$$

at its poles $s=1$ and $s=0$, inside $ABCD$, is $-\dfrac{1}{4\pi a}+\dfrac{1}{2a^2}$.

By Cauchy's theorem,

$$\frac{1}{2\pi i}\int_{DA}\omega(s)\,ds+\frac{1}{2\pi i}\int_{AB}\omega(s)\,ds+\frac{1}{2\pi i}\int_{CD}\omega(s)\,ds = -\frac{1}{4\pi a}+\frac{1}{2a^2}+\frac{1}{2\pi i}\int_{CB}\omega(s)\,ds.$$

The integrals

$$\int_{AB}\omega(s)\,ds \quad \text{and} \quad \int_{CD}\omega(s)\,ds$$

tend to zero as $t\to\infty$, and so

$$\int_0^\infty xJ_0(ax)\,\sigma(x)\,dx=-\frac{1}{4\pi a}+\frac{1}{2a^2}+\frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}\omega(s)\,ds.$$

Writing $s=1-\sigma$, we see that

$$\frac{1}{4\pi a}+\int_0^\infty xJ_0(ax)\,\sigma(x)\,dx = \frac{1}{2a^2}+\left(\frac{2\pi}{a}\right)^3\frac{1}{2\pi i}\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty}\frac{\pi\zeta^2(1-\sigma)\,d\sigma}{\sin\pi\sigma\,2^{\sigma-1}\Gamma^2(\frac{1}{2}\sigma)(4\pi^2/a)^{2-\sigma}}, \quad (14)$$

and so

$$\frac{1}{4\pi a}+\int_0^\infty xJ_0(ax)\,\sigma(x)\,dx=\frac{1}{2a^2}+\left(\frac{2\pi}{a}\right)^3\int_0^\infty xJ_0\left(\frac{4\pi^2}{a}x\right)\sigma(x)\,dx.$$

Therefore

$$\sqrt{a^3}\int_0^\infty xJ_0(ax)\left\{\sigma(x)+\frac{1}{4\pi x}\right\}dx=\sqrt{b^3}\int_0^\infty xJ_0(bx)\left\{\sigma(x)+\frac{1}{4\pi x}\right\}dx, \quad (15)$$

where $$ab=4\pi^2.$$

This formula is analogous to (2), when we express it in the form

$$\sqrt{a^3}\int_0^\infty xe^{-a^2x^2}\left\{\sigma_0(x)-\frac{1}{2\pi x}\right\}dx=\sqrt{b^3}\int_0^\infty xe^{-b^2x^2}\left\{\sigma_0(x)-\frac{1}{2\pi x}\right\}dx, \quad (16)$$

$$ab=\pi.$$

It is not difficult to prove that

$$\int_0^\infty J_0\{4\pi\sqrt{(ax)}\}\left\{\sigma(x)+\frac{1}{4\pi x}\right\}dx=\frac{1}{2\pi}\left\{\sigma(a)+\frac{1}{4\pi a}\right\}. \quad (17)$$

This result is analogous to the well-known Legendre's integral

$$\int_0^\infty \sin 2\pi ax\left\{\sigma_0(x)-\frac{1}{2\pi x}\right\}dx=\tfrac{1}{2}\left\{\sigma_0(a)-\frac{1}{2\pi a}\right\}. \quad (18)$$

4. Now let us suppose that

$$\left.\begin{aligned}\phi(a)&=\int_0^\infty \frac{J_0(\bar\epsilon a\sqrt{x})+J_0(\epsilon a\sqrt{x})}{4}\,\sigma(\sqrt{x})\,dx,\\ \psi(a)&=\int_0^\infty \frac{J_0(\bar\epsilon a\sqrt{x})-J_0(\epsilon a\sqrt{x})}{4i}\,\sigma(\sqrt{x})\,dx-\frac{1}{a^2}.\end{aligned}\right\} \quad (19)$$

From (12) it follows that

$$\phi(a)=\frac{1}{2\pi i}\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty}\frac{\pi\zeta^2(1-s)\,ds}{2^s\cos\frac{1}{2}\pi s\cos\frac{1}{4}\pi s\,\Gamma^2(\frac{1}{2}s)\,a^{2-s}}, \quad (20)$$

$$\psi(a)=\frac{1}{2\pi i}\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty}\frac{\pi\zeta^2(1-s)\,ds}{2^s\cos\frac{1}{2}\pi s\sin\frac{1}{4}\pi s\,\Gamma^2(\frac{1}{2}s)\,a^{2-s}}-\frac{1}{a^2}. \quad (21)$$

Hence

$$\psi(a)-\phi(a)=\frac{1}{2\pi i}\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty}\frac{\pi(\cos\frac{1}{4}\pi s-\sin\frac{1}{4}\pi s)\,s^2(s)\,ds}{2^{s-2}\pi^{2s-1}\sin\pi s\,\Gamma^2(\frac{1}{2}-\frac{1}{2}s)\,a^{2-s}}-\frac{1}{a^2}.$$

Write $$\omega(s)=\frac{\pi(\cos\frac{1}{4}\pi s-\sin\frac{1}{4}\pi s)\,\zeta^2(s)}{2^{s-2}\pi^{2s-1}\sin\pi s\,\Gamma^2(\frac{1}{2}-\frac{1}{2}s)\,a^{2-s}}.$$

By Cauchy's theorem,

$$\frac{1}{2\pi i}\int_{ABCD}\omega(s)\,ds=\frac{1}{a^2}.$$

The integrals along AB and CD tend to zero as $t \to \infty$; therefore

$$\psi(a)-\phi(a) = \frac{8\sqrt{2}}{a^3}\pi^3 \frac{1}{2\pi i}\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \frac{\pi\zeta^2(1-\sigma)\,d\sigma}{2^\sigma \cos\frac{1}{2}\pi\sigma \cos\frac{1}{4}\pi\sigma\,\Gamma^2(\frac{1}{2}\sigma)(4\pi^2/a)^{2-\sigma}},$$

and so

$$\psi(a)-\phi(a) = 8\sqrt{2}\left(\frac{\pi}{a}\right)^3 \phi\left(\frac{4\pi^2}{a}\right). \tag{22}$$

Similarly,

$$\psi(a)+\phi(a) = 8\sqrt{2}\left(\frac{\pi}{a}\right)^3 \psi\left(\frac{4\pi^2}{a}\right). \tag{23}$$

These equalities may be written as follows:

$$\left.\begin{aligned} \phi(a) &= \sqrt{\left(\frac{2b^3}{a^3}\right)}\,\psi(b)-\psi(a), \\ \psi(a) &= \sqrt{\left(\frac{2b^3}{a^3}\right)}\,\phi(b)+\phi(a), \end{aligned}\right\} \tag{24}$$

where

$$ab = 4\pi^2.$$

5. Let us establish the last formula analogous to (1). It is easy to see that

$$\sum_{n=1}^{\infty} \tau(n)\, n^{4m+1}\sigma(n) = \frac{1}{2\pi i}\int_{ai+\infty}^{ai-\infty} \frac{\zeta^2(1-s)\,\zeta^2(s-4m-1)}{2\cos\frac{1}{2}\pi s}\,ds, \tag{25}$$

where

$$4m+2 < a < 4m+3.$$

By integrating the function

$$\omega(s) = \frac{\zeta^2(1-s)\,\zeta^2(s-4m-1)}{2\cos\frac{1}{2}\pi s}$$

along the contour $ABCD$,

$$A(a+it), \quad B(-a+it), \quad C(-a-it), \quad D(a-it),$$

where

$$0 < \alpha = a-4m-2 < 1,$$

we obtain

$$\frac{1}{2\pi i}\int_{ABCD}\omega(s)ds = \frac{d}{ds}\left[\frac{s^2\,\zeta^2(1-s)\,\zeta^2(s-4m-1)}{2\cos\frac{1}{2}\pi s}\right]_{s=0} + \frac{d}{ds}\left[\frac{(s-4m-2)\,\zeta^2(s-4m-1)\,\zeta^2(1-s)}{2\cos\frac{1}{2}\pi s}\right]_{s=4m+2}$$

$$(m > 0).$$

Hence

$$\sum_{n=1}^{\infty} \tau(n)\, n^{4m+1}\, \sigma(n) = -2\gamma\, \zeta^2(-4m-1) + 2\zeta(-4m-1)\, \zeta'(-4m-1) + \frac{1}{2\pi i} \int_{-a-i\infty}^{-a+i\infty} \frac{\zeta^2(1-s)\, \zeta^2(s-4m-1)}{2\cos \frac{1}{2}\pi s}\, ds.$$

Putting $s = 4m+2-\sigma$, we obtain

$$\frac{1}{2\pi i} \int_{-a-i\infty}^{-a+i\infty} \frac{\zeta^2(1-s)\, \zeta^2(s-4m-1)}{2\cos \frac{1}{2}\pi s}\, ds = -\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{\zeta^2(1-\sigma)\, \zeta^2(\sigma-4m-1)}{2\cos \frac{1}{2}\pi \sigma}\, d\sigma = -\sum_{n=1}^{\infty} \tau(n)\, n^{4m+1}\, \sigma(n),$$

and so

$$\sum_{n=1}^{\infty} \tau(n)\, n^{4m+1}\, \sigma(n) = \zeta^2(-4m-1) \left\{ \frac{\zeta'(-4m-1)}{\zeta(-4m-1)} - \gamma \right\} \quad (m > 0). \tag{26}$$

Since

$$\frac{\zeta'(1-s)}{\zeta(1-s)} = \log 2\pi + \frac{1}{s} - \frac{\Gamma'(1+s)}{\Gamma(1+s)} - \frac{\pi}{2} \tan \tfrac{1}{2}\pi s - \frac{\zeta'(s)}{\zeta(s)},$$

and

$$\frac{\Gamma'(4m+3)}{\Gamma(4m+3)} = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \frac{1}{4m+2} - \gamma,$$

we have

$$\sum_{n=1}^{\infty} \tau(n)\, n^{4m+1}\, \sigma(n) = \zeta^2(-4m-1) \left\{ \log 2\pi - 1 - \tfrac{1}{2} - \tfrac{1}{3} - \ldots - \frac{1}{4m+1} - \frac{\zeta'(4m+2)}{\zeta(4m+2)} \right\} \tag{27}$$

$(m > 0)$,

where

$$\zeta(-4m-1) = -\frac{B_{2m+1}}{4m+2}, \quad \zeta(4m+2) = \frac{(2\pi)^{4m+2}\, B_{2m+1}}{2\,(4m+2)!},$$

and B_{2m+1} denotes a Bernoullian number.

This formula is analogous to Watson's result,

$$\sum_{n=1}^{\infty} n^{4m+1}\, \sigma_0(n) = \frac{B_{2m+1}}{8m+4} \quad (m > 0). \tag{28}$$

For $m = 0$ the function $\omega(s)$ has one more simple pole $s = 1$ with the residue $-1/(16\pi)$. In this case we have

$$\sum_{n=1}^{\infty} \tau(n)\, n\sigma(n) = \frac{1}{144}\left\{\log 2\pi - 1 - \frac{6}{\pi^2}\,\zeta'(2)\right\} - \frac{1}{32\pi}, \tag{29}$$

analogous to

$$\sum_{n=1}^{\infty} n\sigma_0(n) = \frac{1}{24} - \frac{1}{8\pi}. \tag{30}$$

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SOME UNIQUENESS THEOREMS

By M. L. CARTWRIGHT.

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1. Let $\phi(z)$ be regular for $|z|<1$, and $\psi(z)$ for $|z|>1$. Suppose that $\psi(z)$ is continuous on $|z|=1$, and that the boundary values of $\phi(z)$ and $\psi(z)$ coincide on a certain arc of $|z|=1$. Then $\psi(z)$ is the analytical continuation of $\phi(z)$ outside $|z|=1$, and if the boundary values coincide at all points on $|z|=1$, the function is regular at all points on $|z|=1$. Since $\psi(z)$ is regular outside $|z|=1$, we have a function which is regular everywhere, and therefore reduces to a constant. The object of this paper is to discuss certain cases in which the coincidence of the boundary values on an arc of $|z|=1$ implies that $\phi(z)=\psi(z)=$ a constant. The theorems were suggested by a theorem of Levinson, and some theorems of Mandelbrojt.

Let $\theta(n)$ denote a non-decreasing function of n such that

$$\sum_1^\infty \frac{\theta(n)}{n^2} \tag{1.1}$$

is divergent. Levinson* proved

THEOREM A. *Suppose that*

$$f(\theta)\sim \sum_{n=-\infty}^{\infty} a_n e^{ni\theta} \tag{1.2}$$

is integrable† in the sense of Lebesgue, and that

$$|a_{-n}|<e^{-\theta(n)} \quad (n=1,\,2,\,\ldots). \tag{1.3}$$

Suppose, further, that $\phi(z)$ is regular for $|z|<1$, and that

$$\int_0^{2\pi} \log^+|\phi(re^{i\theta})|\,d\theta<K \quad (r<1). \tag{1.4}$$

* N. Levinson, "On a class of non-vanishing functions", *Proc. London Math. Soc.* (2) (unpublished). I quote from a MS. which Dr. Levinson showed me about six months ago. The paper also contains important theorems of a similar character on Fourier transforms.

† The proof seems to require that $\Sigma|a_{-n}|$ should be convergent.

Then, if $f(\theta) = \phi(e^{i\theta})$ *everywhere in a certain interval, no matter how small,* $f(\theta) \equiv \phi(e^{i\theta})$.

Mandelbrojt* considers Fourier series in which a_n and a_{-n} satisfy the same condition; and although the hypotheses are much more restrictive than Levinson's in this respect, they suggest that a certain sequence $\{a_{-n_i}\}$ may be exempted from the condition (1.3) without affecting the conclusion, in particular a sequence for which $\Sigma 1/n_i$ is convergent need not satisfy (1.3). In Levinson's theorem, and in these particular theorems of Mandelbrojt, it is essential to know that the boundary values coincide *in a certain interval*, and not merely in a set of positive measure, as is the case in certain theorems about the boundary values of two analytic functions, and in the usual theory of quasi-analytic functions. Levinson's result is deduced from the theory of Fourier transforms, and Mandelbrojt's from the classical theory of quasi-analytic functions.

I shall obtain two results combining the characteristics of Levinson's theorem with the gap series idea of Mandelbrojt. The method used here depends on some theorems on interpolation of integral functions which are of interest in themselves. The hypothesis (1.4) appears to be quite unnecessarily heavy, and it is unnecessary to suppose that $f(\theta)$ is integrable; for the terms in which $n \geqslant 0$ can be absorbed into $\phi(e^{i\theta})$, and the remaining terms form the boundary function of a function $\psi(z)$ regular outside $|z| = 1$.

2. Let $\chi(y)$ be a positive non-decreasing function of y such that

$$\int_1^\infty \chi(y)\,\frac{dy}{y^2} \tag{2.1}$$

is convergent. The main theorem on integral functions which we shall use is

THEOREM I. *Suppose that* $F(z)$ *is an integral function such that*

$$\overline{\lim_{r\to\infty}}\ \frac{\log M(r)}{r} = k < \pi, \quad M(r) = \max_{0\leqslant\theta<2\pi} |F(re^{i\theta})|, \tag{2.2}$$

$$\log|F(x \pm iy)| < \chi(y), \quad -1 \leqslant x \leqslant 1, \quad y > 0, \tag{2.3}$$

$$\log|F(in)| < -\theta(n) \tag{2.4}$$

in a set of positive integers n *of minimal density* d *greater than* k/π.

Then $F(z) \equiv 0$.

* S. Mandelbrojt, *Séries de Fourier et classes quasi-analytiques de fonctions* (Paris, 1935), 141–143.

In particular, the result holds if (2.2) *and* (2.4) *hold and*

$$\overline{\lim_{y\to\infty}} \frac{\log\log|F(iy)|}{\log y} = \tau < 1. \tag{2.5}$$

Let $N(r)$ be the number of numbers n for which (2.4) holds. Then the minimal density d is defined by

$$\lim_{\xi\to 1-0} \underline{\lim_{r\to\infty}} \frac{N(r)-N(r\xi)}{r(1-\xi)} = d.$$

Pólya* has shown that this limit exists, and it is obviously less than or equal to 1.

If (2.2) and (2.5) hold, it is easy to show that

$$F(re^{i\theta}) = O\{\exp(kr|\cos\theta|+r^{\tau'})\} \quad (\tau<\tau'<1),$$

uniformly in θ, and this includes (2.3). Theorem I is obtained from

THEOREM II. *Suppose that $F(z)$ is regular for $R(z)\geqslant 0$ and that*

$$\overline{\lim_{r\to\infty}} \frac{\log M(r,\tfrac{1}{2}\pi)}{r} = k < \infty, \quad M(r,\tfrac{1}{2}\pi) = \max_{-\frac{1}{2}\pi\leqslant\theta\leqslant\frac{1}{2}\pi} |F(re^{i\theta})|, \tag{2.6}$$

that

$$\int_1^\infty \{\log^+|F(iy)|+\log^+|F(-iy)|\}\frac{dy}{y^2} \tag{2.7}$$

is convergent, and that

$$\int_1^\infty \log|F(iy)|\frac{dy}{y^2} \tag{2.8}$$

is divergent. Then $F(z)\equiv 0$.

This theorem can very easily be deduced from Carleman's formula. It is substantially the result used by Levinson† in his proof of Ingham's theorem on Fourier transforms.

3. The uniqueness theorem which we obtain from Theorem I is

THEOREM III. *Let*

$$\psi(z) = \sum_{n=0}^{\infty} a_n z^{-n}, \quad \phi(z) = \sum_{n=0}^{\infty} b_n z^n$$

* G. Pólya, *Math. Zeitschrift*, 29 (1929), 559.

† N. Levinson, *Journal London Math. Soc.*, 11 (1936), 6–7.

be regular for $|z|>1$, *and* $|z|<1$ *respectively. Suppose that*

$$(3.1)\qquad \overline{\lim_{r\to 1+0}}\frac{\log\log M(r,\psi)}{\log\{1/(r-1)\}}<\infty,\quad \overline{\lim_{r\to 1-0}}\frac{\log\log M(r,\phi)}{\log\{1/(1-r)\}}<\infty,$$

where

$$M(r,\psi)=\max_{0\leqslant\theta\leqslant 2\pi}|\psi(re^{i\theta})|,$$

and that

$$(3.2)\qquad \overline{\lim_{r\to 1+0}}\int_\alpha^\beta|\psi(re^{i\theta})|\,d\theta<\infty,\quad \overline{\lim_{r\to 1-0}}\int_\alpha^\beta|\phi(re^{i\theta})|\,d\theta<\infty\quad(\alpha<\beta).$$

Suppose, further, that

$$(3.3)\qquad |a_n|\leqslant e^{-\theta(n)}$$

in a set of n *of minimal density* $d>1-(\beta-\alpha)/(2\pi)$, *and that*

$$(3.4)\qquad \lim_{r\to 1-0}\int_\alpha^\beta\left|\psi\left(\frac{e^{i\theta}}{r}\right)-\phi(re^{i\theta})\right|d\theta=0.$$

Then

$$\psi(z)=\phi(z)\equiv a_0=b_0.$$

If $f(\theta)$ satisfies the hypotheses of Theorem A, and $\Sigma|a_{-n}|$ is convergent, we can write

$$\psi(z)=\sum_1^\infty a_{-n}z^{-n},\quad \phi_1(z)=\sum_0^\infty a_n z^n.$$

Since $\Sigma|a_{-n}|$ is convergent, $\psi(e^{i\theta})$ is continuous, and so

$$\phi_1(e^{i\theta})=f(\theta)-\psi(e^{i\theta})$$

is integrable. It follows that $\psi(z)$ and $\phi(z)-\phi_1(z)$ satisfy the hypotheses of Theorem III for some α,β. Hence $\psi(z)\equiv\phi(z)-\phi_1(z)=0$, which states rather more that the result of Theorem A.

*It follows from** (3.2) *and* (3.4) *that* $\psi(z)$ *is the analytical continuation of* $\phi(z)$ *across the arc* $\alpha<\arg z<\beta$, $|z|=1$, *and that* $\psi(z)$ *is regular on the arc.* If $e^{-\theta(n)}$ in (3.3) is replaced by 0, $\psi(z)$ becomes a gap series, and the result is a corollary of a theorem of Pólya†; for then $\psi(z)$ has at least one singularity on every arc of length $2\pi(1-d)$ on its circle of convergence. This remains true‡ when (3.3) is replaced by $|a_n|\leqslant e^{-\gamma n}$, $\gamma>0$.

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* This is easily proved by the method used in a similar well-known theorem in which the functions are continuous on the arc. See Hurwitz-Courant, *Funktionentheorie* (Berlin, 1929), 372.

† G. Pólya, *Math. Zeitschrift*, 29 (1929), 626.

‡ See §6–end.

4. *Proof of Theorem I.* It will be sufficient to show that (2.8) is divergent. For (2.7) follows from (2.3) and (2.1). It follows from the definition of minimal density that for every $d' < d$ we can find a number ξ_0 such that

$$\lim_{r\to\infty} \frac{N(r)-N(r\xi)}{r(1-\xi)} > d'$$

for $\xi_0 \leqslant \xi < 1$. Since (2.4) holds in a set of minimal density greater than k/π, we can choose $d' > k/\pi$, and then d'', so that

$$k/\pi < d'' < d' < d;$$

and then

$$\frac{N(r)-N(r\xi_0)}{r(1-\xi_0)} > d'' \tag{4.1}$$

for $r > r_0'(\xi_0, d'')$.

Suppose that $F(z) \not\equiv 0$; as we have already observed, (2.7) holds. Hence, by a theorem of Levinson*, for every $\delta > 0$ we have

$$n(r, \delta) \sim Br, \quad B \leqslant k/\pi < 1,$$

where $n(r, \delta)$ is the number of zeros of $f(z)$ for which

$$|z| \leqslant r, \quad |\tfrac{1}{2}\pi - \arg z| \leqslant \delta.$$

It follows that

$$n(r, \delta) - n(r\xi_0, \delta) < Br(1-\xi_0) + \epsilon r$$

for every $\epsilon > 0$, $r > r_0'(\epsilon, \xi_0)$. Since $B \leqslant k/\pi < d''$, we can choose B' and $\epsilon = \epsilon(\xi_0, d'', B')$ so that $B < B' < d''$ and

$$n(r, \delta) - n(r\xi_0, \delta) < B' r(1-\xi_0) - 2, \tag{4.2}$$

for $r > r_0''(\xi_0, B')$.

Let $r_0 = \max\{r_0'(\xi_0, d''), r_0''(\xi_0, B')\}$; let $r = \xi_0^{-m} r_0$, where m is an integer greater than 1, and consider the squares

$$-\tfrac{1}{2} \leqslant x \leqslant \tfrac{1}{2}, \quad n-\tfrac{1}{2} \leqslant y \leqslant n+\tfrac{1}{2}, \quad (z = x+iy),$$

$$n = [\xi_0 r]+1, \ [\xi_0 r]+2, \ \ldots, \ [r]-1,$$

These lie in the region $\xi_0 r \leqslant |z| \leqslant r$, $|\tfrac{1}{2}\pi - \arg z| \leqslant \delta$ for $r > r_0(\delta)$. Hence,

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* N. Levinson, *Proc. Cambridge Phil. Soc.*, 31 (1935), 336. See also M. L. Cartwright, *Proc. Cambridge Phil. Soc.*, 31 (1935), 347–350, and *Proc. London Math. Soc.* (2), 38 (1935), 179.

by (4.2), there are not more than $B'r(1-\xi_0)-2$ of these squares containing zeros, and by (4.1) there are at least

$$(d''-B')r(1-\xi_0) \tag{4.3}$$

squares not containing zeros in which (2.4) holds.

Let n_i denote an integer which determines a square without zeros for which (2.4) holds. Then, either

$$\log|F(iy)| < -K\theta(n_i) \tag{4.4}$$

for $n_i-\frac{1}{6} \leqslant y \leqslant n_i+\frac{1}{6}$, or else

$$\chi(y) > K\theta(n_i) \tag{4.5}$$

for $y \geqslant n_i+\frac{1}{2}$, K being a number independent of n_i in both cases. To see this, suppose that

$$\log|F(iy_0)| \geqslant -K\theta(n_i)$$

for some y_0 such that $n_i-\frac{1}{6} \leqslant y \leqslant n_i+\frac{1}{6}$, and that

$$\chi(n_i+\tfrac{1}{2}) \leqslant K\theta(n_i).$$

Then $\chi(y) \leqslant \chi(n_i+\frac{1}{2})$ for $y \leqslant n_i+\frac{1}{2}$, and so, by (2.3),

$$\log|F(z)| \leqslant K\theta(n_i)$$

for $|z-iy_0| \leqslant \frac{1}{3}$. This circle lies inside the square; and so, applying Carathéodory's theorem to $\log F(z)$ in it, we have

$$\big|\log|F(iy)|\big| \leqslant HK\theta(n_i),$$

where H is a numerical constant, for $|z-iy_0| \leqslant \frac{1}{6}$, *i.e.*

$$\log|F(in)| \geqslant -HK\theta(n_i).$$

Choosing $K = H^{-1}$, we have a contradiction of (2.4); and so either (4.4) or (4.5) holds.

Let Σ' denote summation over numbers n_i, for which (4.5) holds. Then

$$\begin{aligned}\Sigma' \frac{\theta(n_i)}{n_i^2} &< \frac{1}{K}\,\Sigma' \frac{\chi(n_i+\frac{1}{2})}{n_i^2} \\ &< \frac{4}{K}\,\Sigma' \int_{n_i+\frac{1}{2}}^{n_i+1} \chi(y)\frac{dy}{y^2} \\ &< \frac{4}{K}\int_1^{\infty} \chi(y)\frac{dy}{y^2},\end{aligned}$$

which is convergent. Hence the series $\Sigma\theta(n)/n^2$ remains divergent when the terms for which (4.5) holds are omitted. Let Σ'' denote summation over numbers n or n_i, as the case may be, which are not included in Σ'.

Now let $\log^-|F(iy)| = \log|F(iy)|$ when $\log|F(iy)| < 0$, and 0 otherwise. Then, by (4.4),

$$I_m = \int_{\xi_0^{-m+1} r_0}^{\xi_0^{-m} r_0} \log^-|F(iy)| \frac{dy}{y^2} < -K \sum_{\xi_0^{-m+1} r_0 < n_i < \xi_0^{-m} r_0}'' \theta(n_i) \int_{n_i - \frac{1}{6}}^{n_i + \frac{1}{6}} \frac{dy}{y^2}$$

$$< -\frac{K}{3} \sum_{\xi_0^{-m+1} r_0 < n_i < \xi_0^{-m} r_0}'' \frac{\theta(n_i)}{n_i^2}.$$

By (4.3) the number of terms on the right-hand side together with similar terms belonging to Σ' is at least $(d''-B')\,\xi_0^{-m} r_0(1-\xi_0)$. Hence

$$I_m < -\frac{K}{3}(d''-B')\,\xi_0^{-m} r_0(1-\xi_0)\frac{\theta([\xi_0^{-m+1} r_0])}{\xi_0^{-2m} r_0^2} + \frac{K}{3}\Sigma'\frac{\theta(n_i)}{n_i^2}$$

$$= -\frac{K}{3}(d''-B')(1-\xi_0)\frac{\theta([\xi_0^{-m+1} r_0])}{\xi_0^{-m} r_0} + \frac{K}{3}\Sigma'\frac{\theta(n_i)}{n_i^2}.$$

Putting $m = 1, 2, \ldots, M$, we have

$$J = \int_1^\infty \log^-|F(iy)| \frac{dy}{y^2} < \sum_1^M I_m$$

$$< -\frac{K}{3r_0}(d''-B')(1-\xi_0)\sum_2^M \frac{\theta([\xi_0^{-m+1} r_0])}{\xi_0^{-m}} + \frac{K}{3}\sum_1^\infty{}' \frac{\theta(n)}{n^2}.$$

Since $\theta(n)$ is non-decreasing,

$$\sum_{[\xi_0^{-m+2} r_0]+1}^{[\xi_0^{-m+1} r_0]} \frac{\theta(n)}{n^2} < \frac{\theta([\xi_0^{-m+1} r_0])}{\xi_0^{-2m+4} r_0^2}\,\xi_0^{-m+1} r_0(1-\xi_0)$$

$$= \frac{\theta([\xi_0^{-m+1} r_0])}{\xi_0^{-m+3} r_0}(1-\xi_0).$$

Hence $$J < -\frac{K}{3}(d''-B')\,\xi_0^3 \sum_{n=[r_0]+1}^{N} \frac{\theta(n)}{n^2} + \frac{K}{3}\sum_1^\infty{}' \frac{\theta(n)}{n^2},$$

where $N = [\xi_0^{-m+1} r_0]$, and this expression tends to $-\infty$ as M tends to infinity, by (1.1). It now follows from Theorem I that $F(z) \equiv 0$.

5. *Proof of Theorem III.* We may suppose, without real loss of generality, that $\alpha = k$, $\beta = 2\pi - k$, where

(5.1) $$\pi(1-d) < \pi - k < \pi.$$

Let $$F(z)=\frac{1}{2\pi i}\int_{C_1}\psi(u)\,u^{z-1}\,du-\frac{1}{2\pi i}\int_{C_2}\phi(u)\,u^{z-1}\,du,$$

where C_1 is any contour enclosing $|z|=1$, and C_2 is a contour inside $|z|=1$ enclosing the origin. Then

$$F(n)=a_n,\quad F(-n)=b_n\quad(n=1,\,2,\,\ldots),\tag{5.2}$$

$$F(0)=a_0-b_0.$$

Deform C_1 into C_1' which consists of the arcs $|z|=r_1>1$, $-k'\leqslant\arg z\leqslant k'$, where $k'>k$, and $|z|=\rho>1$, $k'\leqslant\arg z\leqslant 2\pi-k'$, and the parts of $\arg z=\pm k'$ joining them. Deform C_2 into C_2' which consists of the arcs $|z|=r_2<1$, $-k'\leqslant\arg z\leqslant k'$, and $|z|=\rho^{-1}<1$, $k'\leqslant\arg z\leqslant 2\pi-k'$, and the parts of $\arg z=\pm k'$ joining them. As $\rho\to 1$, we have

$$|I|=\left|\int_{k'}^{2\pi-k'}\left\{\psi(\rho e^{i\theta})\rho^z-\phi\left(\frac{e^{i\theta}}{\rho}\right)\rho^{-z}\right\}e^{i\theta z}\,d\theta\right|$$

$$\leqslant\int_{k'}^{2\pi-k'}\left|\psi(\rho e^{i\theta})-\phi\left(\frac{e^{i\theta}}{\rho}\right)\right|d\theta+O\left(1-\frac{1}{\rho}\right)\int_{k'}^{2\pi-k'}\left\{|\psi(\rho e^{i\theta})|+\left|\phi\left(\frac{e^{i\theta}}{\rho}\right)\right|\right\}d\theta.$$

Hence I tends to 0 as $\rho\to 1$ in virtue of (3.2) and (3.4); and, making $\rho\to 1$, we have

$$F(z)=\frac{1}{2\pi}\int_{-k'}^{k'}\{\psi(r_1e^{i\theta})r_1^{\,z}e^{i\theta z}-\phi(r_2e^{i\theta})r_2^{\,z}e^{i\theta z}\}\,d\theta\tag{5.3}$$

$$+\frac{1}{2\pi i}\int_1^{r_1}\{\psi(re^{-ik'})r^{z-1}e^{-izk'}-\psi(re^{ik'})r^{z-1}e^{izk'}\}\,dr$$

$$+\frac{1}{2\pi i}\int_{r_2}^{1}\{\phi(re^{-ik'})r^{z-1}e^{-izk'}-\phi(re^{ik'})r^{z-1}e^{izk'}\}\,dr$$

$$=I_1+I_2+I_3.$$

Now, since (3.2) holds,

$$\int_\rho^{r_1}dr\int_k^{2\pi-k}|\psi(re^{i\theta})|\,d\theta=O(1)\tag{5.4}$$

as $\rho\to 1$. Hence we can find $k'>k$, and as near to k as we please, such that

$$\int_\rho^{r_1}|\psi(re^{ik'})|\,d\theta=O(1),\qquad\int_\rho^{r_1}|\psi(re^{-ik'})|\,d\theta=O(1)\tag{5.5}$$

as $\rho\to 1$. For if not, (5.4) is false. Choose k' so that (5.5) is satisfied and also $k'<\pi d$. This is possible in virtue of (5.1).

It follows from (5.5) that, if $z = x+iy$, we have

$$|I_2| < Kr_1^{|x|}\, e^{|y|k'}, \quad |I_3| < Kr_2^{-|x|}\, e^{|y|k'}. \tag{5.6}$$

Suppose that the upper limits in (3.1) are less than $\sigma > 1$. Then

$$|I_1| < K\{e^{(r_1-1)^{-\sigma}} r_1^{|x|} + e^{(1-r_2)^{-\sigma}} r_2^{-|x|}\}\, e^{k'|y|}.$$

Choose r_1 and r_2 so that, as $|z| \to \infty$,

$$e^{(r_1-1)^{-\sigma}} \sim r_1^{|x|} \sim e^{|x|(r_1-1)},$$

$$e^{(1-r_2)^{-\sigma}} \sim r_2^{-|x|} \sim e^{|x|(1-r_2)},$$

i.e.
$$|x| = (r_1-1)^{-1-\sigma} = (1-r_2)^{-1-\sigma}.$$

Then
$$|I_1|+|I_2|+|I_3| < Ke^{2|x|^{\sigma/(\sigma+1)}+k'|y|}.$$

Since $\sigma/(\sigma+1) < 1$, $k' < \pi d$, we see that $F(iz)$ satisfies the hypotheses (2.2) and (2.5) of Theorem I. Hence $F(z) \equiv 0$, and so

$$a_n \equiv 0, \quad b_n \equiv 0 \quad (n = 1, 2, \ldots), \quad \text{and} \quad a_0 = b_0.$$

Hence
$$\phi(z) \equiv a_0 = b_0 \equiv \psi(z).$$

6. By strengthening the hypothesis on a_n we can relax the other conditions, in particular (3.1), and the condition on the density.

THEOREM IV. *Let*

$$\psi(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \phi(z) = \sum_{n=0}^{\infty} b_n z^n$$

be regular for $|z| > 1$, *and* $|z| < 1$ *respectively. Suppose that* (3.2) *and* (3.4) *are satisfied, and that*

$$|a_n| \leqslant e^{-\gamma n} \quad (\gamma > 0) \tag{6.1}$$

in a set of n *of maximal density* $D > 1-(\beta-\alpha)/2\pi$.

Then
$$\psi(z) \equiv \phi(z) \equiv a_0 = b_0.$$

Let Σ' denote summation over integers for which (6.1) holds, and Σ'' summation over the complementary set, and let

$$\psi(z) = \psi_1(z) + \psi_2(z) = \Sigma' a_n z^{-n} + \Sigma'' a_n z^{-n}.$$

Then $\psi_1(z)$ is regular on $|z|=1$, and $\psi_2(z)$ is represented by a gap series whose coefficients have *minimal density* 1. As we observed in § 3, (3.2) and (3.4) imply that $\psi(z)$ is regular on the arc $\alpha \leqslant \arg z \leqslant \beta$, $|z|=1$, and so $\psi_2(z)=\psi(z)-\psi_1(z)$ is regular on the arc. If the coefficients have *maximal density* D, the result follows from Pólya's theorem*. The more general result will be proved here quite independently by extensions of the methods just used. The results for integral functions on which Theorem V depends have some intrinsic interest.

7. The corresponding theorem† on integral functions on which the proof depends is

THEOREM V. *Suppose that* $F(z)$ *is an integral function such that* (2.2) *holds,*

$$\overline{\lim_{y\to\infty}} \frac{\log|F(\pm iy)|}{y}=0, \tag{7.1}$$

and

$$\log|F(in)| \leqslant -\gamma n \quad (\gamma>0), \tag{7.2}$$

in a set of positive integers n *of maximal density* $D>k/\pi$. *Then* $F(z)\equiv 0$.

Using the notation of § 2, we define the maximal density D by

$$\lim_{\xi\to 1-0} \overline{\lim_{r\to\infty}} \frac{N(r)-N(r\xi)}{r(1-\xi)}=D. \tag{7.3}$$

A similar theorem may be deduced from a theorem of V. Bernstein‡ if the set of positive integers has density D.

We need two particular cases of theorems which I have proved elsewhere§.

THEOREM VI. *Suppose that* $F(z)$ *is an integral function satisfying* (2.2) *and such that*

$$h(\theta)=\overline{\lim_{r\to\infty}} \frac{\log|F(re^{i\theta})|}{r}=A\cos\theta \quad (-\tfrac{1}{2}\pi \leqslant \theta \leqslant \tfrac{1}{2}\pi), \tag{7.4}$$

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* G. Pólya, *Math. Zeitschrift*, 29 (1929), 626.

† Compare the theorem of Faber and Pólya, given by J. M. Whittaker, "Interpolatory function theory", *Cambridge Tracts*, No. 33 (1935), 57.

‡ V. Bernstein, *Rendiconti della R. Accad. Naz. d. Lincei. Classe Sci. fis. mat. e nat.* (6), 15 (1932), 30–34.

§ M. L. Cartwright, *Proc. London Math. Soc.* (2), 38 (1934), 507, Theorem I, and 537, Theorem V.

Then, given any $\epsilon, \delta, \eta, \zeta > 0$, *we can choose* R_0 *so that*

$$\log|F(re^{i\theta})| < \{A(R)\cos\theta+\epsilon\}r \tag{7.5}$$

for $|\theta| \leqslant \frac{1}{2}\pi$, $\eta R \leqslant r \leqslant R$, *and*

$$\log|F(re^{i\theta})| > \{A(R)\cos\theta-\epsilon\}r \tag{7.6}$$

for $|\theta| \leqslant \frac{1}{2}\pi-\delta$, $\eta R \leqslant r \leqslant R$, *except perhaps in circles the sum of whose radii is less than* ζR, *provided that* $R > R_0$.

Further, in (7.5) *and* (7.6) $A(R)$ *is such that**

$$\overline{\lim_{r\to\infty}} A(r) = A, \qquad \underline{\lim_{r\to\infty}} A(r) \geqslant \max\{h'(-\tfrac{1}{2}\pi-);\ -h'(\tfrac{1}{2}\pi+)\}.$$

The other result takes the place of the theorem of Levinson which we used in Theorem I.

THEOREM VII. *Suppose that* $F(z)$ *satisfies the hypotheses of Theorem VI, and that*

$$h(\theta) = B\cos\theta, \quad |\theta-\pi| \leqslant \tfrac{1}{2}\pi. \tag{7.7}$$

Then, given any $\epsilon, \delta > 0$ *and any* $\xi < 1$, *we can choose* R_0 *so that*

$$n(R, \delta)-n(R\xi, \delta) < \tfrac{1}{2}\{A(R)-B(R)+\epsilon\}R(1-\xi)\pi,$$

where

$$B(R) \leqslant A(R), \qquad \overline{\lim_{R\to\infty}} A(R) = A, \qquad \underline{\lim_{R\to\infty}} B(R) = B,$$

and $n(r, \delta)$ *denotes the number of zeros of* $F(z)$ *for which*

$$|z| \leqslant r, \quad |\tfrac{1}{2}\pi-\arg z| \leqslant \delta,$$

provided that $R > R_0$. *In particular,*

$$\overline{\lim_{r\to\infty}} \frac{n(r, \delta)-n(r\xi, \delta)}{r(1-\xi)} \leqslant \frac{A-B}{2\pi}.$$

8. From Theorem VI we deduce a result corresponding to Theorem II.

THEOREM VIII. *Suppose that* $F(z)$ *satisfies the hypotheses of Theorem VI, and that, for some* $\xi < 1$, $\zeta > 0$, $\gamma > 0$, *there exists a sequence*

$$R_1, R_2, \ldots, R_n \to \infty$$

* There is a misprint in (3.14) of the original theorem. It should be $h'(-\frac{1}{2}\pi/\rho-)$ not $-h'(-\frac{1}{2}\pi/\rho-)$.

such that

$$\log|F(iy)| < -\gamma y \tag{8.1}$$

for $\xi R_n \leqslant y \leqslant R_n$ *in a set of measure greater than* $\zeta(1-\xi)R_n$. *Then* $F(z) = 0$.

Proof. Suppose that $F(z) \not\equiv 0$. Then it follows from (2.2) and (7.4), by the methods of Phragmén and Lindelöf, that (7.7) holds for some $B \geqslant -k$. Hence $-h'(\frac{1}{2}\pi+) = h'(-\frac{1}{2}\pi-) = B \geqslant -k$; and so it follows from (7.6) that

$$\log|F(re^{i(\frac{1}{2}\pi-\delta)})| > (-k\sin\delta-\epsilon)r \tag{8.2}$$

for $\xi R_n \leqslant r \leqslant R_n$, except perhaps in a set of measure less than $\zeta_1 R_n$, for $n > n_0(\zeta_1, \delta, \xi)$. We also have

$$\log|F(re^{i\theta})| < (k\sin 3\delta+\epsilon)r \tag{8.3}$$

for $|\theta-\frac{1}{2}\pi| \leqslant 3\delta$, $r > r_0(\epsilon)$.

Let $r = r_s$ be a value for which (8.2) holds in the interval $\xi R_n \leqslant r \leqslant R_n$, and consider the circles $|z-r_s e^{i(\frac{1}{2}\pi-\delta)}| \leqslant 2\sin\delta r_s$, $|z-r_s e^{i(\frac{1}{2}\pi-\delta)}| \leqslant 2^{\frac{1}{2}}\sin\delta r_s$. The first lies in the region for which (8.3) holds, and the second has a radius $2^{-\frac{1}{2}}$ times the first. Applying a well known theorem* on the minimum modulus of a function to $F(z)$ in the larger circle, we have

$$\log|F(z)| > -H(\zeta_2)(k\sin 3\delta+\epsilon)r_s \tag{8.4}$$

in the smaller circle, except perhaps in circles the sum of whose radii is less than $\zeta_2 \sin\delta r_s$. The circle $|z-r_s^{i(\frac{1}{2}\pi-\delta)}| \leqslant 2^{\frac{1}{2}}\sin\delta r_s$ covers the interval

$$r_s\cos\delta - r_s\sin\delta \leqslant y \leqslant r_s\cos\delta + r_s\sin\delta$$

of the imaginary axis. Since ζ_1 may be chosen as small as we please, we can find $r_1 < r_2 < \ldots < r_n$ such that

$$\xi R_n \leqslant r_1 \leqslant \xi R_n(1+\sin\delta), \quad R_n(1-\sin\delta) \leqslant r_n \leqslant R_n,$$

$$r_s(1-\tfrac{3}{2}\sin\delta) < r_{s-1} < r_s(1-\sin\delta),$$

provided that $n > n_0(\xi, \delta)$. Then

$$\begin{aligned} r_{s-1}\cos\delta + r_{s-1}\sin\delta &< r_s(\cos\delta+\sin\delta)(1-\sin\delta) \\ &< r_{s+1}(\cos\delta+\sin\delta)(1-\sin\delta)^2 = r_{s+1}\{\cos\delta-\sin\delta-\sin^2\delta+O(\delta^3)\} \\ &< r_{s+1}(\cos\delta-\sin\delta), \end{aligned}$$

* The form used here is due to V. Bernstein, *Annali di Mat.* (4), 12 (1934), 72. See also G. Valiron, *Integral functions* (Toulouse, 1923); M. L. Cartwright, *Proc. London Math. Soc.* (2), 38 (1935), 419, Lemma 1.

and

$$\begin{aligned} r_{s-1}(\cos\delta+\sin\delta) &> r_s(1-\tfrac{3}{2}\sin\delta)(\cos\delta+\sin\delta) \\ &= r_s\{\cos\delta-\tfrac{1}{2}\sin\delta+O(\delta^2)\} \\ &> r_s(\cos\delta-\sin\delta) \end{aligned}$$

for sufficiently small δ. Hence the whole interval $\xi R_n \leqslant y \leqslant R_n$ is covered by the circles $|z-r_s e^{i(\frac{1}{2}\pi-\delta)}| \leqslant 2^{\frac{1}{2}}\sin\delta\, r_s$, $s=1, 2, \ldots, m$, and no part is covered more than twice. The circle determined by r_s covers an interval of length $2\sin\delta r_s$; and so

$$\sin\delta(r_1+r_2+\ldots+r_m) \leqslant R_n(1-\xi) \leqslant 2\sin\delta(r_1+r_2+\ldots+r_m).$$

The interval of length $2\sin\delta r_s$ has an excepted set of measure less than

$$\zeta_2\sin\delta(r_{s-1}+r_s+r_{s+1}) < 4\zeta_2\sin\delta r_s;$$

and so (8.4) holds except in a set of measure less than

$$4\zeta_2\sin\delta(r_1+r_2+\ldots+r_m) < 4\zeta_2 R_n(1-\xi).$$

Choose ζ_2 first so that $4\zeta_2 < 1-\zeta$, and then choose δ and ϵ so that $H(\zeta_2)(k\sin 3\delta+\epsilon) < \gamma$. Then (8.4) gives a contradiction of (8.1) in a set of measure greater than $\zeta(1-\xi)R_m$ in the interval $\xi R_n \leqslant y \leqslant R_n$. Hence $F(z) \equiv 0$.

9. *Proof of Theorem V.* It follows from the definition of maximal density that, for every $D' < D$, we can find ξ_0 such that

$$\overline{\lim_{r\to\infty}}\,\frac{N(r)-N(r\xi)}{r(1-\xi)} > D'$$

for $\xi_0 < \xi < 1$. Since $D > k/\pi$, we can choose D' and D'' so that

$$k/\pi < D'' < D' < D;$$

and then there is a sequence $R_1, R_2, \ldots, R_n \to \infty$ such that

$$\frac{N(R_n)-N(R_n\xi_0)}{R_n(1-\xi_0)} > D'' \tag{9.1}$$

for $n > n_0(\xi_0, D'')$.

Suppose that $F(z) \not\equiv 0$. Then it follows from (7.1) and (2.2) that (7.4) and (7.7) hold for some A and B such that $-k \leqslant B \leqslant A \leqslant k$; and

so, by Theorem VII, we have

$$n(R_n, \delta)-n(R_n\xi_0, \delta) < \left(\frac{A-B}{2\pi}+\epsilon\right)(1-\xi_0)R_n$$

$$< (k+\epsilon)(1-\xi_0)R_n/\pi,$$

where $n(r, \delta)$ is the number of zeros of $F(z)$ for which $|z| \leqslant r$, $|\arg z-\frac{1}{2}\pi| \leqslant \delta$. Hence we can choose C such that

$$k/\pi < C < D'',$$

$$n(R_n, \delta)-n(R_n\xi_0, \delta) < C(1-\xi_0)R_n-2, \tag{9.2}$$

for $n > n_0(C, \delta, \xi_0)$.

Consider the squares $-\frac{1}{2} \leqslant x \leqslant \frac{1}{2}$, $m-\frac{1}{2} \leqslant y \leqslant m+\frac{1}{2}$, $(z=x+iy)$, $m = [\xi_0 R_n]+1, [\xi_0 R_n]+2, \ldots, [R_n]-1$. These lie in the region

$$\xi_0 R_n \leqslant |z| \leqslant R_n, \quad |\arg z-\tfrac{1}{2}\pi| \leqslant \delta$$

for $n > n_0$. Hence, by (9.2), there are not more than $C(1-\xi_0)R_n-2$ of these squares containing zeros of $F(z)$, and by (9.1) there are at least

$$[(D''-C)(1-\xi_0)R_n]$$

squares not containing zeros in which (7.2) holds for $n > n_0$. Let m_i denote an integer which determines a square without zeros in which (7.2) holds, and suppose that for every $\epsilon > 0$ $\log|F(iy_0)| \geqslant -\epsilon R_n$ for some y_0 such that $m_i-\frac{1}{6} \leqslant y_0 \leqslant m_i+\frac{1}{6}$, $\xi_0 R_n \leqslant m_i \leqslant R_n$, $n > n_0$. Then, since

$$h(\theta) = \max(A\cos\theta, B\cos\theta),$$

we have $$\log|F(re^{i\theta}| < (k\sin\delta+\epsilon)r < (k\sin\delta+\epsilon)R_n$$

for $\xi_0 R_n \leqslant r \leqslant R_n$, $|\theta-\frac{1}{2}\pi| \leqslant \delta$, $n > n_0(\epsilon, \xi_9)$. Applying Carathéodory's theorem to $\log F(z)$ in the circle $|z-iy_0| \leqslant \frac{1}{3}$ which lies in this region, and in the square, we have

$$\left|\log|F(z)|\right| \leqslant H(k\sin\delta+2\epsilon)R_n,$$

where H is a numerical constant, for $|z-y_0| \leqslant \frac{1}{6}$,

i.e. $$\log|F(im_i)| \geqslant -H(k\sin\delta+2\epsilon)R_m.$$

By choosing ϵ and δ sufficiently small, we can obtain a contradiction of (7.2). Hence

$$\log|F(iy)| < -\gamma' R_n, \tag{9.3}$$

where γ' depends on γ only, for $m_i - \frac{1}{6} \leqslant y \leqslant m_i + \frac{1}{6}$. This is an interval of length $\frac{1}{3}$, and there are $[(D''-C)(1-\xi_0)R_n]$ such intervals. It follows that (9.3) holds in a set of measure $\frac{1}{3}[(D''-C'')(1-\xi_0)R_n]$ in the interval $\xi_0 R_n \leqslant y \leqslant R_n$ for $n > n_0$. The result now follows from Theorem VIII.

10. *Proof of Theorem IV.* We have (5.2), (5.3), (5.4), (5.5), and (5.6) as before, and it follows from (5.3) that

$$|I_1| < K'(r_1, r_2)(r_1^{|x|} + r_2^{-|x|})\, e^{k'|y|} \quad (z = x+iy),$$

where

$$K'(r_1, r_2) = \max_{-k' \leqslant \theta \leqslant k'} \{|\psi(r_1 e^{i\theta})| + |\phi(r_2 e^{i\theta})|\}.$$

We may choose r_1 and r_2 as near to 1 and k' as near to k as we please. Hence, using (5.6), we see that

$$F(re^{i\theta}) = O\{e^{(k|\sin\theta|+\epsilon)r}\}$$

for every $\epsilon > 0$. The result now follows from (6.1) and (5.2) by Theorem V.

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ON CERTAIN SETS OF ELEMENTS IN A FREE GROUP†

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1. Let $a_1, \ldots, a_p, a_1', \ldots, a_p'$ be the generators of a free group G, a_i' being the inverse of a_i (if x stands for any one of these $2p$ symbols, x' will stand for its inverse, with $x'' = x$). It is customary‡ to use "word" in its ordinary sense, relative to the "alphabet" consisting of the $2p$ letters a_i and a_i'. We shall refer to a sequence of these letters in cyclic order as a *cyclic word*, as apart from an ordinary or *true word*. Unless the contrary is indicated it is to be understood that any statement about a word, or set of words, applies both to true and cyclic words, a set of words consisting entirely of true or entirely of cyclic words.

By an *elementary reduction* of a word we shall mean the omission of consecutive letters xx'. It is well known that any word is reducible by elementary reductions, applied at random, to a unique *reduced* word§ (*i.e.* one to which no elementary reduction is applicable). We shall describe the inverse of an elementary reduction as an *elementary insertion*, an elementary reduction or insertion as an *elementary transformation*, and any two words which are interchangeable by sequences of elementary transformations as *equivalent*. Thus equivalent true words determine the same element of the group and equivalent cyclic words determine the same class of elements.

† I wish to thank Mr. M. H. A. Newman for helping me to clarify the exposition of this note.

‡ *Cf.* K. Reidemeister, *Einführung in die kombinatorische Topologie* (Brunswick, 1932), 31.

§ *Cf.* Reidemeister, *loc. cit.*, 33–35. Our term reduced word, applied to cyclic words, is equivalent to Reidemeister's "short word".

By a *simple transformation* of a set of words $(W) = (W_1, \ldots, W_k)$, we shall mean either

(1) replacing x by xy and x' by $y'x'$ whenever x and x' occur in (W), x and y ($y \neq x$ or x') being two of the generating symbols†, or

(2) an elementary transformation applied to some word in (W).

By a *simple set*‡ of words we shall mean one which is derived from an independent set of letters by a sequence of simple transformations. Any sub-set of a simple set of words is also a simple set, and if $k < p$ any simple set of k words is a sub-set of a simple set of p words. For if T carries the given set into an independent set of letters, T also carries any sub-set of the former into a sub-set of the latter. Moreover, the set of k independent letters belongs to some set of p independent letters, which are transformed by the inverse of T into a simple set of p words containing the given set as a sub-set.

If t is any simple transformation of the form $x \to xy$ and if u is any elementary reduction (insertion) it is easy to verify that $tu = vt$ (u applied first), where v is u itself or the resultant of two elementary reductions (insertions). Therefore $tU = Vt$, where U is any sequence of elementary transformations and V is a sequence of elementary transformations which depends on U and t. If W and W^* are equivalent words it follows that tW and tW^* are equivalent, and hence that $T^* W^*$ is equivalent to TW, where T and T^* are sequences of simple transformations which differ at most in their component elementary transformations. Thus T, regarded as a transformation of elements rather than of true words, or of classes of elements rather than of cyclic words, coincides with T^*.

If a set of true words (W) is derived from (W^0) by a simple transformation of the first type, say $a_i \to a_i a_j$ ($j \neq i$), the elements (W^0) correspond to the elements (W) in the automorphism $a_i \to a_i a_j$, $a_k \to a_k$ ($k \neq i$). Any automorphism of G is the resultant of such automorphisms and, possibly, of§ $a_1 \to a_1'$. Therefore (W) is a simple set of words if, and only if, the

† It is to be understood that the transformation $x \to xy$ is applicable to (W) even if no word in the set contains x or x', when (W) is left unaltered.

‡ That is to say, one whose members are distinct and which does not contain two mutually inverse letters.

§ J. Nielsen, *Math. Annalen*, 79 (1919), 269–272. *These transformations are, in a sense, contragredient to those considered by Nielsen. To deduce the result quoted here from the result which he proves notice that the automorphism* $(a_1, a_2, \ldots, a_p) \to (W_1 W_2, W_2, \ldots, W_p)$ *is the resultant of* $(a_1, \ldots, a_p) \to (a_1 a_2, \ldots, a_p)$ *followed by* $(a_1, \ldots, a_p) \to (W_1, \ldots, W_p)$, *where the latter is any automorphism.*

elements (W) correspond to an independent set of generators in some automorphism of G.

The main object of this note is to give a process of reduction which, applied at random, will reduce any simple set of words to a set of independent letters†. The final section indicates how this process may be applied to the study of three-dimensional manifolds.

2. With any set of words (W) we associate a graph $g(W)$, as follows. With the group G we associate $2p$ vertices $A_1, \ldots, A_p, A_1', \ldots, A_p'$ (as with small letters, if X stands for A_i or A_i', X' will stand for A_i' or A_i). With each pair of consecutive letters xy, in each word in (W), we associate an oriented segment XY' (if $y = x'$ the segment XY' will be singular in that it will begin and end at X). We now distinguish between a set of true words and a set of cyclic words. In the first case we introduce an "origin" O, and associate an oriented segment OX' with the first letter x, in each word and an oriented segment YO with the last letter y. In the graph of a set of cyclic words there is no origin, the last letter of each word preceding the first and corresponding to a segment YX'. It will be convenient to assume that each segment XY' contains at least three edges. Then each edge incident with X is *adjacent* to an edge incident with X'.

If some sub-graph‡ g_1, of $g = g(W)$, contains a vertex X but not X' ($X = A_i$ or A_i' for some i) and meets $g - g_1$ in the vertex X, if at all, it will appear that the set (W) is "reducible", in the sense that the number

$$L(W) = \sum_{i=1}^{p} n_i$$

is reduced by a certain set of simple transformations which are determined by the graph g_1, n_i being the number of times that a_i and a_i' occur in (W).

Let g contain such a sub-graph, and suppose first that g_1 contains no vertex A_i^{ϵ} ($= A_i$ or A_i') other than X. We omit from (W) all the syllables xx' corresponding to the singular segments XX' in g_1, if there are any. This is the reduction in question unless (W) is a set of true words and g_1 contains the vertex O. In that case g_1 now consists entirely of segments OX, XO, and

$$W_\lambda = x' W_\lambda^0 x \quad (\lambda = 1, \ldots, k).$$

We then transform (W) into (W^0) by the resultant of the transformations $a_i \to x a_i x'$ $(i = 1, \ldots, p)$, followed by suitable omissions of $x'x$.

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† In a subsequent paper it will be shown that we can discover by means of this process whether or no two given sets of words are interchangeable by simple transformations.

‡ It is to be understood that none of the graphs to which we refer contains an isolated vertex.

In general, let $X_0 = X, X_1, \ldots, X_r$ be the vertices A_i^ϵ which are contained in g_1, where $r > 0$. Then each letter x_α in (W) is followed by an x_σ', where x_ρ is the letter corresponding to X_ρ, and each x_α' is preceded by an $x_\sigma (\alpha = 1, \ldots, r; \quad \rho, \sigma = 0, 1, \ldots, r)$, and the resultant of the simple transformations†

$$x_\alpha \to x_\alpha x_0 \quad (\alpha = 1, \ldots, r), \tag{1.1}$$

followed by suitable omissions of xx' and, if g_1 contains O, by the transformation

$$x' W_\lambda^0 x \to W_\lambda^0,$$

reduces $L(W)$ by precisely the number of edges in g_1 having X as an end point. For

$$x_\lambda x_\mu' \to x_\lambda x x' x_\mu' \to x_\lambda x_\mu' \quad (\lambda, \mu = 1, \ldots, r),$$

while

$$(x_\lambda x')^\epsilon \to (x_\lambda x x')^\epsilon \to x_\lambda^\epsilon \quad (W^\epsilon = W \text{ or } W'),$$

and any letters x^ϵ which correspond to singular segments XX, or to segments OX, XO, are omitted as before.

If (W^0) is the set of words obtained from (W) by the above process, it is clear that a representative graph $g(W^0)$ is obtained from $g(W)$ if we omit the edges of g_1 which are incident with X, also the adjacent edges incident with X', say XB_ρ and $X'B_\rho'$ $(\rho = 1, \ldots, m)$, and identify B_ρ with B_ρ'.

If (W) is any simple set of words our goal is to show that $g(W)$ contains such a sub-graph g_1. Let us, for the moment, admit the truth of the lemma:

LEMMA. *If (W) is a simple set of words, $g(W)$ contains acut vertex‡, other than O.*

It follows from this lemma, and induction on the number of words in (W), that $g(W)$ contains a sub-graph g_1, satisfying the required conditions. For g contains a cut vertex P. If P is an A_i^ϵ not more than one of the maximal connected sub-graphs separated from each other by P contains P', and any one of the others may be taken as g_1 with $X = P$. If P is interior to a segment XY, either X or Y is a cut vertex if XY meets the rest of g. If it does not we may take XY to be g_1 unless $Y = X'$. If $Y = X'$ and XX' does not meet the rest of g, the set (W) contains the one-

† It may happen that $x_\lambda = a_i$, $x_\mu = a_i'$, in which case $a_i \to x' a_i x$.

‡ If g consists of two sub-graphs, not necessarily connected, with a single vertex in common, the latter is called a cut vertex [*cf.* Hassler Whitney, *Trans. American Math. Soc.*, 34 (1932), 342]. The condition " other than O ", which is, of course, vacuous if the words in (W) are cyclic, must not be taken to imply that O is necessarily a cut vertex when (W) consists of true words.

letter word x^ϵ. Therefore (W) is fully reduced if it consists of a single word. Otherwise the assertion follows from the hypothesis of the induction, applied to the remaining words. If P is interior to YO, (W) being a set of true words, Y is a cut vertex if other segments of g are incident with Y. If no other segment of g is incident with Y the segment incident with Y' may be taken as g_1 unless it is OY'. If it is OY', (W) contains the word y and, as before, the assertion follows by induction.

If (W) is a simple set, it is obvious that $L(W) \geqslant k$, where k is the number of words in (W), and that (W) is a set of independent letters if $L(W) = k$. Therefore the lemma and induction on $L(W)-k$ allow us to conclude:

THEOREM. *A simple set of k words is reducible at random to a set of k independent letters in a number of steps which does not exceed $L(W)-k$.*

Thus, assuming the truth of the lemma, we have a criterion for deciding whether or no (W) is a simple set of words.

3. To prove the lemma we construct a slightly different model for $g(W)$. Let M be an orientable, three-dimensional manifold represented by an oriented region R, in a three-sphere H, bounded by $2p$ oriented two-spheres C_i, C_i' $(i = 1, \ldots, p)$, which are identified in pairs, C_i with C_i', to give M. The notation is to be such that

$$\widehat{R} = \Sigma(C_i - C_i'),$$

$\widehat{R}$ being the boundary of R. If Γ_i is the two-sphere in M corresponding to C_i, we shall refer to the side of Γ_i corresponding to C_i as the positive side, and to the one corresponding to C_i' as the negative side. Then (W) can be represented by a set of oriented circuits $w_1, \ldots, w_k$, in M, a positive intersection $w_\lambda . \Gamma_i$, that is to say a vertex at which w_λ crosses Γ_i from the negative to the positive side, being marked a_i and a negative intersection being marked a_i'. If (W) is a set of cyclic words no two of the circuits meet each other, and if it is a set of true words the circuits all meet in a single vertex O, separating the first intersections $w_\lambda . \Gamma_i$ from the last. The graph $g(W)$ is obtained from the image of $w_1, \ldots, w_k$ in R if we identify C_i and C_i' with the vertices A_i and A_i' of $g(W)$. Therefore we shall use $g(W)$ to stand for the image of $w_1, \ldots, w_k$ in R.

Let $\Sigma_1, \ldots, \Sigma_k$ be non-singular and non-intersecting surfaces in M, such that Σ_λ meets Γ_i, if at all, in a set of non-singular circuits†, each of which

† It is to be assumed throughout that the chains under consideration are "general" relative to slight deformations. Thus no Σ contains O or has an edge in common with a w, and no $\Sigma . \Gamma$ meets a w.

bounds on Σ_λ. The surfaces Σ correspond to a set of connected surfaces $S_1, \ldots, S_q$, in R, which, in general, have boundary circuits on C_i and C_i'. I say that the surfaces Σ may be replaced by a set which satisfy the same condition relative to the intersections $\Sigma \,.\, \Gamma$, which have the same intersections with the circuits w and which have *normal* intersections with the two-spheres Γ, meaning that each surface S meets each of C_i and C_i' in a single circuit if at all. For let $\widehat{S}_\alpha$ contain more than one circuit on C_i and let s be a non-singular segment on C_i joining vertices on two circuits of $\widehat{S}_\alpha$. After certain segments of s have been replaced, if necessary, by suitable segments of $\widehat{S}_\beta$, and slightly deformed, s will not meet any one circuit belonging to any $\widehat{S}_\beta$ more than once. Then a segment s_0 of s, possibly s itself, will join two circuits of some $\widehat{S}_\beta$ without meeting $\widehat{S}_\gamma$ more than once if $\gamma \neq \beta$, and without meeting $\widehat{S}_\beta$ except in its end points. If P and Q are the end points of s_0 there is a segment σ_1, joining P to Q on S_β. Holding P and Q fixed, let s_0 be slightly deformed into a segment σ_2, which is inside R except for its end points. If s_0 were to cut $\widehat{S}_\gamma$, the segment σ_2, and therefore the circuit $\sigma_1+\sigma_2$, would intersect S_γ algebraically and $\sigma_1+\sigma_2$ would link $\widehat{S}_\gamma$, which it does not. Therefore s_0 does not meet any $\widehat{S}$ except in P and Q. At the negligible expense of increasing the genus of Σ_λ, the surface corresponding to S_β, the two circuits of $\widehat{S}_\beta$ joined by s_0 can, therefore, be united by shrinking the image of s_0 into a point on Γ_i, removing the conical neighbourhood of this point from the new Σ_λ, and identifying the two boundary circuits so formed. The new surface Σ_λ obviously satisfies the same condition relative to $\Sigma_\lambda \,.\, \Gamma_i$, the intersections $\Sigma_\lambda \,.\, w_\mu$ are the same as before, and the number of circuits in $\widehat{S}_\beta$ is reduced. Therefore a repetition of this process leads to a set of surfaces Σ satisfying the required conditions.

Now let (W) be a given set of words represented by oriented circuits $w_1, \ldots, w_k$ in M. Let us assume the existence of non-singular and non-intersecting surfaces $\Sigma_1, \ldots, \Sigma_k$, which satisfy the above condition† relative to $\Sigma \,.\, \Gamma$ and which are *conjugate* to $w_1, \ldots, w_k$, in the sense that Σ_λ intersects w_λ in a single vertex and does not meet w_μ if $\mu \neq \lambda$. Let t be a simple transformation of (W). If t is an elementary insertion, the appropriate segment of w, and the surfaces Σ, can obviously be deformed in such a way that the same conditions are satisfied relative to $t(W)$. If t consists of removing xx' let PQ be the corresponding segment of $g(W)$, P and Q being points on some C_i or C_i', say C_i. We now replace the Σ's by conjugate surfaces which have normal intersections with the Γ's. Under this condition an argument similar to one used before shows that P is joined to Q by a segment s, on

† As yet we do not assume the intersections $\Sigma \,.\, \Gamma$ to be normal.

C_i, which has at most one intersection with the circuits $\widehat{S}$. For the segment PQ of $g(W)$ has at most one intersection with the surfaces S, and s will intersect an $\widehat{S}$ or not according to whether PQ does or does not intersect the corresponding S. In either case PQ may be replaced by the segment on C_i' corresponding to s and deformed clear of C_i'. Again the same conditions are satisfied relative to $t(W)$. Finally, let t consist of $b \to bc$. Four possibilities present themselves, according to whether b is an a_i or an a_i' and c an a_j or a_j'. It will be sufficient to consider one of these, say $a_i \to a_i a_j$, the others involving no more than a change of notation. In this case we leave the circuits w and the surfaces Σ unaltered and modify the two-spheres Γ by a transformation similar to a move of the type $I \,.\, C$, described in an article by J. Singer†, our two-spheres C playing the part of the circuits ϵ in J.S. If we join two vertices on C_i and C_j by a non-singular segment s, in R, a neighbourhood in R of $C_i + C_j + s$ will be bounded by C_i, C_j and a two-sphere C^*, whose image Γ^*, in M is to replace Γ_j. We orient C^* so that it covers C_j positively, that is to say its positive side is the one remote from C_j, cut the region bounded by C_i, C_j and C^* away from R, and reattach it by identifying C_j with C_j'. A positive intersection $w_\lambda \,.\, \Gamma_i$ or $w_\lambda \,.\, \Gamma_j$ is immediately followed by a positive $w_\lambda \,.\, \Gamma^*$, and a negative $w_\lambda \,.\, \Gamma_i$ or $w_\lambda \,.\, \Gamma_j$ preceded by a negative $w_\lambda \,.\, \Gamma^*$. If we rewrite Γ^* as Γ_j, with the corresponding change of notation in R, the new C_j being the positive side of C^*, the same circuits w serve as a representation of $t(W)$. The circuits in $\Sigma_\lambda \,.\, \Gamma^*$ all bound on Σ_λ. For they are either homologous to circuits in $\Sigma_\lambda \,.\, \Gamma_i$ or $\Sigma_\lambda \,.\, \Gamma_j$, or bound two-elements corresponding to neighbourhoods on S_α of the intersections $s \,.\, S_\alpha$.

If (W) is a set of independent letters the surfaces Σ may be obtained by slight deformation of appropriate Γ's. If (W) is any simple set of words, it follows, from induction on the number of simple transformations necessary to derive (W) from an independent set of letters, that (W) can be represented by a set of circuits in M which are conjugate to a set of surfaces Σ, satisfying the above conditions.

With the notation used in the last two paragraphs, let (W) be any simple set of words. First assume that none of the surfaces Σ meets a two-sphere Γ. Then Σ_λ corresponds to a surface which bounds in the three-sphere H and meets $g(W)$ in a single vertex other than O. Therefore $g(W)$ has a cut vertex other than O. In general, the individual circuits in the intersections $\Sigma_\lambda \,.\, \Gamma$ bound a set of regions on Σ_λ whose boundaries do not meet each other. It follows that one, at least, of these regions contains neither the vertex $\Sigma_\lambda \,.\, w_\lambda$

† *Trans. American Math. Soc.*, 35 (1933), 88–111. This paper will be referred to as J.S.

nor any circuit belonging to $\Sigma_\lambda . \Gamma$ other than its boundary. The image S_α of this region in R is a surface bounded by a circuit on a two-sphere C_j or C_j', say C_j. Let E_1 and E_2 be the two two-elements on C_j bounded by $\widehat{S}_\alpha$. Since $g(W)$ does not meet S_α, it follows that A_j is a cut vertex if E_1 and E_2 both contain vertices of $g(W)$. If one of them, say E_1, does not meet $g(W)$ let E_0 be a two-element in E_1, possibly E_1 itself, which is bounded by a circuit of some $\widehat{S}_\beta$, and which contains no other circuit of an $\widehat{S}$. Let Σ_μ be the surface corresponding to S_β. Since each circuit in $\Sigma_\mu . \Gamma_j$ bounds on Σ_μ, the image of $\widehat{E}_0$ in M separates Σ_μ into two bounded surfaces, one of which, say Σ_0, does not meet any of the circuits w. Since E_1 does not meet $g(W)$ the same is true of E_0, and we replace Σ_0 by the image of E_0 in M. After a slight deformation of the new Σ_μ the new surfaces Σ satisfy the same conditions as before and the number of circuits in $\Sigma . \Gamma$ has been reduced. The lemma now follows from induction on the latter.

4. With the apparatus of J.S., these results are readily applicable to the study of three-dimensional manifolds. For example, let us ask, as many people must have done before: *is there a process of random reduction by which some standard representation of a three-dimensional manifold is reducible to a normal, or "semi-normal", form?* Can a Heegard diagram, for example, be reduced at random to a diagram on a canonical surface of minimum genus?

Let $\{e^1\}$ and $\{e^2\}$ be canonical† sets of curves for the two regions into which a given three-dimensional manifold M is separated by a canonical surface. Then M is represented by either one of two "dual" diagrams, according to whether one takes $\{e^1\}$ or $\{e^2\}$ to determine the "circles" of the diagram. Associated with these diagrams are dual sets of generators and sets of relations (W^1) and (W^2), for the group of M. As in § 3, with the dimensions of H, C_i and C_i' reduced by unity, each diagram provides a graph for one of the sets of cyclic words (W^1) and (W^2). According to‡ J.S. (Theorem 7), one can pass from either one of two diagrams for M to the other by a series of moves which, so far as they affect these generators and

† *Cf.* J.S.

‡ Theorem 7, p. 111. Notice that the wording of § 5 on p. 107 of J.S. needs qualification. For in order to interchange the dual diagrams determined by $\{e^1\}$ and $\{e^2\}$ one has to interchange the two canonical regions. The subsequent, but not the preceding, arguments show that one can do this by deformat on and, if necessary, by the creation and subsequent destruction of handles.

words, are of four types:

(1) the addition of a new generator and one-letter word a_0, by the creation of a handle;

(2) simple transformations† of (W^1), or replacing some word in (W^1) by its "cyclic" inverse;

(3) similar transformations of (W^2);

(4) the inverse of a transformation of the first type.

Moreover, it is not difficult to show that the required transformation may be exhibited in the form $T_4 \dots T_1$, where T_i, if it is not vacuous, consists of moves of the i-th type. If only one of the types 2 and 3, say 2, occurs in the transformation from a diagram on a canonical surface of minimum genus to a given diagram on a surface L, the latter can be "steadily" reduced to a canonical surface of minimum genus. For, after testing each sub-set of the words (W^1), a maximal simple sub-set can be reduced to an independent set of generators and then eliminated by moves of type 3 and 4.

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† Elementary transformations are important here, since a simple transformation of the type $x \to xy$ may not be geometrically possible unless accompanied by elementary insertions.

HEREDITARY MATRICES

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The following investigation originated in a reconsideration of the factor types* of the prepared systems for two quadratic forms in 4, 5, and 6 variables by symbolic methods. It is characteristic of the theory that symbolic expressions for invariants of non-linear ground forms consist of factors which have an actual interpretation as irreducible invariants of certain linear forms. Such invariants are associated with, and often completely characterized by, matrices of positive integers.

In § 1 these matrices are introduced, and the relation between them and the invariants of linear forms is described. Later, in § 3, it is shown how the matrices of a field $[n]$ are derived from those of an earlier field $[n-1]$, and thereafter how the projective invariants of systems of linear forms may be divided into families, all members of one family having certain properties in common. In the concluding sections complete lists of the invariants in particular cases are obtained, and from these the prepared systems of two quadratics in 4, 5, and 6 variables may be written down immediately. One surprising result is that all irreducible invariants corresponding to a prepared system for 4 or 5 variables belong to one and the same family, while, even in the last case, that of 6 variables, only two more families appear.

1. Let p positive integers, which are not all necessarily distinct, be arranged in a matrix H, consisting of m columns, in which the first row and column is fully occupied and the length of each subsequent row or column is either equal to or less than that of its predecessor. Suppose, therefore, that

$$H=\begin{bmatrix} a_1 & \beta_1 & \cdots & \gamma_1 \\ a_2 & \beta_2 & \cdots & \gamma_2 \\ a_3 & \beta_3 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}=[\theta_{ij}], \tag{1}$$

* *Cf.* J. Williamson, *American J. of Math.*, 52 (1930), 863–876; 53 (1931), 343–357.

where θ_j denotes the typical element, a positive integer, in column θ and row j.

If a partition $\alpha, \beta, \ldots, \gamma$ of the integer p into m terms is made, let $\alpha, \beta, \ldots, \gamma$ respectively denote the number of elements in the corresponding columns of H. Accordingly,

$$\alpha \geqslant \beta \geqslant \ldots \geqslant \gamma. \tag{2}$$

Further, let the elements of each column separately be arranged in descending order so that

$$\theta_1 \geqslant \theta_2 \geqslant \theta_3 \geqslant \ldots. \tag{3}$$

Each element θ_j of H will denote a linear subspace of $\theta_j - 1$ dimensions situated in a space of $n-1$ dimensions, so that the element 1 will denote a point, 2 a line, 3 a plane, and so on. Such a linear subspace of dimension $\theta_j - 1$ will here be called a *figure of currency* θ_j, since it is determined by θ_j linearly independent points. Accordingly the matrix $[n]$ of one row and one column will indicate a space of currency n, *i.e.* a space of dimension $n-1$, and, since each linear space is a subspace of this space,

$$n \geqslant \theta_j > 0. \tag{4}$$

If n_1, n_2 is a partition of n into two non-zero parts, the matrix $[n_1 n_2]$ will indicate the same space, this time defined by two dual subspaces in general position; for example, [1 1] denotes a line defined by the two points, [2 1] a plane defined by a line and a point not on the line. Corresponding to any partition of n into m non-zero parts, $n_1, n_2, \ldots, n_m$, there will be a matrix consisting of a single row

$$[n_1, \ n_2, \ \ldots, \ n_m],$$

which will indicate the space of currency n defined by the m subspaces $[n_i]$ in general position.

When the matrix H has more than one row, each column, which contains two or more elements, will specify a *nest* of subspaces; that is, each smaller element will denote a subspace of each greater element, and each group of equal elements coincident subspaces. For instance, the column {3, 2, 2, 1} denotes a plane, a line within it counted twice, and a point within the line. (Here the notation { } is used conventionally to denote the column of integers enclosed, in order to save space.) The significance of the repetition of an element within a column will be apparent later.

If H has m columns it indicates that the corresponding geometrical figure consists of m distinct nests in general position. Beyond this nesting condition nothing is specified concerning the relative positions of the different geometrical elements.

The valency condition. In the sequel the elements of H will not be quite arbitrary but will be such that their sum N is a multiple of n. This restriction, which is called the *valency condition*, defines a positive integer w such that

$$N = nw = \Sigma\,\Sigma\,\theta_j. \tag{5}$$

In this case the matrix H is conveniently written as

$$H = {}_w[\theta_j]_n,$$

and is said to characterize a geometrical figure belonging to the n-ary *field* and having a *weight* w. If the integer matrix is given, the external suffixes w and n can always be supplied, and in some cases uniquely. Consider, for example, the matrix

$$H = \begin{bmatrix} 4 & 4 & 4 \\ 1 & 2 & 2 \\ 1 & & \end{bmatrix}.$$

This specifies a figure of three three-dimensional spaces containing respectively a point, specified twice, a line, and a line. Here $N = 18$ and $n \geqslant 4$, the highest element. Hence, either $n = 6$, $w = 3$ or $n = 9$, $w = 2$.

A second restriction is now introduced, again for reasons which will shortly appear: *the number of elements in a column must never exceed the weight.* With this restriction, in the above example, it is impossible for n to be 9 and w to be 2, since there are three elements in the first column of H. Even with these restrictions a matrix may still belong to two fields of different orders; for example,

$${}_3\begin{bmatrix} 4 & 4 & 4 \\ 2 & 2 & 2 \end{bmatrix}_6 \quad \text{and} \quad {}_2\begin{bmatrix} 4 & 4 & 4 \\ 2 & 2 & 2 \end{bmatrix}_9.$$

Dual configurations. Corresponding to any matrix H there is a dual matrix obtained by replacing each element θ_j by $n-\theta_j$ and then reversing the order in each column so as to retain condition (3). If $\theta_j = n$, then $\theta_j - n = 0$ and specifies no geometrical element at all. Consequently it is best to set aside the exceptional matrix $[n]$, which denotes the field, and to stipulate that, *if H contains two or more elements, each element is less than n.*

Evidently H', the dual of H, is another such matrix in the n-ary field, containing the same number p of elements and the same number of repeated elements in a column but usually possessing a different weight w'. If N' denotes the sum of all its elements, we evidently have

$$N' = \Sigma\Sigma(n-\theta_j) = np-N,$$

and, as a consequence of (5), $N' = nw'$, so that

$$w+w' = p, \tag{6}$$

or *the weights of two dual matrices are dual with respect to the number of their elements.*

The relation of duality, which is obviously symmetrical, will be denoted by the sign $\simeq$, so that

$$H \simeq H' \quad \text{and} \quad H' \simeq H.$$

This algebraic process of duality is consistent with the fundamental geometric duality, or reciprocation, where a point is the dual of a prime, a line of a subprime, etc. Moreover, since the dual of a space $[r]$ contained in a space $[t]$ is a space $[n-r]$ containing a space $[n-t]$, it follows that the nesting condition pertaining to each column of the dual matrix is preserved.

2. *Representation of invariants.* An H matrix may be regarded as prescribing a possible projective invariant in the field $[n]$ of the geometrical elements which it indicates. If G denotes the system of such elements, supposed to be q in number ($q \leqslant p$), then H will be said to typify every possible invariant of weight w in the *ground forms* G, of total degree p, and of partial degree p_{ij} in each separate geometrical element, where p_{ij} is the number of repetitions of that particular element in its own column of H. Conversely, any rational integral invariant of weight w and of prescribed partial degrees p_{ij} in any system of linear ground forms G in the n-ary field is specified by such a matrix H. These facts are an immediate consequence of the fundamental theorem of symbolic forms in the theory of invariants*.

To see this, consider first the invariants of unit weight. They are known to be of the following type:

$$\Delta = (a_1 a_2 \dots a_{n_1} b_1 b_2 \dots b_{n_2} \dots c_1 c_2 \dots c_{n_m}) = (A_{n_1} B_{n_2} \dots C_{n_m}),$$

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* H. W. Turnbull, *The theory of determinants, matrices, and invariants*, 211.

where

(i) $n_1+n_2+\ldots+n_m=n$,

(ii) each element $a_j, b_j, \ldots, c_j$ denotes a point whose coordinates are $\{a_{1j}, a_{2j}, \ldots, a_{nj}\}$ for a_j, etc.,

(iii) Δ is a determinant of order n,

(iv) the a_j, $j=1, 2, \ldots, n_1$, denote n_1 linearly independent points in $[n]$, defining the linear space A_{n_1} of currency n_1, and similarly for $B_{n_2}, \ldots, C_{n_m}$. Such an invariant belongs to these ground forms and is linear in each. This fact is naturally characterized by the H matrix of unit weight $[n_1, n_2, \ldots, n_m]$. The valency condition is reflected in the abstract theory by the fact that a system of ground forms G possesses an invariant of unit weight, if and only if the sum of the currencies is n. More generally, every irreducible invariant of weight w is expressible as

$$\phi=\Sigma\Delta_1\Delta_2\ldots\Delta_w, \tag{7}$$

where each of $\Delta_1, \Delta_2, \ldots, \Delta_w$ is a determinant of order n of the same nature as Δ, so that each term on the right of (7) contains nw symbols $a_j, b_j, \ldots, c_j$. Moreover, each term contains the same symbols but in a different order. The symbols $a_j, b_j, \ldots, c_j$ belong respectively to the ground forms $A_{n_1}, B_{n_2}, \ldots, C_{n_m}$ and the summation Σ means that each of the m ground forms is convolved implicitly or explicitly, so that ϕ is an alternating function in each of the sets $a_j, b_j, \ldots, c_j$ respectively. Further, the sum of the currencies gives

$$n_1+n_2+\ldots+n_m=nw.$$

It is now clear that to every invariant ϕ one such ordered matrix H can be attached, complying with the valency condition (5), where each element θ_j denotes the currency of a ground form. But the converse is not necessarily true—to each ordered matrix H there may correspond one or more invariants.

Since one ground form may occur to any degree k in an invariant ϕ, a particular element θ_j of H may occur k times in H. Further, one ground form may enclose or be enclosed by another; in other words, the ground forms may be classified into nests and this is indicated by the separate columns of H. It is evident that two ground forms may have a subspace in common without belonging to the same nest, but of this no account is taken here.

It is now possible to explain why we impose the restriction that the number of elements in a column of H should never exceed the weight. It is a necessary condition for H to characterize an invariant not identically zero. If H contains a column containing θ elements, since these elements form a nest, all of these θ elements have at least one point x in common. This point x must occur θ times in each term $\Delta_1 \Delta_2 \ldots \Delta_u$ of the corresponding invariant ϕ. If θ is greater than w, at least one determinantal factor Δ of each term contains x twice, and accordingly ϕ is identically zero.

For example, the matrix $H_6 = [\{4\ 1\ 1\}, \{4\ 2\}, \{4\ 2\}]$ specifies an invariant such as

$$\phi_6 = \Sigma(A_4 bc)(B_4 c' a)(C_4 b' a), \tag{8}$$

where the summation denotes four terms, due to the independent interchange of b with b' and c with c', each interchange being accompanied by a change in sign. Here A_4 denotes a three space containing the point a while B_4 and C_4 denote two other such spaces containing the line through the points b and b' and the line through the points c and c' respectively. It happens that for this matrix H_6 the invariant ϕ is unique, since any other arrangement of the five points a, b, b', c, c' within the three factors (A_4), (B_4), (C_4) makes one or other of them zero.

3. *The Clebsch transference principle.* Whenever an invariant ϕ_n in the field $[n]$ is such that the same point x occurs exactly w times, the invariant may be regarded as an extensional of a certain invariant ϕ_{n-1} belonging to the field $[n-1]$. For if $\prod_{i=1}^{w} \Delta_i$ is a typical term of ϕ_n, so that each factor Δ_i is an n-rowed determinant $(d_1\ d_2 \ldots d_{n-1}\, x)$, which necessarily contains the column x, together with $n-1$ further independent columns d_k, then ϕ_{n-1} is formally constructed from ϕ_n merely by suppressing x in each factor Δ_i and treating the resultant expression $\Delta_i = (d_1 d_2 \ldots d_{n-1})$ as a determinant of order $n-1$. For example, if we write the invariant ϕ_6 of (8) in the form

$$\phi_6 = \Sigma(bc\, a'\, a''\, a'''\, a)(B_4 c' a)(C_4 b' a),$$

where the space A_4 has been resolved into four linearly independent points, one of which must be a, we may consider ϕ_6 to be an extensional of

$$\phi_5 = \Sigma(bc\, a'\, a''\, a''')(B_4 c')(C_4 b'),$$

which belongs to the field [5]. All this is an illustration of the principle of

Clebsch, well known in the theory of invariants, and the corresponding principle of extensionals in the theory of determinants*.

The corresponding H matrices of ϕ_6 and ϕ_5 are manifestly

$$H_6 = {}_3\begin{bmatrix} 4 & 4 & 4 \\ 1 & 2 & 2 \\ 1 & & \end{bmatrix}_6 \quad \text{and} \quad H_5 = {}_3\begin{bmatrix} 4 & 4 & 3 \\ 2 & 2 & \end{bmatrix}_5.$$

Inspection shows at once that H_5 may be derived directly from H_6 by subtracting unity from each element in the column whose length is equal to the weight 3, then omitting zero elements, and finally ordering the columns. Evidently, such a process does not disturb the valency condition.

We may, however, adopt the converse view of obtaining ϕ_6 from ϕ_5, or H_6 from H_5, and so forming the extensional in a higher dimension from one in a lower. For H_5 this can be done in four ways, two of which are virtually alike, in the following manner: to either of the three columns of H_5 three units may be added, giving us

$$\begin{bmatrix} 5 & 4 & 3 \\ 3 & 2 & \\ 1 & & \end{bmatrix}, \quad \begin{bmatrix} 4 & 5 & 3 \\ 2 & 3 & \\ & 1 & \end{bmatrix}, \quad \begin{bmatrix} 4 & 4 & 4 \\ 2 & 2 & 1 \\ & & 1 \end{bmatrix},$$

or a new column of three units may be added to give

$$\begin{bmatrix} 4 & 4 & 3 & 1 \\ 2 & 2 & & 1 \\ & & & 1 \end{bmatrix}.$$

After ordering the columns, we have, in addition to H_6, two other distinct matrices,

$$K_6 = {}_3\begin{bmatrix} 5 & 4 & 3 \\ 3 & 2 & \\ 1 & & \end{bmatrix}_6 \quad \text{and} \quad L_6 = \begin{bmatrix} 1 & 4 & 4 & 3 \\ 1 & 2 & 2 & \\ 1 & & & \end{bmatrix}.$$

Thus there are three distinct invariants ϕ_6, ψ_6, and χ_6 which are direct extensionals of ϕ_5. Obviously ϕ_5 may be recovered from ψ_6, or χ_6, in the same way as from ϕ_6, by striking out a suitable common element from each of the three factors.

This extensional process, in addition to furnishing a method for the construction of more complicated from simpler invariants, has three fundamental properties. In the first place it applies not only to single invariants

* Turnbull, *op. cit.*, 287 and 48.

but also to invariant equations*. If, in fact, ϕ_{n-1} vanishes owing to a special configuration of the geometrical elements involved, so also will its extensional ϕ_n. Correlatively, if ϕ_{n-1} does not vanish identically, owing to a sufficiently general choice of ground elements, neither can ϕ_n; but then ϕ_n will vanish if, and only if, the added point x lies on a primal of order w, whose equation in point coordinates is $\phi_n = 0$. In the second place the passage from ϕ_{n-1} to ϕ_n has a direct geometrical interpretation by replacing each part of the geometric figure representative of ϕ_{n-1} by a corresponding linear figure of one higher dimension, including the said part and the new point x. Conversely, when x occurs exactly w times in ϕ_n, the figure of ϕ_{n-1} is obtained by taking an arbitrary section of the figure ϕ_n by a prime not containing x.

In the third place, if ϕ_{n-1} is algebraically irreducible, so also is ϕ_n. For, if not, either ϕ_n is reducible or identically zero. It cannot be the latter without making ϕ_{n-1} so, which is not the case. Nor can it be reducible; for x would then necessarily continue to appear in each factor Δ of each term in the reduced parts. This would imply an analogous expression furnishing a reduction of ϕ_{n-1}, which is impossible.

4. *Generalized extensionals.* Instead of adding the same x to each of the w factors of ϕ_{n-1} to obtain its extensional ϕ_n, we may add w different points x, or again x to the first i, y to the next j factors, and so on. This, which may be called the generalized extensional of ϕ_{n-1}, continues to have the property that if ϕ_{n-1} is irreducible so is ϕ_n. For if not, let ϕ_n reduce, as it must do, to

$$\Sigma\phi(x)\psi(y)+\Sigma\chi(xy),$$

the first series indicating all terms wherein the w points $x, y, \ldots$ are entirely separated, and the second where at least two of x, y are convolved in a factor of the term. Then, if we equate all the points to x, each term of the latter series vanishes; and, unless the former also vanishes, the original ϕ_n has been reduced, which is impossible. Hence $\phi_n = \Sigma\chi(xy)$ only, and again $\Sigma\chi$ vanishes by equating all x, y, whereas the original ϕ_n does not. Hence ϕ_n is irreducible with ϕ_{n-1} in the generalized case.

To see how this applies to H matrices, let the symbols of the invariant ϕ_{n-1} answering to the α elements of the first column of H_{n-1} be $A_1, A_2, \ldots, A_\alpha$. Since† in ϕ_{n-1} every symbol A_k is explicitly or implicitly convolved, we

* Turnbull, *op. cit.*, 271.

† *Cf.* Turnbull, *op. cit.*, 304.

may convolve A_1 in the first, A_2 in the second, ..., and A_i in the i-th factor (and if $\alpha < i$ the symbols A will be confined to the first α factors only). And, since $i < w$, we may continue by convolving B_1 in the $(i+1)$-th, ..., and B_j in the $(i+j)$-th factor, from symbols of the second column, and so on. To the ϕ_{n-1}, prepared in this way, let the generalized extension be made, as above, by affixing x to the first i, y to the next j factors, and so on: and, finally, let this x be included, as before, in the extended nest A, but let y now belong to the extended nest B, and so on, it being apparent that the characteristic property of the nests is still preserved. The result is evidently an irreducible form ϕ_n with a matrix H_n, obtained from H_{n-1} (the matrix of ϕ_{n-1}) by adding unity to each of the first i elements of the first column, the first j elements of the next column, and so on $(i+j+\ldots = w)$.

We shall call an H matrix irreducible if there exists at least one irreducible invariant belonging to it. We have, therefore, shown that

If H_n, of weight w, is obtained from an irreducible matrix H_{n-1} by adding unity to w of its elements in any manner, provided that no element is increased by unity unless all its predecessors in the same column are, then H_n also is irreducible.

Whereas the irreducibility of H_n follows from that of H_{n-1}, the converse is not true, as a subsequent example will show. On the other hand, in the case of simple extensionals the converse is true. It is for this reason that simple extensionals are of more fundamental importance.

5. *Principle of hereditary reduction.* This principle consists in combining the processes of reciprocation and extensionals with a view to discovering the essentially distinct configurations of given ground forms and their invariants. The reciprocation implied by the relation

$$_{w'}[\theta_j']_n \simeq {}_w[\theta_j]_n$$

between dual matrices H and H' evidently leaves the field $[n]$ unaltered. In contrast to this a matrix of the field $[n]$ may be deduced from one of the field $[n-1]$ by the simple extensional process, namely *a matrix H_n of weight w is said to be derived from a matrix H_{n-1} if it is formed by adding unity exactly w times to an old or to a fresh column of the latter.* This relation is written

$$H_n \sim' H_{n-1}, \quad H_{n-1} \sim H_n.$$

In particular, if H_{n-1} possesses h distinct columns, there are $h+1$ distinct matrices H_n derivable in this manner, and the process in this direction is

always possible whatever H_{n-1} may be. On the other hand, the process of recovering H_{n-1} from H_n is possible only if the latter has $r > 0$ distinct columns containing w elements, in which case it is possible in r ways.

Suppose now that either process is repeated; then in the first case, that of duality, nothing is gained, since $H_n \simeq H_n' \simeq H_n$. In the second, however, a chain can be formed and continued indefinitely by extension,

$$H_n \sim H_{n+1} \sim H_{n+2} \cdots$$

or, in the other direction,

$$H_n \sim H_{n-1} \sim \cdots \sim H_{n-v},$$

until finally in H_{n-v} a column equal in length to the weight no longer exists. In all such chains the weight is constant; but, if the reciprocation process is also admitted, change of weight may occur and retrogression to still lower fields be possible. All H matrices which can be linked by a succession of one or other processes $\simeq$ or $\sim$ are said to belong to the same family. It follows from what has been said that certain properties, such as irreducibility, will be shared by all members of the family*; these properties will therefore be called *hereditary properties*. The process of linking together two H matrices, which belong to the same family, will be called that of *hereditary reduction*.

The chief family is that which starts with a single point. One branch of the family runs as follows:

$$[1]_1 \sim [1\ 1]_2 \sim [1\ 1\ 1]_3 \simeq [2\ 2\ 2_3] \begin{cases} \sim \begin{bmatrix} 1\,2\,2\,2 \\ 1 \end{bmatrix}_4 \begin{cases} \sim \cdots \\ \sim \cdots \\ \sim \cdots \\ \simeq \begin{bmatrix} 3\,2\,2\,2 \\ 3 \end{bmatrix}_4 \end{cases} \begin{cases} \sim \begin{bmatrix} 4\,2\,2\,2 \\ 4 \\ 1 \end{bmatrix}_5 \\ \sim \begin{bmatrix} 3\,3\,2\,2 \\ 1\,3 \\ 1 \end{bmatrix}_5 \end{cases} \\ \sim \begin{bmatrix} 3\,2\,2 \\ 1 \end{bmatrix}_4 \begin{cases} \sim \cdots \\ \sim \cdots \\ \simeq \begin{bmatrix} 3\,2\,2 \\ 1 \end{bmatrix}_4 \begin{cases} \sim \begin{bmatrix} 4\,2\,2 \\ 2 \end{bmatrix}_5 \\ \sim \cdots \end{cases} \end{cases} \end{cases}$$

* The dual of an invariant ϕ is obtained by replacing each point by a prime α and then removing a factor uniquely determined by the currencies of the ground forms. Such a process obviously preserves the property of irreducibility. *Cf.* Turnbull, *Proc. London Math. Soc.* (2), 22 (1923), 495–507 (503).

The various members have been entered in the tree according to the rules already given. The links may be inserted in every possible way, except that $\approx$ may not occur twice in succession. The weights have not been inserted since they may readily be ascertained. It is always apparent, by noticing which column has been altered, how a later has been derived from an earlier entry. It follows at once that all the invariants of the branch, and they are many, are irreducible, since the reduction of any one would imply that of all its predecessors, whereas those of weight 1 are certainly irreducible.

For example, there must be an irreducible invariant corresponding to the matrix

$$\begin{bmatrix} 5 & 4 & 3 \\ 3 & 2 & \\ 1 & & \end{bmatrix}_6,$$

that is, one in space of currency [6] involving a prime, a plane, and a point in one nest, a subprime and included line, and another plane. The sum of the elements being 18, w must be 3. Hence

$$\begin{bmatrix} 5 & 4 & 3 \\ 3 & 2 & \\ 1 & & \end{bmatrix}_6 \sim \begin{bmatrix} 4 & 4 & 3 \\ 2 & 2 & \end{bmatrix}_5 \approx \begin{bmatrix} 3 & 3 & 2 \\ 1 & 1 & \end{bmatrix}_5 \sim \begin{bmatrix} 3 & 2 & 2 \\ 1 & & \end{bmatrix}_4 \sim [2\ 2\ 2]_3$$
$$\approx [1\ 1\ 1]_3 \sim [1\ 1]_2 \sim 1,$$

which proves the irreducibility. It will be seen that the duality process comes effectively into play whenever the number of rows has fallen below the weight. Again

$$\begin{bmatrix} 5 & 4 & 4 \\ 3 & 2 & 2 \\ 1 & & \end{bmatrix}_7 \sim \begin{bmatrix} 4 & 4 & 4 \\ 2 & 2 & 2 \end{bmatrix}_6 \approx \begin{bmatrix} 4 & 4 & 4 \\ 2 & 2 & 2 \end{bmatrix}_6.$$

Here the weight $(21 \div 7 = 3)$ cannot be reduced, and the process towards lower space fails. More refined examination shows that the invariant

$$\begin{bmatrix} 4 & 4 & 4 \\ 2 & 2 & 2 \end{bmatrix}_6$$

is reducible.

6. *Categories of invariants.* Let an invariant corresponding to an H matrix of r columns be said to belong to the r-th category. We proceed

to examine the irreducible types for values $r = 1, 2, 3$. By the methods employed in dealing with quadratics* it is readily found that such irreducibles, for $r = 1$ or 2, possess an H matrix of only one row. For $r = 1$ this is obvious; for $r = 2$ it follows by first convolving the symbols A (of the first column) as in § 4, and then the symbols B in the reverse order throughout the factors. The summary of the results, in so far as irreducible invariants (or H matrices) are concerned, is therefore as follows:

First category, one column only,

$$[1]_1 \sim [2]_2 \sim [3]_3 \sim \ldots \sim [n]_n.$$

Second category, two columns only,

$$[1\ 1]_2 \sim [2\ 1]_3 \lesssim [1\ 2]_3 \sim [2\ 2]_4 \ldots \sim [i,\ n-i]_n \lesssim [n-i,\ i]_n.$$

We may express this result otherwise. *All irreducible H matrices of categories one or two belong to the family whose first member is* $[1]_1$.

Invariants of the third category. In the third category, in which H has three columns and the corresponding invariant has three nests of ground forms, the situation is much more complicated. Accordingly we are obliged to study the H matrices of this type, consisting of p integers in three columns,

$$H = \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \vdots & \vdots & \vdots \\ \alpha_i & \beta_j & \gamma_k \end{bmatrix}_n, \tag{9}$$

in greater detail. By the valency condition,

$$\alpha_1+\alpha_2+\ldots+\alpha_i+\beta_1+\beta_2+\ldots+\beta_j+\gamma_1+\ldots+\gamma_k = nw. \tag{10}$$

Also

$$i+j+k = p, \quad w \geqslant i \geqslant j \geqslant k, \tag{11}$$

so that $3i \geqslant p$. Likewise, if w' is the weight of the dual matrix, $w' \geqslant i$. But $w'+w = p$, so that $p \geqslant 2i$. Now if w or $w' = i$ the matrix descends at once to a lower field and this certainly happens if $p = 2i$ or $2i+1$. Hence the search for irreducibles may be confined to cases for which

$$3i \geqslant p > 2i+1. \tag{12}$$

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* Turnbull, *op. cit.*, 304–306.

To simplify the notation we introduce further elements

$$\alpha_{i+1} = \alpha_{i+2} = \ldots = \alpha_w = \beta_{j+1} = \ldots = \beta_w = \gamma_{k+1} = \ldots = \gamma_w = 0.$$

The typical invariant may then be written

$$\phi = \Sigma(A_1 B_w \ldots)(A_2 B_{w-1} \ldots) \ldots (A_w B_1 \ldots), \tag{13}$$

where, in each factor Δ_r, the symbol A_r answers to α_r of H, B to β, and the remaining $n-\alpha_r-\beta_{w-r+1}$ symbols are composed of C symbols only. Such symbols of C are implicitly convolved. It follows at once that, unless ϕ is reducible,

$$\alpha_r+\beta_{w-r+1} < n \quad (r = 1, 2, \ldots, w), \tag{14}$$

and, similarly,

$$\beta_r+\gamma_{w-r+1} < n, \quad \gamma_r+\alpha_{w-r+1} < n. \tag{15}$$

Further, the number of c symbols in any factor Δ_r cannot exceed γ_1; hence

$$\alpha_r+\beta_{w-r+1}+\gamma_1 > n, \tag{16}$$

and, similarly,

$$\beta_r+\gamma_{w-r+1}+\alpha_1 > n, \quad \gamma_r+\alpha_{w-r+1}+\beta_1 > n \tag{17}$$

when $r = 1, 2, \ldots, w > 1$. These six inequalities show that

$$\alpha_1 > 1, \quad \beta_1 > 1, \quad \gamma_1 > 1, \quad \text{when} \quad w > 1, \tag{18}$$

and, by duality, that

$$\alpha_i < n-1, \quad \beta_j < n-1, \quad \gamma_k < n-1 \quad \text{when} \quad w > 0. \tag{19}$$

Moreover, if $\alpha_r = 1$,

$$\beta_{w-r+1} > 1; \tag{20}$$

otherwise, ϕ is reduced by convolving C_1 in the factor Δ_r. In particular cases, further restrictions are placed upon the integers α_r, β_r, γ_r by considering the corresponding inequalities for the dual matrix H'.

We now prove two lemmas which are of considerable use in the calculation of irreducibles for low values of n.

LEMMA I. *If α_1 or β_1 or γ_1 is less than 3, so is the weight w.*

Proof. By (18), $w=1$ if $\gamma_1=1$. If $\gamma_1=2$ and $\alpha_r+\beta_{w-r+1}<n-1$ for one value of r, then ϕ is identically zero or else reducible unless $r=w=1$, in which case $\alpha_1+\beta_1=n-2$. Otherwise, when $\gamma_1=2$, ϕ is identically zero or reducible unless $w=2$ (with the two c symbols of C_1 in the separate factors).

LEMMA II. *If α_1 or β_1 or γ_1 is less than 4, so is w.*

Proof. Let $\gamma_1=3$, so that $w>2$, and let the positive integers $n-\alpha_r-\beta_{w-r+1}$, which give the number of unfilled columns of the factors Δ_r in (13), be denoted by $s_1, s_2, \ldots, s_w$, where (after perhaps rearranging the factors Δ in ϕ)

$$s_1 \geqslant s_2 \geqslant \ldots \geqslant s_w.$$

By (16), ϕ is zero if $s_1>3$. Also, if $s_1=1$ and $w>3$, ϕ certainly has an invariant factor of weight 3, containing C_1 implicitly, so that ϕ reduces. Similarly, if $s_1=2$, $s_2=1$, ϕ reduces. There remains the case $s_1=2=s_2$, with $\gamma_1=3$ as always. Let the factor corresponding to s_1 be written

$$(Dcc'),$$

where D consists of $n-2$ symbols a or b. By convolving $C_1 \equiv c_1c_2c_3$ into this factor and then re-convolving D in it, we effectively replace c, c' by two of c_1, c_2, c_3. With $s_2=2$ the only new possibility is

$$\Sigma(Dc_1c_2)(\ldots c_3c)\ldots.$$

If $\gamma_2=1$, these first two factors form an invariant. If $\gamma_2=2$, the only new possibility is

$$\Sigma(Dc_1c_2)(\ldots c_3c_1')(c_2'\ldots)\ldots,$$

where Σ denotes a sequence of six terms obtained by separate interchange of c_1, c_2, c_3 and c_1', c_2', with $c_1=c_1'$, $c_2=c_2'$. In effect c_1, c_2 always appear convolved in one or other factor; hence ϕ is reducible. Lastly, let $\gamma_2=3$: then the case

$$\Sigma(\ldots c_1c_2)(\ldots c_3c_1')(\ldots c_2')(\ldots c_3')$$

reduces by convolving $c_1'c_2'$ in the second factor, and the only new possibility is

$$\Sigma(\ldots c_1c_2)(\ldots c_3c_1')(\ldots c_2'c_3'),$$

which indicates that ϕ has an invariant factor of weight 3. Our lemma is accordingly proved.

COROLLARY I. *If $n=3$ the maximum value of w is 2. If $n=4$ the maximum value of w is 3.*

LEMMA 3. *Any H matrix of category 3 whose weight is 1 or 2 is a member of the family* [1].

This is obviously true if $w=1$. If $w=2$ and H is a matrix of one row, its dual H' is of weight 1 and belongs to the family [1]. If $w=2$ and H is a matrix of two rows, H is a simple extensional of a matrix of weight 2 in the field $[n-1]$. The truth of the lemma follows immediately by induction.

COROLLARY. *All invariants of the third category and field* [3] *belong to the family* [1].

If H is of weight 3 but is not derivable from a matrix in a lower field, then $i=2$ and it follows from the inequalities (12) that $p=6$, so that $j=k=2$ and

$$H=\begin{bmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\end{bmatrix}. \tag{21}$$

By Lemma 1, $\alpha_1, \beta_1, \gamma_1 > 2$ and, by duality, $\alpha_2, \beta_2, \gamma_2 < n-2$. If $n=4$, in (21), $\alpha_1=\beta_1=\gamma_1=3$, $\alpha_2=\beta_2=\gamma_2=1$, and H is reducible as a consequence of (20). If $n=5$, $\alpha_1, \beta_1, \gamma_1$ may have the values 3 or 4 and $\alpha_2, \beta_2, \gamma_2$ the values 1 or 2. By (20), however, at most one of $\alpha_2, \beta_2, \gamma_2$ may have the value 1 and, by duality, at most one of $\alpha_1, \beta_1, \gamma_1$ have the value 4. The matrix H in (21) may, therefore, be one of the three matrices

$$\begin{bmatrix}4 & 3 & 3\\ 2 & 2 & 1\end{bmatrix}, \quad \begin{bmatrix}4 & 3 & 3\\ 1 & 2 & 2\end{bmatrix}, \quad \begin{bmatrix}3 & 3 & 3\\ 2 & 2 & 2\end{bmatrix},$$

each of which, on trial, is found to be reducible. Moreover, it can be shown (by continuing the above methods) that no irreducible matrices whose weight is greater than 3 exist in the field [5]. *Hence all irreducible H matrices of category 3 in the fields* [3], [4], *or* [5] *belong to the family* [1].

The case $n=6$ is interesting, since now for the first time irreducible H matrices appear which are not members of the family [1]. If $n=6$, the elements of the matrix H in (21) of weight 3 are restricted in the following manner: $6>\alpha_1, \beta_1, \gamma_1>2$; $0<\alpha_2, \beta_2, \gamma_2<4$; no two of $\alpha_2, \beta_2, \gamma_2$ have the value 1; no two of $\alpha_1, \beta_1, \gamma_1$ have the value 5. The possible values of H are

easily written down; those which are not obviously reducible are

$$\begin{bmatrix} 5 & 4 & 4 \\ 2 & ? & 1 \end{bmatrix}, \quad \begin{bmatrix} 4 & 4 & 4 \\ 2 & 2 & 2 \end{bmatrix}, \quad \begin{bmatrix} 4 & 4 & 3 \\ 2 & 2 & 3 \end{bmatrix},$$

all of which are self dual. The second of these is reducible since the corresponding invariant may be written as $\Sigma(A_4 c_1' c_2')(A_2 B_2 c_3' c_1'')(B_4 c_4' c_2'')$, where c_1', c_2', c_3', c_4', and c_1'', c_2'' are permuted independently. Each non-zero term of Σ has a factor $(A_4 C_2)$ or $(A_2 B_2 C_2)$ or $(B_4 C_2)$, where $C_2 = c_1 c_2$. Hence $\begin{bmatrix} 4 & 4 & 4 \\ 2 & 2 & 2 \end{bmatrix}$ reduces. The first is a generalized extensional of an irreducible in [5], and hence is irreducible. In fact, if $\overset{\sim}{\sim}$ denotes the generalized extension,

$$\begin{bmatrix} 5 & 4 & 4 \\ 2 & 2 & 1 \end{bmatrix}_6 = \begin{bmatrix} 4+1 & 4 & 3+1 \\ 2 & 2 & 1 \end{bmatrix}_6 \overset{\sim}{\sim} \begin{bmatrix} 4 & 4 & 3 \\ 2 & 2 & \end{bmatrix}_5 \overset{\sim}{\sim} \begin{bmatrix} 3 & 3 & 2 \\ 1 & 1 & \end{bmatrix}_5 \sim \begin{bmatrix} 3 & 2 & 2 \\ 1 & & \end{bmatrix}_4 \sim \ldots \sim [1].$$

The third $H = \begin{bmatrix} 4 & 4 & 3 \\ 2 & 2 & 3 \end{bmatrix}$ is irreducible or zero, since the invariant $\begin{bmatrix} 4 & 2 & 3 \\ & & 3 \end{bmatrix}$ is obviously zero. But trial, with C_3 and $aa'a''$ forming the frame of reference, shows that H does not vanish identically. Hence H is irreducible. Yet

$$H = \begin{bmatrix} 3+1 & 3+1 & 3 \\ 1+1 & 2 & 3 \end{bmatrix}_6 \overset{\sim}{\sim} \begin{bmatrix} 3 & 3 & 3 \\ 1 & 2 & 3 \end{bmatrix}_5,$$

and this latter easily reduces to $\begin{bmatrix} 3 & 3 & 3 \\ 1 & & \end{bmatrix}[.\ 2\ 3]$, so that *an irreducible H is here derived by generalized extension from a reducible matrix*, illustrating the remark at the end of §4. On the other hand, this H does not arise by any of the previous methods from an irreducible matrix in the field [5]. It can be shown that no irreducible matrix of weight greater than 4 exists and that all of weight 4 belong to the family [1].

Thus, for $n = 6$, all but two irreducible matrices belong to the family [1]. The two exceptions, which may be called *primary invariants* or *primary matrices*, are represented by

$$\begin{bmatrix} 4 & 5 & 4 \\ 1 & 2 & 2 \end{bmatrix}, \quad \begin{bmatrix} 4 & 4 & 3 \\ 2 & 2 & 3 \end{bmatrix}.$$

These two primary invariants are founders of families of invariants for values of $n = 6$. For $n = 7$ new primary invariants also arise, but all of them have not yet been determined.

7. *List of irreducible H matrices of three or less columns for* $n \leqslant 6$.

$n = 1$: [1].

$n = 2$: [2], [1 1].

$n = 3$: [3], [2 1], [1 1 1] $\approx$ [2 2 2].

$n = 4$: [4], [3 1], [2 2], [2 1 1] $\approx$ [3 3 2], $\begin{bmatrix} 3 & 2 & 2 \\ 1 \end{bmatrix}$.

$n = 5$: [5], [4 1], [3 2], [3 1 1] $\approx$ [4 4 2], [2 2 1] $\approx$ [4 3 3],

$$\begin{bmatrix} 3 & 3 & 2 \\ 1 & 1 \end{bmatrix} \approx \begin{bmatrix} 4 & 4 & 3 \\ 2 & 2 \end{bmatrix}, \quad \begin{bmatrix} 3 & 3 & 3 \\ 1 \end{bmatrix} \approx \begin{bmatrix} 4 & 2 & 2 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 4 & 3 & 2 \\ 1 \end{bmatrix}.$$

$n = 6$: [6], [5 1], [4 2], [3 3], [4 1 1] $\approx$ [5 5 2], [3 2 1] $\approx$ [5 4 3],

$$[2\ 2\ 2] \approx [4\ 4\ 4], \quad \begin{bmatrix} 3 & 3 & 3 \\ 1 & 1 & 1 \end{bmatrix} \approx \begin{bmatrix} 5 & 5 & 5 \\ 3 & 3 & 3 \end{bmatrix}, \quad \begin{bmatrix} 4 & 3 & 3 \\ 1 & 1 \end{bmatrix} \approx \begin{bmatrix} 5 & 5 & 3 \\ 3 & 2 \end{bmatrix},$$

$$\begin{bmatrix} 4 & 4 & 2 \\ 1 & 1 \end{bmatrix} \approx \begin{bmatrix} 5 & 5 & 4 \\ 2 & 2 \end{bmatrix}, \quad \begin{bmatrix} 5 & 3 & 2 \\ 2 \end{bmatrix} \approx \begin{bmatrix} 4 & 4 & 3 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 4 & 3 & 2 \\ 2 & 1 \end{bmatrix} \approx \begin{bmatrix} 5 & 4 & 4 \\ 3 & 2 \end{bmatrix},$$

$$\begin{bmatrix} 3 & 4 & 4 \\ 1 \end{bmatrix} \approx \begin{bmatrix} 5 & 2 & 2 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 4 & 3 & 3 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 5 & 4 & 2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 5 & 3 & 3 \\ 1 \end{bmatrix},$$

$$\begin{bmatrix} 5 & 4 & 3 \\ 3 & 2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 4 & 4 & 4 \\ 1 & 2 & 2 \\ 1 \end{bmatrix} \approx \begin{bmatrix} 5 & 4 & 4 \\ 5 & 2 & 2 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 5 & 4 & 4 \\ 2 & 2 & 1 \end{bmatrix}, \quad \begin{bmatrix} 4 & 4 & 3 \\ 2 & 2 & 3 \end{bmatrix}.$$

Number of forms (or matrices).

	$w = 1$	2	3	4	Total
$n = 1$	1				1
$= 2$	2				2
$= 3$	3	1			4
$= 4$	4	2			6
$= 5$	5	6	1		12
$= 6$	7	14	7	2	30

Except $\begin{bmatrix} 5 & 4 & 4 \\ 2 & 2 & 1 \end{bmatrix}$ and $\begin{bmatrix} 4 & 4 & 3 \\ 2 & 2 & 3 \end{bmatrix}$ each of the above is easily traced back in the hereditary system to [1], and is therefore irreducible. Again, with these two exceptions, the above list has been constructed from [1] by performing the operations $\sim$, $\approx$ in every possible way up to $n = 6$.

Corresponding to each of the above matrices there is an irreducible invariant ϕ, unique in the sense that two representative invariants of the same matrix differ from each other by reducible terms. For example, the irreducible invariant corresponding to

$$\begin{bmatrix} 5 & 4 & 3 \\ 3 & 2 & \\ 1 & & \end{bmatrix}$$

may be taken as

$$\phi = \Sigma\,(A_5 c_1)(A_3 B c_2)(a B_4 c_3), \tag{22}$$

a sum of six terms whose C_3 is determined by the three independent points c_1, c_2, c_3. Another such invariant is

$$\phi' = \Sigma(A_5 b_1')(A_3 b_1 b_2 b_3)(a C_3 b_4 b_2'),$$

where b_1' and b_2' and b_1, b_2, b_3, b_4 are permuted independently. The two invariants ϕ and ϕ' are congruent modulo reducible terms (in this case terms which are identically zero), as is evident by convolving $B_4 = b_1 b_2 b_3 b_4$ in the last factor of ϕ' and then $B_2 = b_1' b_2'$ in the second factor. There are many other representations of this matrix, but all such representations are equivalent in the sense outlined above. In fact, even for $n = 7$, $m = 3$ no matrix has yet been discovered to which correspond more than one irreducible invariant. It is, however, probable that for higher values of n this is no longer the case.

It is easy to interpret the vanishing of one of these invariants geometrically. In fact, if in (22) ϕ is zero, the prime A_5, the prime through the plane A_3 and the line B_2, the prime through B_4 and the point a, and the plane C_3 have a point in common.

8. *Prepared systems for two quadratics.* In the previous section we have confined our investigation to the invariants of categories 1, 2, and 3. We have done so for the reason that they are intimately connected with another branch of invariant theory, the concomitant system of two quadratics. For if $f = a_x^2$ and $g = b_x^2$ are two quadratics in symbolic form, every rational integral concomitant of the two quadratics may be

represented as a sum of terms where each term is a product of simple factors of the type

$$(d_1 d_2 \dots d_k p_{n-k}).$$

In such a factor p_{n-k} is a compound coordinate such that

$$p_1 = u, \quad p_2 = uv, \quad \dots, \quad p_{n-1} = uv \dots w = x, \tag{23}$$

and each symbol d_j is equivalent to a or b and must occur an even number of times in each product*. A prepared system for the two quadratics is a system of factor types in terms of which every concomitant of the two quadratics may be expressed and such that equivalent symbols may be regarded as the same. Such a system is composed of factors involving the symbols A_j, B_j, $j = 1, 2, \dots, n$, where A_j is a convolution of j equivalent symbols a and B_j of j equivalent symbols b. A concomitant expressed in terms of such a system is reducible if at any stage it is possible to convolve a symbol from A_i with A_j or a symbol from B_i with B_j when $i \leqslant j$. Thus, if a is a symbol from A_j,

$$aA_i \equiv 0 \text{ (is reducible) } i \leqslant j.$$

But this is analogous to supposing that the symbols A_j, B_j form a nest in the sense of our previous work. Since the symbols p_j, by virtue of (23), already form a nest, *a prepared system for two quadratics in n variables can be obtained from a system of H matrices of categories* 1, 2, 3 *in the field n.*

For example, if $n = 4$ and

$$A_2 = A, \quad A_3 = \alpha, \quad B_2 = B, \quad B_3 = \beta, \quad C_3 = p_3 = x, \quad C_2 = p_2 = p, \quad c = u,$$

we obtain the prepared system as follows:

$$a_\alpha, b_\beta, u_x \text{ from } [4],$$

$$b_\alpha, a_\beta, u_\alpha, u_\beta, a_x, b_x \text{ from } [3, 1],$$

$$(AB), (Ap), (Bp) \text{ from } [2, 2],$$

$$(Abu), (Bau), (abp) \text{ from } [2, 1, 1],$$

$$(a\alpha Bp), (b\beta Ap), (ABux) = (a_1 Bu)a_{2x} - (a_2 Bu)a_{1x} \text{ from } \begin{bmatrix} 3 & 2 & 2 \\ 1 & & \end{bmatrix},$$

$$(A\beta x), (Bax), (\alpha\beta p) = u_\alpha v_\beta - u_\beta v_\alpha \text{ from } [3\ 3\ 2].$$

* Turnbull, *op. cit.*, 329, 330; J. Williamson, "A special prepared system for two quadratics in n variables", *American J. of Math.*, 52 (1930), 863–876.

Of these the first three members need not be considered further since u_x is the absolute concomitant and the presence of a_α (or b_β) in a concomitant implies its reducibility, since it has the invariant factor $a_\alpha{}^2$ (or $b_\beta{}^2$). The remaining eighteen factors form the complete system*.

In a similar manner, the forty-two factors of the prepared system for $n = 5$ and the 118 factors of the prepared system for $n = 6$ may at once be written down. The prepared systems when $n = 5$ or $n = 6$ have been determined previously by another, more complicated, method†.

The previous determination, for $n = 6$, contained a factor which is redundant. With the notation of the previous paper that factor is written

$$(12,\ 34,\ 65)$$

and corresponds to the H matrix $\begin{bmatrix} 4 & 4 & 4 \\ 2 & 2 & 2 \end{bmatrix}$, which was shown in § 6 to be reducible. The reduction of (12, 34, 65), using the methods and notation of the previous paper, is, for interest, here appended:

$$\begin{aligned}(12,\ 34,\ 65) &= (1\ 2\ 3'\ 6'')\,(4'\ 5'') \\ &= (1\ 2\ 3'\ 6')\,(4'\ 5') + (1\ 2\ 3\ 4)\,(6\ 5) + (1\ 2\ 5\ 6)\,(4\ 3) \\ &= (4\ 5\ 3\ 6)\,(1\ 2) + (1\ 2\ 3\ 4)\,(6\ 5) + (1\ 2\ 5\ 6)\,(4\ 3).\end{aligned}$$

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* *Cf.* H. W. Turnbull, *Proc. London Math. Soc.* (2), 18 (1919), 69–94.

† J. Williamson, *American J. of Math.*, 53 (1931), 343–357.

A PROPERTY OF A REAL BOUNDED SEQUENCE

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1. There is a property of a real bounded sequence which presents itself in connection with a process of iterated division of an integral function. Let $f(x)$ and $\phi_1(x)$ be integral functions for which $f(0) = \phi_1(0) = 1$. We can write

$$f(x) = (1-gx)\,\phi_1(x) - mx^2\psi_1(x), \tag{1}$$

determining g and m so that $\psi_1(x)$ is an integral function, and $\psi_1(0) = 1$, assuming that $m \neq 0$. We now take as a new divisor, in place of $\phi_1(x)$, the integral function $\phi_2(x)$, where

$$\phi_2(x) = (1-gx)\,\psi_1(x), \tag{2}$$

and determine a new g and m, and so on indefinitely. It can be shown* that, under certain conditions, when $f(x)$ has only real zeros, the values of m and the alternate values of g tend to definite limits which are related in a simple way to the numerically smallest positive and negative zeros of $f(x)$.

To show how this process is related to the property of a sequence which will be discussed in this paper, it will be sufficient to take the simple case in which $f(x)$ is of genus zero, and all its zeros are positive. Then we write

$$f(x) = \prod_{s=1}^{\infty} (1-a_s x) = 1 + a_1 x + a_2 x^2 + \ldots,$$

where $\sum_{s=1}^{\infty} a_s$ is convergent, and $\sum_{s=1}^{\infty} a_s = -a_1$. Also

$$f'(x) = -\sum_{s=1}^{\infty} a_s (1-a_s x)^{-1} f(x) = a_1 \phi_1(x),$$

where $\phi_1(x)$ is an integral function, which we take as the first divisor in (1).

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* See the following paper, *Integral functions with real zeros.*

Writing $q_s = \alpha_s(\Sigma\alpha_s)^{-1}$, so that $\sum_{s=1}^{\infty} q_s$ is a convergent series of positive terms, and $\sum_{s=1}^{\infty} q_s = 1$, we get $\phi_1(x) = \Sigma q_s(1-\alpha_s x)^{-1}f(x)$.

If now $\phi_2(x) = (1-gx)\psi_1(x) = \sum_{s=1}^{\infty} q_s'(1-\alpha_s x)^{-1}f(x)$, we get, from (1) and (2), that

$$1-gx = (1-gx)^2 \sum_{s=1}^{\infty} q_s(1-\alpha_s x)^{-1} - mx^2 \sum_{s=1}^{\infty} q_s'(1-\alpha_s x)^{-1},$$

and it easily follows that

$$g = \sum_{s=1}^{\infty} q_s\alpha_s, \quad m = \sum_{s=1}^{\infty} q_s(\alpha_s-g)^2, \quad q_s' = m^{-1}q_s(\alpha_s-g)^2, \tag{3}$$

and we can next calculate the new values of g and m, and so on.

We are thus led to consider the sets of values of g, m, and q_s ($s = 1, 2, 3, \ldots$) determined in succession in this way. The property of a sequence (α_s), which will be established, holds good in much wider conditions than those in which the values of α_s are the reciprocals of the zeros of an integral function; and for this reason the theorem will be enunciated and proved in the more general form..

2. Let $a_1, a_2, a_3, \ldots, a_s, \ldots$ be an infinite sequence of real numbers, bounded above and below, the upper and lower bounds being α and β. Let $\sum_{s=1}^{\infty} q_{rs}$ be a convergent series, whose terms are non-negative, and whose sum is unity. Let

$$g_r = \sum_{s=1}^{\infty} q_{rs}a_s, \tag{4}$$

and

$$m_r = \sum_{s=1}^{\infty} q_{rs}(a_s-g_r)^2 = \sum_{s,t} q_{rs}q_{rt}(a_s-a_t)^2, \tag{5}$$

so that, if κ is the greater of $|\alpha|$ and $|\beta|$,

$$|g_r| < \kappa, \quad \text{and} \quad 0 < m_r < 4\kappa^2. \tag{6}$$

We form a new series of weights $\sum_{s=1}^{\infty} q_{r+1,s}$, where

$$q_{r+1,s} = q_{rs}(a_s-g_r)^2 m_r^{-1} \quad (s = 1, 2, 3, \ldots,), \tag{7}$$

so that $q_{r+1,s} \geqslant 0$, $\sum_{s=1}^{\infty} q_{r+1,s}$ is convergent, and $\sum_{s=1}^{\infty} q_{r+1,s} = 1$. We can now find g_{r+1}, m_{r+1}, and repeat the operations. Thus, starting with any convergent series $\sum_{s=1}^{\infty} q_{1s}$, whose terms are assumed to be all positive,

and whose sum is unity, we can calculate the sets q_{rs} $(s = 1, 2, 3, \ldots)$ for successive values of r.

We shall prove that, when $r \to \infty$, m_r tends to a limit m, and g_r (r odd) and g_r (r even) tend to limits g, g' respectively, and, further, that the zeros of the quadratic $(x-g)(x-g')-m$ are α and β.

We shall write $\sum_{s=1}^{\infty} q_{rs} a_s^{\,k} = s_{rk}$ $(k \geqslant 0)$, so that $s_{r0} = 1$, and $s_{r1} = g_r$.

We also note that it follows from (4) and (5) that

$$\sum_{s=1}^{\infty} q_{rs}(a_s-g_r)(a_s-g_{r+1}) = \sum_{s=1}^{\infty} q_{rs}(a_s-g_r)^2 + \sum_{s=1}^{\infty} q_{rs}(a_s-g_r)(g_r-g_{r+1}) = m_r. \quad (8)$$

3. We first prove that m_r tends to a limit. From (4) and (5),

$$m_{r+1} = \sum_s q_{r+1,\,s}\, a_s^{\,2} - g_{r+1}^2,$$

and, using (7), we have

$$m_r^{\,2} m_{r+1} = m_r \sum_s q_{rs}\, a_s^{\,2}(a_s-g_r)^2 - \left[\sum_s q_{rs}\, a_s (a_s-g_r)^2\right]^2$$

$$= (s_{r2}-g_r^{\,2})(s_{r4}-2g_r s_{r3}+g_r^{\,2} s_{r2}) - (s_{r3}-2g_r s_{r2}+g_r^{\,3})^2$$

and

$$m_r^{\,3} = (s_{r2}-g_r^{\,2})^3.$$

Hence

$$m_r^{\,2}(m_{r+1}-m_r) = s_{r2}\, s_{r4} - s_{r3}^2 - s_{r2}^3 + 2g_r\, s_{r2}\, s_{r3} - g_r^{\,2}\, s_{r4},$$

and

$$m_{r+1}-m_r = m_r^{-2} \begin{vmatrix} s_{r0}, & s_{r1}, & s_{r2} \\ s_{r1}, & s_{r2}, & s_{r3} \\ s_{r2}, & s_{r3}, & s_{r4} \end{vmatrix}$$

$$= m_r^{-2} \sum_{stu} q_{rs}\, q_{rt}\, q_{ru} (a_s-a_t)^2 (a_t-a_u)^2 (a_u-a_s)^2 \quad (9)$$

and is positive, so that m_r increases with r. Hence, since $m_r < 4\kappa^2$, by (6), it follows that m_r tends to a limit m when $r \to \infty$.

This result also follows from the identity, which is easily proved,

$$m_r(m_{r+1}-m_r) = \sum_{st} q_{rs}\, q_{rt} [(a_s-g_r)(a_s-g_{r+1}) - (a_t-g_r)(a_t-g_{r+1})]^2.$$

It follows from (9) that the series

$$\sum_r m_r^{-2} \sum_{stu} q_{rs}\, q_{rt}\, q_{ru} (a_s-a_t)^2 (a_t-a_u)^2 (a_u-a_r)^2$$

is convergent. Also, for every set of values of s, t, u, the series $\sum_r q_{rs}\, q_{rt}\, q_{ru}$ is convergent, and hence is also convergent when r is restricted to odd or to

even values. Also $q_{rs}q_{rt}q_{ru} \to 0$ when $r \to \infty$; and the series

$$\sum_r \sum_{stu} q_{rs}q_{rt}q_{ru} \tag{10}$$

is convergent when r is restricted to odd or to even values.

4. We next prove that the alternate values of g_r tend to limits g, g', say. For, by (4) and (7),

$$g_{r+1} = \sum_s q_{r+1,s}a_s = m_r^{-1}\sum_s q_{rs}a_s(a_s-g_r)^2,$$

so that
$$m_r g_{r+1} = s_{r3} - 2g_r s_{r2} + g_r^3,$$

and
$$g_{r+1}+g_r = (s_{r3}-g_r s_{r2})m_r^{-1}$$

$$= \begin{vmatrix} s_{r0}, & s_{r1} \\ s_{r2}, & s_{r3} \end{vmatrix} \Big/ \begin{vmatrix} s_{r0}, & s_{r1} \\ s_{r1}, & s_{r2} \end{vmatrix}. \tag{11}$$

When the terms q_{rs} are replaced by the terms $q_{r+1,s}$, then s_{rk} becomes

$$m_r^{-1}\sum_s q_{rs}a_s^k(a_s-g_r)^2 = m_r^{-1}(s_{r,k+2} - 2g_r s_{r,k+1} + g_r^2 s_{rk}).$$

Hence, from (11), we get that $g_{r+2}+g_{r+1}$ is

$$\frac{\begin{vmatrix} s_{r2}-2g_r s_{r1}+g_r^2 s_{r0}, & s_{r3}-2g_r s_{r2}+g_r^2 s_{r1} \\ s_{r4}-2g_r s_{r3}+g_r^2 s_{r2}, & s_{r5}-2g_r s_{r4}+g_r^2 s_{r3} \end{vmatrix}}{\begin{vmatrix} s_{r2}-2g_r s_{r1}+g_r^2 s_{r0}, & s_{r3}-2g_r s_{r2}+g_r^2 s_{r1} \\ s_{r3}-2g_r s_{r2}+g_r^2 s_{r1}, & s_{r4}-2g_r s_{r3}+g_r^2 s_{r2} \end{vmatrix}},$$

where the denominator is $m_r^2 m_{r+1}$. Hence $m_r^3 m_{r+1}(g_{r+2}-g_r)$ is equal to

$$\begin{vmatrix} s_{r2}-2g_r s_{r1}+g_r^2 s_{r0}, & s_{r3}-2g_r s_{r2}+g_r^2 s_{r1} \\ s_{r4}-2g_r s_{r3}+g_r^2 s_{r2}, & s_{r5}-2g_r s_{r4}+g_r^2 s_{r3} \end{vmatrix} \begin{vmatrix} s_{r0}, & s_{r1} \\ s_{r1}, & s_{r2} \end{vmatrix}$$

$$- \begin{vmatrix} s_{r2}-2g_r s_{r1}+g_r^2 s_{r0}, & s_{r3}-2g_r s_{r2}+g_r^2 s_{r1} \\ s_{r3}-2g_r s_{r2}+g_r^2 s_{r1}, & s_{r4}-2g_r s_{r3}+g_r^2 s_{r2} \end{vmatrix} \begin{vmatrix} s_{r0}, & s_{r1} \\ s_{r2}, & s_{r3} \end{vmatrix},$$

which reduces to

$$\begin{vmatrix} s_{r0}, & s_{r1}, & s_{r2}, & s_{r3} \\ s_{r1}, & s_{r2}, & s_{r3}, & s_{r4} \\ s_{r2}, & s_{r3}, & s_{r4}, & s_{r5} \\ s_{r0}, & s_{r1}, & s_{r1}^2, & s_{r1}^3 \end{vmatrix}$$

$$= \sum_{stu} q_{rs}q_{rt}q_{ru}(a_s-a_t)^2(a_t-a_u)^2(a_u-a_s)^2(g_r-a_s)(g_r-a_t)(g_r-a_u).$$

Hence from (6) we get

$$|g_{r+2}-g_r| < \lambda \sum_{stu} q_{rs}q_{rt}q_{ru},$$

where λ is assignable independently of r, and from the convergency of the series $\sum_r \sum_{stu} q_{rs}q_{rt}q_{ru}$ in (10), for even or for odd values of r, it follows that g_r (r odd) tends to a limit g, say, and g_r (r even) to a limit g', when $r \to \infty$.

5. We define c_{rs} by the relation

$$c_{rs} = q_{rs}(a_s-g_r)\,m_r^{-1}, \tag{12}$$

so that $\sum_{s=1}^{\infty} c_{rs}a_s^{\,k}$ converges ($k = 1, 2, 3, \ldots$), and, from (4) and (5),

$$\sum_{s=1}^{\infty} c_{rs} = 0, \quad \sum_{s=1}^{\infty} c_{rs}a_s = 1. \tag{13}$$

Since $m_r \geqslant m_1$, which is positive, and $|a_s-g_r| < 2\kappa$, by (6), it follows from (12) that

$$\sum_s |c_{rs}| < M, \tag{14}$$

where the summation includes any number of terms of the series $\sum_s |c_{rs}|$, and M is assignable independently of r.

Also $$|c_{rs}c_{rt}c_{ru}| \leqslant \mu\, q_{rs}q_{rt}q_{ru},$$

where μ is assignable independently of r, so that $c_{rs}c_{rt}c_{ru} \to 0$ when $r \to \infty$, for every set of values of s, t, u.

From (12) we get

$$\frac{c_{r+1,\,s}}{c_{rs}} = \frac{q_{r+1,\,s}(a_s-g_{r+1})\,m_r}{q_{rs}(a_s-g_r)\,m_{r+1}}$$

$$= \frac{(a_s-g_r)(a_s-g_{r+1})}{m_{r+1}}, \tag{15}$$

using (7), and we get also

$$\frac{|c_{r+1,\,s}|}{|c_{rs}|} \to \frac{|(a_s-g)(a_s-g')|}{m} \tag{16}$$

when $r \to \infty$.

Again, we have

$$g_r \neq \alpha, \quad g_r \neq \beta. \tag{17}$$

For, if $g_r = \alpha$, then $\alpha \sum_s q_{rs} = \sum_s q_{rs}a_s$, and $\sum_s q_{rs}(\alpha-a_s) = 0$, and, since

$\alpha \geqslant a_s$, we have $q_{rs}(\alpha - a_s) = 0$ for every value of s. Hence $q_{rs} = 0$ for every value of s except those, if any, for which $a_s = \alpha = g_r$. We shall, therefore, have $m_r = \sum_s q_{rs}(a_s - g_r)^2 = 0$, which is impossible, since m_r increases with r and m_1 is positive. Hence $g_r \neq \alpha$, and similarly $g_r \neq \beta$.

6. Since $\alpha > g_r > \beta$, $\alpha > g_{r+1} > \beta$, $\alpha \geqslant a_s \geqslant \beta$, it follows that, if

$$(\alpha - g_r)(\alpha - g_{r+1}) \geqslant (\beta - g_r)(\beta - g_{r+1}),$$

then

$$(\alpha - g_r)(\alpha - g_{r+1}) \geqslant (a_s - g_r)(a_s - g_{r+1})$$

for every value of s. Hence, using (8), we have

$$(\alpha - g_r)(\alpha - g_{r+1}) - m_r = \sum_s q_{rs}[(\alpha - g_r)(\alpha - g_{r+1}) - (a_s - g_r)(a_s - g_{r+1})]$$

$$\geqslant 0.$$

Thus m_r cannot exceed both $(\alpha - g_r)(\alpha - g_{r+1})$ and $(\beta - g_r)(\beta - g_{r+1})$, so that at least one of the relations

$$(\alpha - g)(\alpha - g') \geqslant m, \quad (\beta - g)(\beta - g') \geqslant m \tag{18}$$

is true.

Now $(\alpha - g)(\alpha - g') \leqslant m$. For suppose that

$$(\alpha - g)(\alpha - g')\, m^{-1} = 1 + \delta \quad (\delta > 0).$$

We can choose λ so that, for

$$\alpha - \lambda \leqslant x \leqslant \alpha, \tag{19}$$

$$\left| \frac{(x-g)(x-g')}{m} - \frac{(\alpha-g)(\alpha-g')}{m} \right| < \tfrac{1}{3}\delta,$$

and we can find p so that, for $r \geqslant p$ and for all such values of x,

$$\left| \frac{(x-g)(x-g')}{m} - \frac{(x-g_r)(x-g_{r+1})}{m_{r+1}} \right| < \tfrac{1}{3}\delta,$$

whence

$$\frac{|(x-g_r)(x-g_{r+1})|}{m_{r+1}} > 1 + \tfrac{1}{3}\delta,$$

and, provided that $c_{rs} \neq 0$, we deduce from (15) that

$$\frac{|c_{r+1,\,s}|}{|c_{rs}|} > 1 + \tfrac{1}{3}\delta, \tag{20}$$

when $r \geqslant p$, for all values of s for which a_s lies in the interval (19). Now from (7) and (12) it follows that $c_{ps} = 0$ only if a_s is equal to one of $g_1, g_2, \ldots, g_p$.

If, then, α is a limit-point of the sequence (a_s), there are an unlimited number of values of s for which a_s lies in the interval (19) and $c_{ps} \neq 0$. Hence, by (20), c_{rs} increases indefinitely with r; and this is impossible, since c_{rs} is bounded, by (14).

If, however, α is an isolated term (whether single or repeated) of the sequence, say $\alpha = a_n$, the same argument shows that, for $r \geqslant p$,

$$\frac{|c_{r+1,n}|}{|c_{rn}|} > 1+\tfrac{1}{3}\delta,$$

and in this case $c_{pn} \neq 0$, since no value of g_r can be equal to α. Hence c_{rn} increases indefinitely with r, which is impossible.

Thus $(\alpha-g)(\alpha-g') \leqslant m$, and similarly $(\beta-g)(\beta-g') \leqslant m$.

Hence, if α and β do not both satisfy the equation $(x-g)(x-g')=m$, it follows from (18) that one of them, say α, satisfies it, and then

$$(\alpha-g)(\alpha-g') = m > (\beta-g)(\beta-g') \geqslant 0.$$

We shall now show that this is impossible.

7. For suppose that $(\alpha-g)(\alpha-g') = m > (\beta-g)(\beta-g')$. Then, for $\alpha > x \geqslant \beta$,

$$m^{-1}(x-g)(x-g') < 1.$$

If, for a real value of l,

$$m^{-1}(l-g)(l-g') < -1,$$

we see, by the argument of § 6, that l cannot be a limit point of the sequence; moreover, if l is an isolated point, say a_n, of the sequence, the same argument shows that, if $|m^{-1}(l-g)(l-g')| = 1+\delta$, we can find p so that, for $r \geqslant p$,

$$\frac{|(a_n-g_r)(a_n-g_{r+1})|}{m_{r+1}} > 1+\tfrac{1}{2}\delta,$$

and hence, unless $c_{pn}=0$, $|c_{rn}|$ increases indefinitely with r. Hence $c_{pn}=0$, and so $c_{rn}=0$, for $r \geqslant p$.

If the roots of $(x-g)(x-g')+m=0$ are real, we denote them by γ_1, γ_2 $(\gamma_1 > \gamma_2)$. We take an arbitrarily small positive λ, and consider the intervals (i) $\alpha \geqslant x \geqslant \alpha-\lambda$, (ii) $\gamma_1+\lambda \geqslant x \geqslant \gamma_1-\lambda$, (iii) $\gamma_1-\lambda \geqslant x \geqslant \gamma_2+\lambda$, (iv) $\gamma_2+\lambda \geqslant x \geqslant \gamma_2-\lambda$, (v) the remainder of the interval $\alpha \geqslant x \geqslant \beta$. If the roots of $(x-g)(x-g')+m=0$ are unreal, we have only the intervals (i) and (v).

Since $(\alpha-g)(\alpha-g')=m=|(\gamma_1-g)(\gamma_1-g')|=|(\gamma_2-g)(\gamma_2-g')|$, then, for every value of x in the intervals (i), (ii), and (iv),

$$|(x-g)(x-g')|\,m^{-1}\sim 1<\tfrac{1}{2}\delta,$$

where δ can be made as small as we please by sufficiently diminishing λ. Also we can choose p so that, for $r\geqslant p$ and for every value of x in the interval $\alpha\geqslant x\geqslant\beta$,

$$|(x-g)(x-g')\,m^{-1}-(x-g_r)(x-g_{r+1})\,m_{r+1}^{-1}|<\tfrac{1}{2}\delta.$$

Hence, for $r\geqslant p$ and for every x in the intervals (i), (ii), and (iv),

$$|(x-g_r)(x-g_{r+1})|\,m_{r+1}^{-1}\sim 1\leqslant\delta.$$

Thus, if a_s lies in the interval (i),

$$(a_s-g_r)(a_s-g_{r+1})\,m_{r+1}^{-1}=1+\delta_{rs}, \tag{21}$$

and if a_s' lies in the intervals (ii) or (iv),

$$(a_s'-g_r)(a_s'-g_{r+1})\,m_{r+1}^{-1}=-(1+\delta_{rs}'), \tag{22}$$

where

$$|\delta_{rs}|<\delta,\quad |\delta_{rs}'|<\delta. \tag{23}$$

As regards the interval (iii), we have already seen above that this interval can contain no limit point of the sequence, since within it $(x-g)(x-g')\,m^{-1}<-1$, and, further, that, if an isolated term a_s lies in this interval, then $c_{rs}=0$ when r exceeds an assignable p'. Thus $\Sigma c_{rs}=0$, and $\Sigma c_{rs}a_s=0$, the summation referring to all such terms.

Finally, in the interval (v),

$$|(x-g)(x-g')|\,m^{-1}<1-\eta,$$

where $\eta\,(0<\eta<1)$ can be assigned when λ has been assigned. Also we can choose ρ so that, if $\sigma>\rho$, then, for all points of the interval $\alpha\geqslant x\geqslant\beta$,

$$|(x-g)(x-g')\,m^{-1}-(x-g_\sigma)(x-g_{\sigma+1})\,m_{\sigma+1}^{-1}|<\tfrac{1}{2}\eta,$$

whence

$$|x-g_\sigma||x-g_{\sigma+1}|\,m_{\sigma+1}^{-1}<1-\tfrac{1}{2}\eta. \tag{24}$$

Now from (14) it follows that the sum of any number of terms of the series $\Sigma_s|c_{rs}|$ is less than a fixed M. If, then, K_σ denotes the sum of all terms of the series $\Sigma_s c_{\sigma s}$ for which the corresponding a_s lies in the interval (v), we have from (15) and (24) that, if $\sigma>\rho$,

$$\begin{aligned}|K_\sigma|&<M(1-\tfrac{1}{2}\eta)^{\sigma-\rho}\\&<\delta,\end{aligned} \tag{25}$$

if σ exceeds an assignable τ.

Since $|a_s| \leqslant \kappa$, where κ is the greater of $|\alpha|$ and $|\beta|$, we also have, for these values of s, $\left|\sum_s c_{\sigma s} a_{\sigma s}\right| \leqslant \kappa |K_\sigma|$; hence $\sum_s c_{\sigma s} a_{\sigma s} = \mu K_\sigma$, where $|\mu| \leqslant \kappa$.

We shall now denote c_{rs} by C_{rs} if a_s lies in the interval (i), and by C'_{rs} if it lies in the intervals (ii) or (iv). Then it follows from the preceding results, and the equations (13), that, if r exceeds an assignable n,

$$\sum_s C_{rs} + \sum_s C'_{rs} + K_r = 0, \tag{26}$$

$$\sum_s C_{rs} a_s + \sum_s C'_{rs} a_s' + \mu K_r = 1, \tag{27}$$

and, changing r into $r+1$, and using (21) and (22), we also have

$$\sum_s C_{rs}(1+\delta_{rs}) - \sum_s C'_{rs}(1+\delta'_{rs}) + K_{r+1} = 0, \tag{28}$$

$$\sum_s C_{rs} a_s (1+\delta_{rs}) - \sum_s C'_{rs} a_s' (1+\delta'_{rs}) + \mu' K_{r+1} = 1, \tag{29}$$

where $|\mu| < \kappa$, $|\mu'| < \kappa$.

Adding (26) and (28), and also (27) and (29), we get, using (23) and (25),

$$\Sigma C_{rs} = \eta_r, \quad \Sigma C_{rs} a_s = 1 + \eta_r',$$

where $|\eta_r| < N\delta$, $|\eta_r'| < N\delta$, and N is assignable independently of δ.

Now, since a_s is in the interval (i), we have $\alpha \geqslant a_s \geqslant \alpha - \lambda$. Hence

$$|\Sigma C_{rs}| < N\delta, \quad |\alpha \Sigma C_{rs} - 1| < N\delta + M\lambda.$$

Since δ can be made as small as we please by diminishing λ sufficiently, these inequalities are inconsistent. Hence both α and β are zeros of $(x-g)(x-g') - m$.

I wish to thank Mr. W. L. Ferrar for corrections to the argument in paragraphs 5 and 6.

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INTEGRAL FUNCTIONS WITH REAL ZEROS

By G. S. Le Beau.

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1. Let $f(x) \equiv 1+a_1x+a_2x^2+\ldots+a_nx^n+\ldots$ be a real integral function of finite genus, whose zeros are all real, and let

$$1+b_{r1}x+b_{r2}x^2+\ldots+b_{rn}x^n+\ldots$$

be another integral function. We write

$$f(x) = (1-g_rx)(1+b_{r1}x+b_{r2}x^2+\ldots)-m_rx^2(1+c_{r1}x+c_{r2}x^2+\ldots), \quad (1)$$

where $g_r = b_{r1}-a_1$, and $m_r = b_{r2}-a_2-b_{r1}g_r$. We assume that $m_r \neq 0$, and it will subsequently be shown that this condition holds in all the cases that will be considered. Then $1+c_{r1}x+c_{r2}x^2+\ldots$ will also be an integral function, as well as

$$(1-g_rx)(1+c_{r1}x+c_{r2}x^2+\ldots), \quad (2)$$

which we rewrite as

$$1+b_{r+1,1}x+b_{r+1,2}x^2+b_{r+1,3}x^3+\ldots,$$

and we take this latter function as a new divisor of $f(x)$, determining g_{r+1}, m_{r+1}, and $c_{r+1,s}$ $(s = 1, 2, 3, \ldots)$ as before. Thus, starting with an initial divisor $1+\sum_{s=1}^{\infty} b_{1s}x^s$, we can determine the values of g_r and m_r for successive values of r, these values of g_r and m_r being rational functions of the coefficients of $f(x)$ and the initial divisor.

We shall prove that, if the initial divisor is suitably chosen, m_r is always positive, and increases steadily to a limit m, that g_r (r odd) and g_r (r even) tend to limits g, g', and that the roots of the quadratic equation

$$(x-g)(x-g') = m$$

are the reciprocals of the least positive and the least negative zeros of the function $f(x)$. If the zeros of $f(x)$ are all positive or all negative, one root of the quadratic is zero.

2. We can suppose the zeros of $f(x)$ to be arranged in non-descending order of absolute value, and let their reciprocals be $a_1, a_2, a_3, \ldots$. Let us assume that the integral function

$$1+b_{r1}x+b_{r2}x^2+b_{r3}x^3+\ldots,$$

used as a divisor at any stage, can be expressed in the form

$$\sum_{s=1}^{\infty} \frac{q_{rs}f(x)}{1-xa_s} \quad (|x|<|a_1^{-1}|), \tag{3}$$

where the coefficients q_{rs} are not negative, $\sum_{s=1}^{\infty} q_{rs}$ converges, and $\sum_{s=1}^{\infty} q_{rs}=1$. Then we have, from (1),

$$m_r x^2(1+c_{r1}x+c_{r2}x^2+\ldots) = (1-xg_r)\sum_{s=1}^{\infty}\frac{q_{rs}f(x)}{1-xa_s} - f(x)$$

$$= \sum_{s=1}^{\infty}\frac{q_{rs}(a_s-g_r)xf(x)}{1-xa_s}, \tag{4}$$

so that

$$0 = \sum_{s=1}^{\infty} q_{rs}(a_s-g_r),$$

whence

$$g_r = \sum_{s=1}^{\infty} q_{rs}a_s, \tag{5}$$

and

$$m_r = \sum_{s=1}^{\infty} q_{rs}a_s(a_s-g_r) = \sum_{s=1}^{\infty} q_{rs}(a_s-g_r)^2. \tag{6}$$

Hence from (4) we have

$$(1-g_rx)(1+c_{r1}x+c_{r2}x^2+c_{r3}x^3+\ldots)$$

$$= \sum_{s=1}^{\infty}\frac{(1-xg_r)q_{rs}a_s(a_s-g_r)f(x)}{m_r(1-xa_s)}$$

$$= \sum_{s=1}^{\infty}\frac{q_{rs}(a_s-g_r)[a_s-g_r+g_r(1-xa_s)]f(x)}{m_r(1-xa_s)}$$

$$= \sum_{s=1}^{\infty}\left[\frac{q_{rs}(a_s-g_r)^2}{m_r(1-xa_s)}+\frac{g_r}{m_r}(a_s-g_r)q_{rs}\right]f(x)$$

$$= \sum_{s=1}^{\infty}\frac{q_{rs}(a_s-g_r)^2f(x)}{m_r(1-xa_s)} = \sum_{s=1}^{\infty}\frac{q_{r+1,s}f(x)}{1-xa_s}, \tag{7}$$

if

$$q_{r+1,s} = q_{rs}(a_s-g_r)^2m_r^{-1}, \tag{8}$$

and g_r and m_r are determined by (5) and (6). Thus $q_{r+1,s}$ is not negative, and $\sum_{s=1}^{\infty} q_{r+1,s}$ is convergent, and $\sum_{s=1}^{\infty} q_{r+1,s}=1$. Also we infer from (2) and

(7) that the new divisor, $1+\sum_{s=1}^{\infty} b_{r+1,\,s}\,x^s$, is also expressible in the form

$$\sum_{s=1}^{\infty} \frac{q_{r+1,\,s}\,f(x)}{1-xa_s} \quad (|x|<|a_1^{-1}|),$$

similar to the expression (3) for the previous divisor.

3. Now let the genus of $f(x)$ be $n-2$ or $n-1$, where n is even. Then we may write

$$f(x)=\exp(\beta_1 x+\beta_2 x^2+\ldots+\beta_{n-1}x^{n-1}) \times \prod_{s=1}^{\infty}\left[(1-xa_s)\exp\left(xa_s+\frac{x^2 a_s^{\,2}}{2}+\ldots+\frac{x^{n-1}a_s^{n-1}}{n-1}\right)\right],$$

where $\beta_1, \beta_2, \ldots, \beta_{n-1}$ are real, and $\sum_{n=1}^{\infty}|a_s|^n$ converges.

Hence

$$\phi(x)\equiv f'(x)-[\beta_1+2\beta_2 x+\ldots+(n-1)\beta_{n-1}x^{n-2}]f(x)=-\sum_{s=1}^{\infty}\frac{x^{n-1}a_s^{\,n}f(x)}{1-xa_s},$$

and $\phi(x)$ is an integral function divisible by x^{n-1}, a circumstance which, of course, determines $\beta_1, \beta_2, \ldots, \beta_{n-1}$ if n is known. Thus

$$\frac{\phi(x)}{f(x)}=-x^{n-1}\sum_{s=1}^{\infty}a_s^{\,n}\sum_{s=1}^{\infty}\frac{q_{1s}}{1-xa_s},$$

where $q_{1s}=a_s^{\,n}\left(\sum_{s=1}^{\infty}a_s^{\,n}\right)^{-1}$, so that q_{1s} is positive, and $\sum_{s=1}^{\infty}q_{1s}=1$. We can thus write $\phi(x)=-ax^{n-1}\left[1+\sum_{s=1}^{\infty}b_{1s}x^s\right]$, where

$$1+\sum_{s=1}^{\infty}b_{1s}x^s=\sum_{s=1}^{\infty}\frac{q_{1s}f(x)}{1-xa_s},$$

and is an integral function which, by (3), can be taken as the first divisor.

Since the sequence $\alpha_1, \alpha_2, \alpha_3, \ldots$ is bounded above and below, and the relations connecting $q_{r+1,\,s}$ and $q_{r,\,s}$, and determining g_r and m_r, are given by the equations (5), (6), and (7), it follows* that m_r is always positive, and increases to a limit m, that g_r (r odd) tends to a limit g, and g_r (r even) to a limit g', and that the roots of the equation $(x-g)(x-g')=m$ are the upper and lower bounds of the sequence of the reciprocals of the zeros of $f(x)$; this establishes the results required.

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* See the preceding paper.

THE BENDING OF MODERATELY THICK RECTANGULAR PLATES

By J. VINT.

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In this paper we consider the problem of the bending of a moderately thick uniform rectangular plate when normal forces are applied at its faces.

In the first section we obtain a normal form of the plane stress solution*, and in the second we work out a general normal solution which satisfies certain assumed conditions with regard to stress at the faces of the plate.

In § 3 we consider certain cases of the problem of a plate subject to constant normal traction c_0 on one face and c_1 on the opposite face.

The cases considered in § 4 are those in which the plate is subject to an arbitrary normal *pressure* over one face and the same distribution of arbitrary normal *pull* over the other face.

By reference to § 2 we see that the general cases of arbitrary normal pressure over each face depend on the solutions of § 4 and those for which there is the *same* arbitrary distribution of *pressure* over both faces†, but the latter do not seem to lend themselves to satisfactory treatment by the method of this paper.

The equations‡ governing the problem are:

$$\frac{\partial}{\partial x}\widehat{xx}+\frac{\partial}{\partial y}\widehat{xy}+\frac{\partial}{\partial z}\widehat{xz}=0, \quad \frac{\partial}{\partial x}\widehat{xy}+\frac{\partial}{\partial y}\widehat{yy}+\frac{\partial}{\partial z}\widehat{yz}=0,$$

$$\frac{\partial}{\partial x}\widehat{xz}+\frac{\partial}{\partial y}\widehat{yz}+\frac{\partial}{\partial z}\widehat{zz}=0,$$

* Love, *Elasticity*, 4th edition, § 300.

† *E.g.*, $c_0=c_1$ or $c_0=-c_1$.

‡ Love, *loc. cit.*, 466.

together with

$$\nabla^2 \widehat{xx} = -\frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial x^2}, \quad \nabla^2 \widehat{yy} = -\frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial y^2}, \quad \nabla^2 \widehat{zz} = -\frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial z^2},$$

and

$$\nabla^2 \widehat{yz} = -\frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial y\,\partial z}, \quad \nabla^2 \widehat{zx} = -\frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial z\,\partial x}, \quad \nabla^2 \widehat{xy} = -\frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial x\,\partial y},$$

where $$\Theta = \widehat{xx}+\widehat{yy}+\widehat{zz}.$$

1. *The normal form of the plane stress solution.*

The plane stress solution obtained in this section is that which arises from the analysis given by Love† when we take

$$\Theta = -(1+\sigma)(\gamma_1 z+\gamma_2)\cosh(nx+\epsilon)\cos(ny+\eta) = z\Theta_1+\Theta_0,$$

Θ_1 and Θ_0 being harmonic functions in x and y.

Then $\widehat{zz} = 0$ everywhere, while

$$\left.\begin{aligned} \widehat{xz} &= -\frac{1}{2(1+\sigma)}(z^2-h^2)\frac{\partial\Theta_1}{\partial x} \\ &= \tfrac{1}{2}n\gamma_1(z^2-h^2)\sinh(nx+\epsilon)\cos(ny+\eta), \\ \widehat{yz} &= -\frac{1}{2(1+\sigma)}(z^2-h^2)\frac{\partial\Theta_1}{\partial y} \\ &= -\tfrac{1}{2}n\gamma_1(z^2-h^2)\cosh(nx+\epsilon)\sin(ny+\eta), \end{aligned}\right\} \qquad (1)$$

$\widehat{xz}$ and $\widehat{yz}$ both vanishing at $z = \pm h$, the faces of the plate.

$\widehat{xx}$, $\widehat{xy}$, $\widehat{yy}$ then satisfy the equations

$$\frac{\partial}{\partial x}\widehat{xx}+\frac{\partial}{\partial y}\widehat{xy} = \frac{1}{1+\sigma}z\frac{\partial\Theta_1}{\partial x},$$

$$\frac{\partial}{\partial x}\widehat{xy}+\frac{\partial}{\partial y}\widehat{yy} = \frac{1}{1+\sigma}z\frac{\partial\Theta_1}{\partial y},$$

and so we may write

$$\widehat{xx} = \frac{1}{1+\sigma}z\Theta_1+\frac{\partial^2\chi}{\partial y^2}, \quad \widehat{xy} = -\frac{\partial^2\chi}{\partial x\,\partial y}, \quad \widehat{yy} = \frac{1}{1+\sigma}z\Theta_1+\frac{\partial^2\chi}{\partial x^2},$$

---

* Love, *loc. cit.*, §§ 299–303.

in which case it follows that

$$\chi = \tfrac{1}{6}\frac{2-\sigma}{1+\sigma} z^3 \Theta_1 - \tfrac{1}{2}\frac{\sigma}{1+\sigma} z^2 \Theta_0 + z\zeta_1 + \zeta_0,$$

where ζ_1 and ζ_0 are functions of x and y only. Also, since

$$\widehat{xx}+\widehat{yy}+\widehat{zz} = z\Theta_1+\Theta_0,$$

χ satisfies

$$\nabla_1^2 \chi = -\frac{1-\sigma}{1+\sigma} z\,\Theta_1 + \Theta_0,$$

where

$$\nabla_1^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

It now follows readily that

$$\begin{aligned}\nabla_1^2 \zeta_1 &= -\frac{1-\sigma}{1+\sigma}\Theta_1 \\ &= (1-\sigma)\gamma_1 \cosh(nx+\epsilon)\cos(ny+\eta), \\ \nabla_1^2 \zeta_0 &= \Theta_0 \\ &= -(1+\sigma)\gamma_2 \cosh(nx+\epsilon)\cos(ny+\eta),\end{aligned}$$

and so we may take

$$\zeta_1 = \frac{1}{2n}(1-\sigma)\gamma_1 x \sinh(nx+\epsilon)\cos(ny+\eta) + F_1,$$

$$\zeta_0 = -\frac{1}{2n}(1+\sigma)\gamma_2 x \sinh(nx+\epsilon)\cos(ny+\eta) + F_2,$$

where F_1 and F_0 are harmonic functions in x and y. By assuming

$$F_1 = \gamma_3 \cosh(nx+\epsilon)\cos(ny+\eta),$$

$$F_0 = \gamma_4 \cosh(nx+\epsilon)\cos(ny+\eta),$$

we have, finally,

$$\chi = \left\{ \begin{aligned} &[-\tfrac{1}{6}(2-\sigma)\gamma_1 z^3 + \tfrac{1}{2}\sigma\gamma_2 z^2 + \gamma_3 z + \gamma_4]\cosh(nx+\epsilon) \\ &\quad + \frac{1}{2n}[(1-\sigma)\gamma_1 z - (1+\sigma)\gamma_2] x \sinh(nx+\epsilon) \end{aligned} \right\} \cos(ny+\eta).$$

The final form of the solution is given by (1) and

$$\widehat{xx} = \left\{ \begin{array}{r} [\tfrac{1}{6}(2-\sigma)n^2\gamma_1 z^3 - \tfrac{1}{2}\sigma n^2\gamma_2 z^2 \\ -(\gamma_1+n^2\gamma_3)z - n^2\gamma_4]\cosh(nx+\epsilon) \\ -\tfrac{1}{2}n[(1-\sigma)\gamma_1 z-(1+\sigma)\gamma_2]\, x\sinh(nx+\epsilon) \end{array} \right\} \cos(ny+\eta),$$

$$\widehat{xy} = \left\{ \begin{array}{r} [-\tfrac{1}{6}(2-\sigma)n^2\gamma_1 z^3 + \tfrac{1}{2}\sigma n^2\gamma_2 z^2 + \{n^2\gamma_3 + \tfrac{1}{2}(1-\sigma\gamma_1)\}z \\ +\{n^2\gamma_4 - \tfrac{1}{2}(1+\sigma\gamma_2)\}]\sinh(nx+\epsilon) \\ +\tfrac{1}{2}n[(1-\sigma)\gamma_1 z-(1+\sigma)\gamma_2]\, x\cosh(nx+\epsilon) \end{array} \right\} \sin(ny+\eta),$$

$$\widehat{yy} = \left\{ \begin{array}{r} [-\tfrac{1}{6}(2-\sigma)n^2\gamma_1 z^3 + \tfrac{1}{2}\sigma n^2\gamma_2 z^2 - (\sigma\gamma_1 - n^2\gamma_3)z \\ -\{(1+\sigma)\gamma_2 - n^2\gamma_4\}]\cosh(nx+\epsilon) \\ +\tfrac{1}{2}n[(1-\sigma)\gamma_1 z-(1+\sigma)\gamma_2]\, x\sinh(nx+\epsilon) \end{array} \right\} \cos(ny+\eta),$$

$$\frac{Eu}{1+\sigma} = \left\{ \begin{array}{r} \left[\tfrac{1}{6}(2-\sigma)n\gamma_1 z^3 - \tfrac{1}{2}\sigma n\gamma_2 z^2 - \left(\dfrac{1-\sigma}{2n}\gamma_1 + n\gamma_3\right)z \right. \\ \left. -\left(\dfrac{1-\sigma}{2n}\gamma_2 + n\gamma_4\right)\right]\sinh(nx+\epsilon) \\ -\tfrac{1}{2}[(1-\sigma)\gamma_1 z-(1+\sigma)\gamma_2]\, x\cosh(nx+\epsilon) \end{array} \right\} \cos(ny+\eta),$$

$$\frac{Ev}{1+\sigma} = \left\{ \begin{array}{r} \left[-\tfrac{1}{6}(2-\sigma)n\gamma_1 z^3 + \tfrac{1}{2}\sigma n\gamma_2 z^2 + n\gamma_3 z \right. \\ \left. -\dfrac{1}{n}(\gamma_2 - n^2\gamma_4)\right]\cosh(nx+\epsilon) \\ +\tfrac{1}{2}[(1-\sigma)\gamma_1 z-(1+\sigma)\gamma_2]\, x\sinh(nx+\epsilon) \end{array} \right\} \sin(ny+\eta),$$

$$\frac{Ew}{1+\sigma} = \left\{ \begin{array}{r} [\tfrac{1}{2}\sigma\gamma_1 z^3 + \sigma\gamma_2 z - (\gamma_1 h^2 - \gamma_3)]\cosh(nx+\epsilon) \\ +\tfrac{1}{2}(1-\sigma)\gamma_1 x\sinh(nx+\epsilon) \end{array} \right\} \cos(ny+\eta).$$

2. *The general normal solution.*

We now proceed to obtain a solution of the equations when $\widehat{xz}$ and $\widehat{yz}$ vanish at the faces $z = \pm h$ of the plate, and $\widehat{zz}$ assumes the prescribed

values

$$\widehat{zz} = c_0 \cos(mx+\epsilon)\cos(ny+\eta) \quad \text{at} \quad z=-h,$$

$$= c_1 \cos(mx+\epsilon)\cos(ny+\eta) \quad \text{at} \quad z=+h.$$

Since $$\nabla^4 \widehat{zz} = 0,$$

let us assume a solution of the form

$$\widehat{zz} = \phi \cos(mx+\epsilon)\cos(ny+\eta),$$

where ϕ is a function of z alone. Then ϕ must satisfy the equation

$$\phi^{\text{iv}} - 2p^2\phi'' + p^4\phi = 0,$$

where $$p^2 = m^2+n^2.$$

Consequently

$$\widehat{zz} = [(A_1+B_1 z)\cosh pz + (A_2+B_2 z)\sinh pz]\cos(mx+\epsilon)\cos(ny+\eta),$$

A_1, B_1, A_2, B_2 being constants.

From the equation

$$-\frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial z^2} = \nabla^2\widehat{zz},$$

it follows that

$$-\frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial z^2} = 2p(B_1\sinh pz + B_2\cosh pz)\cos(mx+\epsilon)\cos(ny+\eta),$$

and so

$$-\frac{1}{1+\sigma}\Theta = \frac{2}{p}(B_1\sinh pz + B_2\cosh pz)\cos(mx+\epsilon)\cos(ny+\eta).$$

where we omit a function which contributes only to the plane stress solutions.

On substituting this expression for Θ in the equation

$$\nabla^2\widehat{xz} = -\frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial x\,\partial z},$$

we find that

$$\nabla^2\widehat{xz} = -2m(B_1\cosh pz + B_2\sinh pz)\sin(mx+\epsilon)\cos(ny+\eta),$$

and so we may take*

$$\widehat{xz} = -\frac{m}{p} z(B_2 \cosh pz + B_1 \sinh pz) \sin(mx+\epsilon) \cos(ny+\eta) + \text{a harmonic function.}$$

The form of this harmonic function is evidently

$$(A_3 \cosh pz + A_4 \sinh pz) \sin(mx+\epsilon) \cos(ny+\eta)$$

in order to fit the equations, and then

$$\widehat{xz} = \left[\left(A_3 - \frac{m}{p} B_2 z\right) \cosh pz + \left(A_4 - \frac{m}{p} B_1 z\right) \sinh pz\right] \times \sin(mx+\epsilon) \cos(ny+\eta).$$

In similar fashion we have

$$\widehat{yz} = \left[\left(A_5 - \frac{n}{p} B_2 z\right) \cosh pz + \left(A_6 - \frac{n}{p} B_1 z\right) \sinh pz\right] \times \cos(mx+\epsilon) \sin(ny+\eta).$$

When we substitute these expressions for $\widehat{xy}$, $\widehat{yz}$, $\widehat{zz}$ in the stress equation

$$\frac{\partial}{\partial x} \widehat{xz} + \frac{\partial}{\partial y} \widehat{yz} + \frac{\partial}{\partial z} \widehat{zz} = 0,$$

we must have

$$\left\{\begin{array}{l} m\left(A_3 - \frac{m}{p} B_2 z\right) \cosh pz + m\left(A_4 - \frac{m}{p} B_1 z\right) \sinh pz \\ n\left(A_5 - \frac{n}{p} B_2 z\right) \cosh pz + n\left(A_6 - \frac{n}{p} B_1 z\right) \sinh pz \\ +p(A_1 + B_1 z) \sinh pz + p(A_2 + B_2 z) \cosh pz \\ \qquad + B_1 \cosh pz + B_2 \sinh pz \end{array}\right\} \cos(mx+\epsilon) \times \cos(ny+\eta) = 0$$

identically. The coefficients of $z \cosh pz$ and $z \sinh pz$ are zero, and so the required condition is satisfied when

$$mA_3 + nA_5 + pA_2 + B_1 = 0,$$

$$mA_4 + nA_6 + pA_1 + B_2 = 0.$$

* The contributions to the plane stress solutions are omitted throughout this section.

The conditions* to be satisfied at $z = -h$ yield the following equations:

$$\left(A_3+\frac{m}{p}B_2h\right)\cosh ph-\left(A_4+\frac{m}{p}B_1h\right)\sinh ph=0,$$

$$\left(A_5+\frac{n}{p}B_2h\right)\cosh ph-\left(A_6+\frac{n}{p}B_1h\right)\sinh ph=0,$$

$$(A_1-B_1h)\cosh ph-(A_2-B_2h)\sinh ph=c_0,$$

and there are three similar equations from the conditions at $z = +h$.

On solving these equations we have

$$B_1=-\tfrac{1}{2}(c_0-c_1)\frac{p\cosh ph}{ph-\sinh ph\cosh ph},$$

$$B_2=-\tfrac{1}{2}(c_0+c_1)\frac{p\sinh ph}{ph+\sinh ph\cosh ph},$$

$$A_1=-\frac{1}{p}B_2(ph\coth ph-1),$$

$$A_2=-\frac{1}{p}B_1(ph\tanh ph-1),$$

$$A_3=\frac{m}{p}B_1h\tanh ph,$$

$$A_4=\frac{m}{p}B_2h\coth ph,$$

$$A_5=\frac{n}{p}B_1h\tanh ph,$$

$$A_6=\frac{n}{p}B_2h\coth ph,$$

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* The solution may be made to comply with the following conditions:

$$\left.\begin{aligned}\widehat{xz}&=d_0\sin(mx+\epsilon)\cos(ny+\eta)\\ \widehat{yz}&=e_0\cos(mx+\epsilon)\sin(ny+\eta)\\ \widehat{zz}&=c_0\cos(mx+\epsilon)\cos(ny+\eta)\end{aligned}\right\}\quad\text{at } z=-h,$$

and three similar conditions at $z = +h$.

and so

$$\left.\begin{aligned}
\widehat{xz} &= \frac{m}{p}\,[(B_1 h \tanh ph - B_2 z)\cosh pz \\
&\qquad + (B_2 h \coth ph - B_1 z)\sinh pz]\sin(mx+\epsilon)\cos(ny+\eta), \\
\widehat{yz} &= \frac{n}{p}\,[(B_1 h \tanh ph - B_2 z)\cosh pz \\
&\qquad + (B_2 h \coth ph - B_1 z)\sinh pz]\cos(mx+\epsilon)\sin(ny+\eta), \\
\widehat{zz} &= \frac{1}{p}\,\{[-B_2(ph\coth ph+1)+pB_1 z]\cosh pz \\
&\quad + [-B_1(ph\tanh ph+1)+pB_2 z]\sinh pz\}\cos(mx+\epsilon)\cos(ny+\eta).
\end{aligned}\right\} \tag{2}$$

The two stress equations

$$\frac{\partial}{\partial x}\widehat{xx}+\frac{\partial}{\partial y}\widehat{xy}+\frac{\partial}{\partial z}\widehat{xz}=0,$$

$$\frac{\partial}{\partial x}\widehat{xy}+\frac{\partial}{\partial y}\widehat{yy}+\frac{\partial}{\partial z}\widehat{yz}=0$$

are now satisfied by introducing* a function χ such that

$$\widehat{xx} = \frac{1}{p}\,\{[B_2(ph\coth ph-1)-pB_1 z]\cosh pz$$
$$+[B_1(ph\tanh ph-1)-pB_2 z]\sinh pz\}\cos(mx+\epsilon)\cos(ny+\eta)+\frac{\partial^2\chi}{\partial y^2},$$

$$\widehat{xy} = -\frac{\partial^2\chi}{\partial x\,\partial y},$$

$$\widehat{yy} = \frac{1}{p}\,\{[B_2(ph\coth ph-1)-pB_1 z]\cosh pz$$
$$+[B_1(ph\tanh ph-1)-pB_2 z]\sinh pz\}\cos(mx+\epsilon)\cos(ny+\eta)+\frac{\partial^2\chi}{\partial x^2}.$$

Proceeding along lines similar to those followed by Love, we find that

$$\frac{\partial^2 H}{\partial x^2}=0,\quad \frac{\partial^2 H}{\partial x\,\partial y}=0,\quad \frac{\partial^2 H}{\partial y^2}=0,$$

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* *Cf.* Love, *loc. cit.*, § 307.

where

$$H=\frac{1}{p}\{[B_2(3-2\sigma-ph\coth ph)+pB_1z]\cosh pz$$
$$+[B_1(3-2\sigma-ph\tanh ph)+pB_2z]\sinh pz\}\cos(mx+\epsilon)\cos(ny+\eta)+\frac{\partial^2\chi}{\partial z^2}.$$

Consequently we may take

$$H=0,$$

and so

$$\chi=\frac{1}{p^3}\{[B_2(ph\coth ph-1+2\sigma)-pB_1z]\cosh pz$$
$$+[B_1(ph\tanh ph-1+2\sigma)-pB_2z]\sinh pz\}\cos(mx+\epsilon)\cos(ny+\eta).$$

It readily follows that the solution is given by (2) and

$$\widehat{xx}=\frac{1}{p^3}\{[B_2\{m^2(ph\coth ph-1)-2\sigma n^2\}-m^2pB_1z]\cosh pz$$
$$+[B_1\{m^2(ph\tanh ph-1)-2\sigma n^2\}$$
$$-m^2pB_2z]\sinh pz\}\cos(mx+\epsilon)\cos(ny+\eta),$$

$$\widehat{xy}=-\frac{mn}{p^3}\{[B_2(ph\coth ph-1+2\sigma)-pB_1z]\cosh pz$$
$$+[B_1(ph\tanh ph-1+2\sigma)-pB_2z]\sinh pz\}\sin(mx+\epsilon)\sin(ny+\eta),$$

$$\widehat{yy}=\frac{1}{p^3}\{[B_2\{n^2(ph\coth ph-1)-2\sigma m^2\}-n^2pB_1z]\cosh pz$$
$$+[B_1\{n^2(ph\tanh ph-1)-2\sigma m^2\}$$
$$-n^2pB_2z]\sinh pz\}\cos(mx+\epsilon)\cos(ny+\eta),$$

$$\frac{Eu}{1+\sigma}=\frac{m}{p^3}\{[B_2(ph\coth ph-1+2\sigma)-pB_1z]\cosh pz$$
$$+[B_1(ph\tanh ph-1+2\sigma)-pB_2z]\sinh pz\}\sin(mx+\epsilon)\cos(ny+\eta),$$

$$\frac{Ev}{1+\sigma}=\frac{n}{p^3}\{[B_2(ph\coth ph-1+2\sigma)-pB_1z]\cosh pz$$
$$+[B_1(ph\tanh ph-1+2\sigma)-pB_2z]\sinh pz\}\cos(mx+\epsilon)\sin(ny+\eta)$$

$$\frac{Ew}{1+\sigma}=\frac{1}{p^2}\{[-B_1(ph\tanh ph+2-2\sigma)+pB_2z]\cosh pz$$
$$+[-B_2(ph\coth ph+2-2\sigma)+pB_1z]\sinh pz\}\cos(mx+\epsilon)\cos(ny+\eta).$$

It is now quite a straightforward matter to show that this solution is self-consistent. In the particular case when m and n are zero, the solution obtained is that given by Love*:

$$\widehat{zz} = a_1 z^3 + a_2 z + a_3,$$

$$\widehat{xz} = -\tfrac{3}{2} a_1 x(z^2 - h^2),$$

$$\widehat{yz} = -\tfrac{3}{2} a_1 y(z^2 - h^2),$$

$$\widehat{xx} = -\tfrac{1}{2}(2+\sigma) a_1 z^3 + [\tfrac{3}{2} a_1(x^2+y^2) - \tfrac{3}{8}(1-\sigma) a_1(x^2+3y^2) - \tfrac{1}{2} a_2] z - \tfrac{1}{2} a_3,$$

$$\widehat{xy} = \tfrac{3}{4}(1-\sigma) a_1 xyz.$$

$$\widehat{yy} = -\tfrac{1}{2}(2+\sigma) a_1 z^3 + [\tfrac{3}{2} a_1(x^2+y^2) - \tfrac{3}{8}(1-\sigma) a_1(3x^2+y^2) - \tfrac{1}{2} a_2] z - \tfrac{1}{2} a_3,$$

$$\frac{Eu}{1+\sigma} = -\tfrac{1}{2} x[(2-\sigma) a_1 z^3 - \tfrac{3}{4}(1-\sigma) a_1(x^2+y^2) z + a_2 z + a_3],$$

$$\frac{Ev}{1+\sigma} = -\tfrac{1}{2} y[(2-\sigma) a_1 z^3 - \tfrac{3}{4}(1-\sigma) a_1(x^2+y^2) z + a_2 z + a_3],$$

$$\frac{Ew}{1+\sigma} = \tfrac{1}{4}(1+\sigma) a_1 z^4 - \tfrac{1}{2}[a_2 - \tfrac{3}{2}\sigma a_1(x^2+y^2)] z^2 + a_3 z$$
$$+ \tfrac{1}{8}(3a_1 h^2 - a_2)(x^2+y^2) - \tfrac{3}{32}(1-\sigma) a_1(x^2+y^2)^2, \qquad (3)$$

where
$$\widehat{zz} = c_0 \quad \text{at} \quad z = -h$$
$$= c_1 \quad \text{at} \quad z = +h.$$
$$\widehat{xz} = \widehat{yz} = 0 \quad \text{at} \quad z = \pm h,$$

and
$$a_1 = \tfrac{1}{4} \frac{c_0 - c_1}{h^4}, \quad a_2 = -\tfrac{3}{4} \frac{c_0 - c_1}{h}, \quad a_3 = \tfrac{1}{2}(c_0 + c_1).$$

3. *Certain cases of a rectangular plate subject to uniform normal tractions on its two faces.*

3.1. In this section we specify the plate by

$$x = \pm a, \quad y = \pm b, \quad z = \pm h,$$

and in the cases under consideration we take

$$\widehat{zz} = c_0 \quad \text{at} \quad z = -h$$
$$= c_1 \quad \text{at} \quad z = +h,$$
$$\widehat{xz} = \widehat{yz} = 0 \quad \text{at} \quad z = \pm h.$$

* Love, *loc. cit.*, § 307.

To build up a solution suitable for these problems we employ

(a) the solution expressed by (3);

(b) that given in Love*,

$$Eu = -\tfrac{1}{2}(1-\sigma^2)\kappa x, \quad Ev = -\tfrac{1}{2}(1-\sigma^2)\kappa y, \quad Ew = \sigma(1+\sigma)\kappa z,$$

where κ is a constant;

(c) a plane stress solution which may be readily written down from § 1.

To obtain (c) we write

$$n = (2r+1)\pi/2b, \quad \gamma_1 = \gamma'_{1r}, \quad \gamma_3 = \gamma'_{3r}, \quad \gamma_2 = \gamma_4 = 0, \quad \epsilon = \eta = 0$$

in the work of § 1, and then combine this with another solution in which the hyperbolic and circular functions are changed round as well as b and a, while new constants γ''_{1r} and γ''_{3r} are introduced. We then have the plane stress solution for which

$$\Theta = -(1+\sigma)z\left[\gamma'_{1r}\cosh\frac{(2r+1)\pi x}{2b}\cos\frac{(2r+1)\pi y}{2b} + \gamma''_{1r}\cos\frac{(2r+1)\pi x}{2a}\cosh\frac{(2r+1)\pi y}{2a}\right]$$

$$= z\Theta_1,$$

$$\chi = \left\{\begin{aligned} &\left[-\tfrac{1}{6}(2-\sigma)\gamma'_{1r}z^3+\gamma'_{3r}z\right]\cosh\frac{(2r+1)\pi x}{2b}\cos\frac{(2r+1)\pi y}{2b} \\ &+\left[-\tfrac{1}{6}(2-\sigma)\gamma''_{1r}z^3+\gamma''_{3r}z\right]\cos\frac{(2r+1)\pi x}{2a}\cosh\frac{(2r+1)\pi y}{2a} \\ &+\frac{1-\sigma}{(2r+1)\pi}z\left[b\gamma'_{1r}x\sinh\frac{(2r+1)\pi x}{2b}\cos\frac{(2r+1)\pi y}{2b} + a\gamma''_{1r}y\sinh\frac{(2r+1)\pi y}{2a}\cos\frac{(2r+1)\pi x}{2a}\right] \end{aligned}\right\},$$

$$\widehat{zz} = 0,$$

$$\widehat{xz} = -\frac{1}{2(1+\sigma)}(z^2-h^2)\frac{\partial\Theta_1}{\partial x}, \quad \widehat{yz} = -\frac{1}{2(1+\sigma)}(z^2-h^2)\frac{\partial\Theta_1}{\partial y},$$

$$\widehat{xx} = \frac{z}{1+\sigma}\Theta_1+\frac{\partial^2\chi}{\partial y^2}, \quad \widehat{xy} = -\frac{\partial^2\chi}{\partial x\,\partial y}, \quad \widehat{yy} = \frac{z}{1+\sigma}\Theta_1+\frac{\partial^2\chi}{\partial x^2},$$

* Love, *loc. cit.*, § 301 (ii), 469.

$$\frac{Eu}{1+\sigma}=\left\{\begin{aligned}&\left[\tfrac{1}{6}(2-\sigma)\,\frac{(2r+1)\pi}{2b}\,\gamma'_{1r}z^3-\frac{b}{(2r+1)\pi}\,(1-\sigma)\,\gamma'_{1r}z-\frac{(2r+1)\pi}{2b}\,\gamma'_{3r}z\right]\\&\qquad\times\sinh\frac{(2r+1)\pi x}{2b}\cos\frac{(2r+1)\pi y}{2b}\\&+\left[-\tfrac{1}{6}(2-\sigma)\,\frac{(2r+1)\pi}{2a}\,\gamma''_{1r}z^3+\frac{(2r+1)\pi}{2a}\,\gamma''_{3r}z\right]\sin\frac{(2r+1)\pi x}{2a}\\&\qquad\times\cosh\frac{(2r+1)\pi y}{2a}\\&-\tfrac{1}{2}(1-\sigma)z\left[\gamma'_{1r}x\cosh\frac{(2r+1)\pi x}{2b}\cos\frac{(2r+1)\pi y}{2b}\right.\\&\qquad\left.-\gamma''_{1r}y\sinh\frac{(2r+1)\pi y}{2a}\sin\frac{(2r+1)\pi x}{2a}\right]\end{aligned}\right\},$$

$$\frac{Ev}{1+\sigma}=\left\{\begin{aligned}&\left[-\tfrac{1}{6}(2-\sigma)\,\frac{(2r+1)\pi}{2b}\,\gamma'_{1r}z^3+\frac{(2r+1)\pi}{2b}\,\gamma'_{3r}z\right]\\&\qquad\times\cosh\frac{(2r+1)\pi x}{2b}\sin\frac{(2r+1)\pi y}{2b}\\&+\left[\tfrac{1}{6}(2-\sigma)\,\frac{(2r+1)\pi}{2a}\,\gamma''_{1r}-\frac{a}{(2r+1)\pi}\,(1-\sigma)\,\gamma''_{1r}z-\frac{(2r+1)\pi}{2a}\,\gamma''_{3r}z\right]\\&\qquad\times\cos\frac{(2r+1)\pi x}{2a}\sinh\frac{(2r+1)\pi y}{2a}\\&+\tfrac{1}{2}(1-\sigma)z\left[\gamma'_{1r}x\sinh\frac{(2r+1)\pi x}{2b}\sin\frac{(2r+1)\pi y}{2b}\right.\\&\qquad\left.-\gamma''_{1r}y\cosh\frac{(2r+1)\pi y}{2a}\cos\frac{(2r+1)\pi x}{2a}\right]\end{aligned}\right\},$$

$$\frac{Ew}{1+\sigma}=\left\{\begin{aligned}&(\tfrac{1}{2}\sigma\gamma'_{1r}z^3-\gamma'_{1r}h^2+\gamma'_{3r})\cosh\frac{(2r+1)\pi x}{2b}\cos\frac{(2r+1)\pi y}{2b}\\&+(\tfrac{1}{2}\sigma\gamma''_{1r}z^3-\gamma''_{1r}h^2+\gamma''_{3r})\cos\frac{(2r+1)\pi x}{2a}\cosh\frac{(2r+1)\pi y}{2a}\\&+\frac{1-\sigma}{(2r+1)\pi}\left[b\gamma'_{1r}x\sinh\frac{(2r+1)\pi x}{2b}\cos\frac{(2r+1)\pi y}{2b}\right.\\&\qquad\left.+a\gamma''_{1r}y\sinh\frac{(2r+1)\pi y}{2a}\cos\frac{(2r+1)\pi x}{2a}\right]\end{aligned}\right\}.$$

The form of solution which we use in this section is then obtained by the superposition of (a), (b), and those of type (c) for which $r = 0, 1, 2, \ldots$, and we now consider its application to certain cases.

Plate supported at its edges.

3.2. The conditions to be satisfied in this case are

$$T_1 = S_1 = G_1 = (w)_{z=0} = 0 \quad \text{at} \quad x = \pm a,$$

$$T_2 = S_2 = G_2 = (w)_{z=0} = 0 \quad \text{at} \quad y = \pm b,$$

where T_1, ... have their usual meanings.

It is easy to show that

$$T_1 = T_2 = -\alpha_3 h - (1+\sigma)\kappa h,$$

$$S_1 = -S_2 = 0$$

throughout the plate, and so T and S vanish provided that

$$\kappa = -\alpha_3/(1+\sigma).$$

The condition

$$G_1 = \int_{-h}^{h} \widehat{xx}\, z dz = 0 \quad \text{at} \quad x = \pm a$$

shows that we must have

$$-\tfrac{1}{5}(2+\sigma)\alpha_1 h^2 + \tfrac{2}{3}\left[\tfrac{3}{2}\alpha_1(a^2+y^2) - \tfrac{3}{8}(1-\sigma)\alpha_1(a^2+3y^2) - \tfrac{1}{2}\alpha_2\right]$$

$$+ \sum_{r=0}^{\infty} \left\{ \left[\tfrac{1}{15}(2-\sigma)\frac{(2r+1)^2\pi^2}{4b^2}\gamma'_{1r} h^2 - \tfrac{2}{3}\gamma'_{1r} - \tfrac{2}{3}\frac{(2r+1)^2\pi^2}{4b^2}\gamma'_{3r} \right] \cosh\frac{(2r+1)\pi a}{2b} \right.$$

$$\left. - \tfrac{1}{3}(1-\sigma)\frac{(2r+1)\pi}{2b}\gamma'_{1r} a \sinh\frac{(2r+1)\pi a}{2b} \right\} \cos\frac{(2r+1)\pi y}{2b} = 0$$

for all values of y in the range $(-b, b)$.

In order to satisfy this condition we write down the Fourier cosine series*

$$\sum_{r=0}^{\infty} f_r' \cos\frac{(2r+1)\pi y}{2b},$$

whose sum is

$$\tfrac{1}{5}(2+\sigma)\alpha_1 h^2 - \tfrac{2}{3}\left[\tfrac{3}{2}\alpha_1(a^2+y^2) - \tfrac{3}{8}(1-\sigma)\alpha_1(a^2+3y^2) - \tfrac{1}{2}\alpha_2\right] \quad (0 < y < b)$$

and

$$-\tfrac{1}{5}(2+\sigma)\alpha_1 h^2 + \tfrac{2}{3}\left[\tfrac{3}{2}\alpha_1\{a^2+(2b-y)^2\} - \tfrac{3}{8}(1-\sigma)\alpha_1\{a^2+3(2b-y)^2\} - \tfrac{1}{2}\alpha_2\right]$$

$$(b < y < 2b),$$

---

* *Cf.* Love, *loc. cit.*, § 221 (*c*), 318.

and then the required condition is satisfied when

$$\left[\tfrac{1}{15}(2-\sigma)\frac{(2r+1)^2\pi^2}{4b^2}\gamma'_{1r}h^2-\tfrac{2}{3}\gamma'_{1r}-\tfrac{2}{3}\frac{(2r+1)^2\pi^2}{4b^2}\gamma'_{3r}\right]\cosh\frac{(2r+1)\pi a}{2b}$$

$$-\tfrac{1}{3}(1-\sigma)\frac{(2r+1)\pi}{2b}\gamma'_{1r}a\sinh\frac{(2r+1)\pi a}{2b}=f_r' \quad (4)$$

for all values of r.

The condition $(w)_{z=0}=0$ at $x=\pm a$ requires that

$$\tfrac{1}{8}(3\alpha_1h^2-\alpha_2)(a^2+y^2)-\tfrac{3}{32}(1-\sigma)\alpha_1(a^2+y^2)^2$$

$$+\sum_{r=0}^{\infty}\left[(\gamma'_{3r}-\gamma'_{1r}h^2)\cosh\frac{(2r+1)\pi a}{2b}+\frac{1-\sigma}{(2r+1)\pi}\gamma'_{1r}ab\sinh\frac{(2r+1)\pi a}{2b}\right]$$

$$\times\cos\frac{(2r+1)\pi y}{2b}=0$$

for all values of y in the range $(-b, b)$.

By using the Fourier cosine series

$$\sum_{r=0}^{\infty}g_r'\cos\frac{(2r+1)\pi y}{2b},$$

whose sum is

$$-\tfrac{1}{8}(3\alpha_1h^2-\alpha_2)(a^2+y^2)+\tfrac{3}{32}(1-\sigma)\alpha_1(a^2+y^2)^2 \quad (0<y<b)$$

and

$$\tfrac{1}{8}(3\alpha_1h^2-\alpha_2)[a^2+(2b-y)^2]-\tfrac{3}{32}(1-\sigma)\alpha_1[a^2+(2b-y)^2]^2 \quad (b<y<2b),$$

we see that this condition is satisfied when

$$(\gamma'_{3r}-\gamma'_{1r}h^2)\cosh\frac{(2r+1)\pi a}{2b}+\frac{1-\sigma}{(2r+1)\pi}\gamma'_{1r}ab\sinh\frac{(2r+1)\pi a}{2b}=g_r' \quad (5)$$

for all values of r.

In similar fashion, the conditions

$$G_2=0, \quad (w)_{z=0}=0 \quad \text{at} \quad y=\pm b$$

lead to two equations corresponding to (4) and (5):

$$\left[\tfrac{1}{15}(2-\sigma)\frac{(2r+1)^2\pi^2}{4a^2}\gamma''_{1r}h^2-\tfrac{2}{3}\gamma''_{1r}-\tfrac{2}{3}\frac{(2r+1)^2\pi^2}{4a^2}\gamma''_{3r}\right]\cosh\frac{(2r+1)\pi b}{2a}$$

$$-\tfrac{1}{3}(1-\sigma)\frac{(2r+1)\pi}{2a}\gamma''_{1r}b\sinh\frac{(2r+1)\pi b}{2a}=f_r'' \quad (6)$$

and

$$(\gamma''_{3r}-\gamma''_{1r}h^2)\cosh\frac{(2r+1)\pi b}{2a}+\frac{1-\sigma}{(2r+1)\pi}\gamma''_{1r}ab\sinh\frac{(2r+1)\pi b}{2a}=g_r'', \quad (7)$$

in which f_r'', g_r'' are the analogues of f_r', g_r' respectively.

The value of κ has already been determined, while γ'_{1r} and γ'_{3r} may readily be written down from (4) and (5), and γ''_{1r} and γ''_{3r} from (6) and (7).

This solution for the moderately thick plate will evidently yield that for the thin plate* when we take the limiting case and introduce the flexural rigidity

$$D = \tfrac{2}{3}\frac{Eh^3}{1-\sigma^2}.$$

Plate clamped at its edges.

3.3. Unlike the previous case, the solution in this case is only approximate, but it has a certain interest in that it yields the corresponding known result† for a thin plate.

Since $$(u)_{z=0} = (v)_{z=0} = 0$$

at the edges, the conditions in this case come down to

$$(w)_{z=0} = \left(\frac{\partial w}{\partial x}\right)_{z=0} = 0 \quad \text{at} \quad x = \pm a,$$

$$(w)_{z=0} = \left(\frac{\partial w}{\partial y}\right)_{z=0} = 0 \quad \text{at} \quad y = \pm b.$$

The vanishing of $(w)_{z=0}$ at the edges leads to the equations (5) and (7). The condition

$$\left(\frac{\partial w}{\partial x}\right)_{z=0} = 0 \quad \text{at} \quad x = \pm a$$

requires that

$$\tfrac{1}{4}(3\alpha_1 h^2 - \alpha_2)a - \tfrac{3}{8}(1-\sigma)\alpha_1 a(a^2+y^2)$$

$$+ \sum_{r=0}^{\infty} \left\{ \left[\frac{(2r+1)\pi}{2b} (\gamma'_{3r} - \gamma'_{1r}h^2) \sinh \frac{(2r+1)\pi a}{2b} \right.\right.$$

$$+ \frac{b}{(2r+1)\pi}(1-\sigma)\gamma'_{1r} \left\{ \sinh \frac{(2r+1)\pi a}{2b} \right.$$

$$\left.\left. + \frac{(2r+1)\pi}{2b} a \cosh \frac{(2r+1)\pi a}{2b} \right\} \right] \cos \frac{(2r+1)\pi y}{2b}$$

$$+ (-1)^{r+1} \left[\frac{(2r+1)\pi}{2a} (\gamma''_{3r} - \gamma''_{1r}h^2) \cosh \frac{(2r+1)\pi y}{2a} \right.$$

$$\left. \left. + \tfrac{1}{2}(1-\sigma)\gamma''_{1r}\, y \sinh \frac{(2r+1)\pi y}{2a} \right] \right\} = 0$$

for all values of y in the range $(-b, b)$.

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* Love, *loc. cit.*, § 314 (ii), 492, 493.

† Love, *loc. cit.*, §314 (iii), 494, 495.

We now introduce

(i) the Fourier cosine series

$$\sum_{r=0}^{\infty} k_r' \cos\frac{(2r+1)\pi y}{2b},$$

whose sum is

$$-\tfrac{1}{4}(3\alpha_1 h^2-\alpha_2)a+\tfrac{3}{8}(1-\sigma)\alpha_1 a(a^2+y^2) \qquad (0<y<b)$$

and

$$\tfrac{1}{4}(3\alpha_1 h^2-\alpha_2)a-\tfrac{3}{8}(1-\sigma)\alpha_1 a\{a^2+(2b-y)^2\} \qquad (b<y<2b),$$

(ii) two similarly defined series,

$$\sum_{s=0}^{\infty} p_{rs}' \cos\frac{(2s+1)\pi y}{2b} = \cosh\frac{(2r+1)\pi y}{2a} \qquad (0<y<b),$$

$$\sum_{s=0}^{\infty} q_{rs}' \cos\frac{(2s+1)y}{2b} = y\sinh\frac{(2r+1)\pi y}{2a} \qquad (0<y<b).$$

The condition to be satisfied is then

$$\begin{aligned}
&\sum_{r=0}^{\infty} k_r' \cos\frac{(2r+1)\pi y}{2b} \\
&= \sum_{r=0}^{\infty}\Bigg\{\Bigg[\frac{(2r+1)\pi}{2b}(\gamma_{3r}'-\gamma_{1r}'h^2)\sinh\frac{(2r+1)\pi a}{2b} \\
&\qquad +\frac{b}{(2r+1)\pi}(1-\sigma)\gamma_{1r}'\Big\{\sinh\frac{(2r+1)\pi a}{2b} \\
&\qquad\qquad +\frac{(2r+1)\pi}{2b}a\cosh\frac{(2r+1)\pi a}{2b}\Big\}\Bigg]\cos\frac{(2r+1)\pi y}{2b} \\
&\qquad +(-1)^{r+1}\Bigg[\frac{(2r+1)\pi}{2a}(\gamma_{3r}''-\gamma_{1r}''h^2)\sum_{s=0}^{\infty}p_{rs}'\cos\frac{(2s+1)\pi y}{2b} \\
&\qquad\qquad +\tfrac{1}{2}(1-\sigma)\gamma_{1r}''\sum_{s=0}^{\infty}q_{rs}'\cos\frac{(2s+1)\pi y}{2b}\Bigg]\Bigg\}
\end{aligned}$$

for all values of y in the interval $(-b, b)$; and in order that this may be true we must have

$$\begin{aligned}
k_m' = &\Bigg\{\Bigg[\frac{(2m+1)\pi}{2b}(\gamma_{3m}'-\gamma_{1m}'h^2)\sinh\frac{(2m+1)\pi a}{2b} \\
&\quad +\frac{b}{(2m+1)\pi}(1-\sigma)\gamma_{1m}'\Big\{\sinh\frac{(2m+1)\pi a}{2b} \\
&\qquad +\frac{(2m+1)\pi}{2b}a\cosh\frac{(2m+1)\pi a}{2b}\Big\}\Bigg] \\
&\quad +\sum_{r=0}^{\infty}(-1)^{r+1}\Bigg[\frac{(2r+1)\pi}{2a}(\gamma_{3r}''-\gamma_{1r}''h^2)p_{rm}'+\tfrac{1}{2}(1-\sigma)\gamma_{1r}''q_{rm}'\Bigg]\Bigg\} \qquad (8)
\end{aligned}$$

for $m = 0, 1, 2, \ldots$.

In similar manner, the condition

$$\left(\frac{\partial w}{\partial y}\right)_{z=0} = 0 \quad \text{at} \quad y = \pm b$$

leads to

$$k_m'' = \left\{\left[\frac{(2m+1)\pi}{2a}(\gamma_{3m}''-\gamma_{1m}''h^2)\sinh\frac{(2m+1)\pi b}{2a} + \frac{a}{(2m+1)\pi}(1-\sigma)\gamma_{1m}''\left\{\sinh\frac{(2m+1)\pi b}{2a} + \frac{(2m+1)\pi}{2a}b\cosh\frac{(2m+1)\pi b}{2a}\right\}\right] + \sum_{r=0}^{\infty}(-1)^{r+1}\left[\frac{(2r+1)\pi}{2b}(\gamma_{3r}'-\gamma_{1r}'h^2)p_{rm}''+\tfrac{1}{2}(1-\sigma)\gamma_{1r}'q_{rm}''\right]\right\} \quad (9)$$

for $m = 0, 1, 2, \ldots$.

By means of (5) and (7) we can remove γ_{3m}', γ_{3m}'', γ_{3r}', γ_{3r}'' from (8) and (9), and so obtain two sets of relations connecting γ_{1r}' and γ_{1r}''. Evidently the resulting equations may be solved approximately by assuming $\gamma_{1r}' = \gamma_{1r}'' = 0$ for r greater than a fixed value.

As already noted, this solution is of the same approximate type as that for the thin plate, and the latter solution will come by a limiting process.

(*a*) *Plate with two opposite edges supported and the other two free.*

(*b*) *Plate with two opposite edges supported and the other two clamped.*

3.4. Approximate solutions for these two cases may be worked out on lines similar to those of § 3.3.

4. *Certain cases of a rectangular plate subject to an arbitrary distribution of normal pressure over one face and the same arbitrary distribution of normal pull over the other face.*

4.1. In this section we take the plate to be bounded by

$$x = 0, \quad x = 2a, \quad y = 0, \quad y = 2b, \quad z = \pm h,$$

and the conditions at the faces of the plate to be

$$\widehat{zz} = -f(x, y) \quad \text{at} \quad z = -h$$
$$= +f(x, y) \quad \text{at} \quad z = +h,$$
$$\widehat{xz} = \widehat{yz} = 0 \quad \text{at} \quad z = \pm h.$$

We also take $f(x, y)$ to be expressed by the double Fourier series

$$f(x, y) = \sum_{r, s=1}^{\infty} c_{rs} \sin \frac{r\pi x}{2a} \sin \frac{s\pi y}{2b}.$$

In dealing with these cases we require to use solutions of each of the types worked out in §§ 1, 2.

(a) The plane stress solution is that for which

$$\Theta = (1+\sigma) z \left(\gamma'_{1s} \cosh \frac{s\pi x}{2b} + \gamma''_{1s} \sinh \frac{s\pi x}{2b}\right) \sin \frac{s\pi y}{2b},$$

$$\chi = \left\{ \tfrac{1}{6}(2-\sigma) z^3 \left(\gamma'_{1s} \cosh \frac{s\pi x}{2b} + \gamma''_{1s} \sinh \frac{s\pi x}{2b}\right) \right.$$

$$- z \left[(1-\sigma) \frac{b}{s\pi} \left(\gamma'_{1s} x \sinh \frac{s\pi x}{2b} + \gamma''_{1s} x \cosh \frac{s\pi x}{2b}\right) \right.$$

$$\left. \left. + \left(\gamma'_{3s} \cosh \frac{s\pi x}{2b} + \gamma''_{3s} \sinh \frac{s\pi x}{2b}\right) \right] \right\} \sin \frac{s\pi y}{2b},$$

$$\frac{Eu}{1+\sigma} = \left\{ \left[-\tfrac{1}{6}(2-\sigma) \frac{s\pi}{2b} z^3 + (1-\sigma) \frac{b}{s\pi} z \right] \left(\gamma'_{1s} \sinh \frac{s\pi x}{2b} + \gamma''_{1s} \cosh \frac{s\pi x}{2b}\right) \right.$$

$$+ \frac{s\pi}{2b} z \left(\gamma'_{3s} \sinh \frac{s\pi x}{2b} + \gamma''_{3s} \cosh \frac{s\pi x}{2b}\right)$$

$$\left. + \tfrac{1}{2}(1-\sigma) z \left(\gamma'_{1s} x \cosh \frac{s\pi x}{2b} + \gamma''_{1s} x \sinh \frac{s\pi x}{2b}\right) \right\} \sin \frac{s\pi y}{2b}.$$

$$\frac{Ev}{1+\sigma} = \left\{ -\tfrac{1}{6}(2-\sigma) \frac{s\pi}{2b} z^3 \left(\gamma'_{1s} \cosh \frac{s\pi x}{2b} + \gamma''_{1s} \sinh \frac{s\pi x}{2b}\right) \right.$$

$$+ \frac{s\pi}{2b} z \left(\gamma'_{3s} \cosh \frac{s\pi x}{2b} + \gamma''_{3s} \sinh \frac{s\pi x}{2b}\right)$$

$$\left. + \tfrac{1}{2}(1-\sigma) z \left(\gamma'_{1s} x \sinh \frac{s\pi x}{2b} + \gamma''_{1s} x \cosh \frac{s\pi x}{2b}\right) \right\} \cos \frac{s\pi y}{2b}.$$

$$\frac{Ew}{1+\sigma} = \left\{ -\tfrac{1}{2}\sigma z^2 \left(\gamma'_{1s} \cosh \frac{s\pi x}{2b} + \gamma''_{1s} \sinh \frac{s\pi x}{2b}\right) \right.$$

$$+ \left[(\gamma'_{1s} h^2 - \gamma'_{3s}) \cosh \frac{s\pi x}{2b} + (\gamma''_{1s} h^2 - \gamma''_{3s}) \sinh \frac{s\pi x}{2b} \right.$$

$$\left. \left. - (1-\sigma) \frac{b}{s\pi} \left(\gamma'_{1s} x \sinh \frac{s\pi x}{2b} + \gamma''_{1s} x \cosh \frac{\varepsilon\pi x}{2b}\right) \right] \right\} \sin \frac{s\pi y}{2b}.$$

For this solution the resultant stresses, etc., are

$$T_1 = T_2 = S_1 = S_2 = 0,$$

$$N_1 = \tfrac{1}{3}\frac{s\pi}{b}\,h^3\left(\gamma'_{1s}\sinh\frac{s\pi x}{2b}+\gamma''_{1s}\cosh\frac{s\pi x}{2b}\right)\sin\frac{s\pi y}{2b},$$

$$N_2 = \tfrac{1}{3}\frac{s\pi}{b}\,h^3\left(\gamma'_{1s}\cosh\frac{s\pi x}{2b}+\gamma''_{1s}\sinh\frac{s\pi x}{2b}\right)\cos\frac{s\pi y}{2b},$$

$$\begin{aligned}-H_1 = H_2 = \Big\{&\left[-\tfrac{1}{15}(2-\sigma)\frac{s^2\pi^2}{4b^2}\,h^5+\tfrac{1}{3}(1-\sigma)h^3\right]\left(\gamma'_{1s}\sinh\frac{s\pi x}{2b}+\gamma''_{1s}\cosh\frac{s\pi x}{2b}\right)\\ &+\tfrac{1}{3}\frac{s^2\pi^2}{2b^2}\,h^3\left(\gamma'_{3s}\sinh\frac{s\pi x}{2b}+\gamma''_{3s}\cosh\frac{s\pi x}{2b}\right)\\ &+\tfrac{1}{3}(1-\sigma)\frac{s\pi}{2b}\,h^3\left(\gamma'_{1s}x\cosh\frac{s\pi x}{2b}+\gamma''_{1s}x\sinh\frac{s\pi x}{2b}\right)\Big\}\cos\frac{s\pi y}{2b},\end{aligned}$$

$$\begin{aligned}G_1 = \Big\{&\left[-\tfrac{1}{15}(2-\sigma)\frac{s^2\pi^2}{4b^2}\,h^5+\tfrac{2}{3}h^3\right]\left(\gamma'_{1s}\cosh\frac{s\pi x}{2b}+\gamma''_{1s}\sinh\frac{s\pi x}{2b}\right)\\ &+\tfrac{1}{3}\frac{s^2\pi^2}{2b^2}\,h^3\left(\gamma'_{3s}\cosh\frac{s\pi x}{2b}+\gamma''_{3s}\sinh\frac{s\pi x}{2b}\right)\\ &+\tfrac{1}{3}(1-\sigma)\frac{s\pi}{2b}\,h^3\left(\gamma'_{1s}x\sinh\frac{s\pi x}{2b}+\gamma''_{1s}x\cosh\frac{s\pi x}{2b}\right)\Big\}\sin\frac{s\pi y}{2b},\end{aligned}$$

$$\begin{aligned}G_2 = \Big\{&\left[\tfrac{1}{15}(2-\sigma)\frac{s^2\pi^2}{4b^2}\,h^5+\tfrac{2}{3}\sigma h^3\right]\left(\gamma'_{1s}\cosh\frac{s\pi x}{2b}+\gamma''_{1s}\sinh\frac{s\pi x}{2b}\right)\\ &-\tfrac{1}{3}\frac{s^2\pi^2}{2b^2}\,h^3\left(\gamma'_{3s}\cosh\frac{s\pi x}{2b}+\gamma''_{3s}\sinh\frac{s\pi x}{2b}\right)\\ &-\tfrac{1}{3}(1-\sigma)\frac{s\pi}{2b}\,h^3\left(\gamma'_{1s}x\sinh\frac{s\pi x}{2b}+\gamma''_{1s}x\cosh\frac{s\pi x}{2b}\right)\Big\}\sin\frac{s\pi y}{2b}.\end{aligned}$$

(*b*) The other type of solution is obtained from § 2 by replacing c_0, c_1, m, n, B_1, B_2 by $-c_{rs}, c_{rs}, r\pi/2a, s\pi/2b, B_1(r,s), B_2(r,s)$ respectively, and putting $\epsilon = \eta = \tfrac{1}{2}\pi$. Then $B_2(r,s)$ vanishes, and

$$B_1(r,s) = c_{rs}\,\frac{p\cosh ph}{ph-\sinh ph\cosh ph},$$

where

$$p^2 = \frac{\pi^2}{4}\left(\frac{r^2}{a^2}+\frac{s^2}{b^2}\right)$$

throughout this section.

The solution is then

$$\frac{Eu}{1+\sigma} = \frac{r\pi}{2ap^3} B_1(r, s)\{(ph \tanh ph - 1 + 2\sigma) \sinh pz - pz \cosh pz\} \cos \frac{r\pi x}{2a} \sin \frac{s\pi y}{2b},$$

$$\frac{Ev}{1+\sigma} = \frac{s\pi}{2bp^3} B_1(r, s)\{(ph \tanh ph - 1 + 2\sigma) \sinh pz - pz \cosh pz\} \sin \frac{r\pi x}{2a} \cos \frac{s\pi y}{2b},$$

$$\frac{Ew}{1+\sigma} = -\frac{1}{p^2} B_1(r, s)\{(ph \tanh ph + 2 - 2\sigma) \cosh pz - pz \sinh pz\} \sin \frac{r\pi x}{2a} \sin \frac{s\pi y}{2b},$$

for which

$$\widehat{zz} = -c_{rs} \sin \frac{r\pi x}{2a} \sin \frac{s\pi y}{2b} \quad \text{at} \quad z = -h,$$

$$= +c_{rs} \sin \frac{r\pi x}{2a} \sin \frac{s\pi y}{2b} \quad \text{at} \quad z = +h,$$

$$\widehat{xz} = \widehat{yz} = 0 \quad \text{at} \quad z = \pm h.$$

The resultant stresses, etc., in this case are

$$T_1 = T_2 = S_1 = S_2 = 0.$$

$$N_1 = \frac{r\pi}{ap^3} B_1(r, s) \frac{ph - \sinh ph \cosh ph}{\cosh ph} \cos \frac{r\pi x}{2a} \sin \frac{s\pi y}{2b},$$

$$N_2 = \frac{s\pi}{bp^3} B_1(r, s) \frac{ph - \sinh ph \cosh ph}{\cosh ph} \sin \frac{r\pi x}{2a} \cos \frac{s\pi y}{2b}.$$

$$-H_1 = H_2 = -\frac{rs\pi^2}{2abp^5} B_1(r, s)\left\{\frac{ph - \sinh ph \cosh ph}{\cosh ph} - 2\sigma(ph \cosh ph - \sinh ph)\right\} \cos \frac{r\pi x}{2a} \cos \frac{s\pi y}{2b},$$

$$G_1 = \frac{2}{p^5} B_1(r, s)\left\{\frac{r^2\pi^2(ph - \sinh ph \cosh ph)}{4a^2 \cosh ph} - 2\sigma \frac{s^2\pi^2}{4b^2}(ph \cosh ph - \sinh ph)\right\} \sin \frac{r\pi x}{2a} \sin \frac{s\pi y}{2b}.$$

$$G_2 = \frac{2}{p^5} B_1(r, s)\left\{\frac{s^2\pi^2(ph - \sinh ph \cosh ph)}{4b^2 \cosh ph} - 2\sigma \frac{r^2\pi^2}{4a^2}(ph \cosh ph - \sinh ph)\right\} \sin \frac{r\pi x}{2a} \sin \frac{r\pi y}{2b}.$$

Plate supported at its edges.

4.2. For this problem we obtain the solution by superposing those of § 4.1, (*b*) for $r, s = 1, 2, \ldots$.

$$\frac{Eu}{1+\sigma} = \sum_{r,s=1}^{\infty} \frac{r\pi}{2ap^3} B_1(r, s) \{(ph \tanh ph - 1 + 2\sigma) \sinh pz - pz \cosh pz\} \cos \frac{r\pi x}{2a} \sin \frac{s\pi y}{2b}, \quad \ldots,$$

and it is evident that the edge conditions

$$T = S = G = (w)_{z=0} = 0$$

are all satisfied.

The solution for the corresponding case of a thin plate* under variable pressure evidently comes from this one by a limiting process.

Plate with two opposite edges supported and the other two free.

4.3. The solution for this case is obtained by superposing the solutions of § 4.1 (*a*) for $s = 1, 2, \ldots$, and those of § 4.1 (*b*) for $r, s = 1, 2, \ldots$.

Taking the edges $y = 0$ and $y = 2b$ to be supported, we see that the conditions

$$T_2 = S_2 = G_2 = (w)_z = 0$$

hold.

The other edge conditions require

$$T_1 = S_1 = G_1 = N_1 - \frac{\partial H_1}{\partial y} = 0$$

at $x = 0$ and $x = 2a$.

T_1, S_1 both vanish everywhere, and $G_1 = 0$ at $x = 0$ requires that

$$\sum_{s=1}^{\infty} \left\{ \left[-\tfrac{1}{15}(2-\sigma) \frac{s^2\pi^2}{4b^2} h^2 + \tfrac{2}{3} \right] \gamma'_{1s} + \tfrac{1}{3} \frac{s^2\pi^2}{2b^2} \gamma'_{3s} \right\} \sin \frac{s\pi y}{2b} = 0 \quad (0 < y < 2b).$$

We must therefore have

$$\left[-\tfrac{1}{15}(2-\sigma) \frac{s^2\pi^2}{4b^2} h^2 + \tfrac{2}{3} \right] \gamma'_{1s} + \tfrac{1}{3} \frac{s^2\pi^2}{2b^2} \gamma'_{3s} = 0 \qquad (10)$$

for $s = 1, 2, \ldots$.

* Love, *loc. cit.*, § 314 (i).

Similarly $G_1 = 0$ at $x = 2a$ leads to

$$\left\{\left[-\tfrac{1}{15}(2-\sigma)\frac{s^2\pi^2}{4b^2}h^2+\tfrac{2}{3}\right]\left(\gamma'_{1s}\cosh\frac{s\pi a}{b}+\gamma''_{1s}\sinh\frac{s\pi a}{b}\right)\right.$$

$$+\tfrac{1}{3}\frac{s^2\pi^2}{2b^2}\left(\gamma'_{3s}\cosh\frac{s\pi a}{b}+\gamma''_{3s}\sinh\frac{s\pi a}{b}\right)$$

$$\left.+\tfrac{1}{3}(1-\sigma)\frac{s\pi}{b}a\left(\gamma'_{1s}\sinh\frac{s\pi a}{b}+\gamma''_{1s}\cosh\frac{s\pi a}{b}\right)\right\}=0$$

for $s = 1, 2, \ldots$.

It readily follows that the vanishing of $N_1-\frac{\partial H_1}{\partial y}$ at the edges leads to

$$\frac{s\pi}{2b}h^3\left\{\left[-\tfrac{1}{15}(2-\sigma)\frac{s^2\pi^2}{4b^2}h^2-\tfrac{1}{3}(1+\sigma)\right]\gamma''_{1s}+\tfrac{1}{3}\frac{s^2\pi^2}{2b^2}\gamma''_{3s}\right\}$$

$$=\sum_{r=1}^{\infty}\left\{\frac{r\pi}{ap^3}B_1(r,s)\left[\frac{ph-\sinh ph\cosh ph}{\cosh ph}\right.\right.$$

$$\left.\left.+\frac{s^2\pi^2}{4b^2p^2}\left\{\frac{ph-\sinh ph\cosh ph}{\cosh ph}-2\sigma(ph\cosh ph-\sinh ph)\right\}\right]\right\} \quad (11)$$

for $s = 1, 2, \ldots$, and

$$\frac{s\pi}{2b}h^3\left\{\left[-\tfrac{1}{15}(2-\sigma)\frac{s^2\pi^2}{4b^2}h^2-\tfrac{1}{3}(1+\sigma)\right]\left(\gamma'_{1s}\sinh\frac{s\pi a}{b}+\gamma''_{1s}\cosh\frac{s\pi a}{b}\right)\right.$$

$$+\tfrac{1}{3}\frac{s^2\pi^2}{2b^2}\left(\gamma'_{3s}\sinh\frac{s\pi a}{b}+\gamma''_{3s}\cosh\frac{s\pi a}{b}\right)$$

$$\left.+\tfrac{1}{3}(1-\sigma)\frac{s\pi}{b}a\left(\gamma'_{1s}\cosh\frac{s\pi a}{b}+\gamma''_{1s}\sinh\frac{s\pi a}{b}\right)\right\}$$

$$=\sum_{r=1}^{\infty}\left\{(-1)^r\frac{r\pi}{ap^3}B_1(r,s)\left[\frac{ph-\sinh ph\cosh ph}{\cosh ph}\right.\right.$$

$$\left.\left.+\frac{s^2\pi^2}{4b^2p^2}\left\{\frac{ph-\sinh ph\cosh ph}{\cosh ph}-2\sigma(ph\cosh ph-\sinh ph)\right\}\right]\right\}$$

for $s = 1, 2, \ldots$.

When we have determined γ'_{1s}, γ''_{1s}, γ'_{3s}, γ''_{3s} from these four linear equations the solution is complete. As in previous cases, this solution yields the corresponding result for a thin plate under variable pressure.

Plate with two opposite edges supported and the other two clamped.

4.4. For this case, we use the same solution as in §4.3, and the conditions for support at $y = 0$ and $y = 2b$ are again satisfied.

The conditions at $x = 0$ and $x = 2a$ that

$$(u)_{z=0}, \quad (v)_{z=0}, \quad (w)_{z=0}, \quad \left(\frac{\partial w}{\partial x}\right)_{z=0}$$

vanish there, and the first two of these conditions are evidently satisfied. In order that

$$(w)_{z=0} = 0 \quad \text{at} \quad x = 0$$

we must have $\gamma'_{1s} h^2 = \gamma'_{3s}$ for $s = 1, 2, \ldots,$

while for $(w)_{z=0} = 0$ at $x = 2a$

we must have

$$(\gamma''_{1s} h^2 - \gamma''_{3s}) \sinh \frac{s\pi a}{b} - 2(1-\sigma) \frac{ab}{s\pi} \left(\gamma'_{1s} \sinh \frac{s\pi a}{b} + \gamma''_{1s} \cosh \frac{s\pi a}{b}\right) = 0 \quad (12)$$

for $s = 1, 2, \ldots.$

The vanishing of $(\partial w/\partial x)_{z=0}$ at $x = 0$ and $x = 2a$ leads us in similar fashion to the equations

$$\left\{\frac{s\pi}{2b} (\gamma''_{1s} h^2 - \gamma''_{3s}) - (1-\sigma) \frac{b}{s\pi} \gamma''_{1s}\right\} = \sum_{r=1}^{\infty} \left\{\frac{r\pi}{2ap^2} B_1(rs)(ph \tanh ph + 2 - 2\sigma)\right\}$$

for $s = 1, 2, \ldots,$ and

$$\left\{\frac{s\pi}{2b} (\gamma''_{1s} h^2 - \gamma''_{3s}) \cosh \frac{s\pi a}{b} - (1-\sigma) \frac{b}{s\pi} \left[\left(\sinh \frac{s\pi a}{b} + \frac{s\pi}{b} a \cosh \frac{s\pi a}{b}\right) \gamma'_{1s} + \left(\cosh \frac{s\pi a}{b} + \frac{s\pi}{b} a \sinh \frac{s\pi a}{b}\right) \gamma''_{1s}\right]\right\}$$

$$= \sum_{r=1}^{\infty} \left\{(-1)^r \frac{r\pi}{2ap^2} B_1(r, s)(ph \tanh ph + 2 - 2\sigma)\right\} \quad (13)$$

for $s = 1, 2, \ldots.$

We only now require to solve these equations to determine $\gamma'_{1s}\gamma''_{1s}\gamma'_{3s}\gamma''_{3s}$ for this case.

As before, this solution yields the corresponding solution for a thin plate which does not appear to have been obtained by direct means.

Plate supported at two opposite edges, clamped at a third, and free at the fourth.

4.5. This case may be dealt with in similar fashion to the previous ones, and then the constants will be given by the four equations (10), (11), (12), and (13).

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ON MUTUAL CONSISTENCY AND REGULAR T-LIMITS

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1. A Toeplitz or T-matrix is one which satisfies the three conditions†

$$\text{(i)} \quad \sum_{k=1}^{\infty} |a_{n,k}| \leqslant M \quad \textit{independently of } n,$$

$$\text{(ii)} \quad \lim_{n\to\infty} a_{n,k} = 0 \quad \textit{for every fixed } k,$$

$$\text{(iii)} \quad \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} = 1.$$

Two T-matrices $a_{n,k}$, $b_{n,k}$ are said to be *mutually consistent*‡ *for any given class of sequences* z_k if, whenever $\sum_{k=1}^{\infty} a_{n,k} z_k$ tends to a limit as $n\to\infty$, then $\sum_{k=1}^{\infty} b_{n,k} z_k$ tends to the *same* limit, and *vice versa*.

Now it is known that§

If the T-matrices A and B are commutable, i.e., if $AB = BA$, the corresponding T-limits are consistent at least for bounded sequences.

This theorem suggests that, to solve the problem of mutual consistency, it would first be desirable to solve that of commutability of infinite matrices; but it seems that the latter problem is much more difficult than the former‖, and I give below some results which at any rate partially solve the problem of mutual consistency without any reference to commutability.

† P. Dienes, *The Taylor series* (Oxford, 1931), 389.

‡ *Ibid.*, 411–2. Hobson uses "equivalent" with the same meaning [E. W. Hobson, *The theory of functions of a real variable* (2 ed., Cambridge, 1926), 2, 86].

§ P. Dienes, *loc. cit.*, 412.

‖ *Ibid.*, 414–417; see also R. G. Cooke, "On the transformation of some classes of infinite matrices into diagonal matrices", *Journal London Math. Soc.*, 8 (1933), 167–175 (172); references to other relevant papers are there given.

These results are then found to be immediately applicable to *regular T-limits*.

Writing $$z_n' = \sum_{k=1}^{\infty} a_{n,k} z_k, \quad z_n'' = \sum_{k=1}^{\infty} b_{n,k} z_k, \tag{1.1}$$

we shall say that the T-matrices $a_{n,k}$, $b_{n,k}$ are *absolutely equivalent for a given class of sequences* z_k if

$$z_n' - z_n'' \to 0 \quad \text{as} \quad n \to \infty;$$

i.e., either z_n', z_n'' both tend to the same limit, or else neither of them tends to a limit, but *their difference tends to zero*.

Absolute equivalence therefore implies more than mutual consistency (or equivalence).

2. We first consider *bounded* sequences.

We write throughout $c_{n,k} \equiv a_{n,k} - b_{n,k}$.

THEOREM I. *A necessary and sufficient condition that the T-matrices* $a_{n,k}$, $b_{n,k}$ *are absolutely equivalent for all bounded sequences is that*

$$S_n \equiv \sum_{k=1}^{\infty} |c_{n,k}| \to 0 \quad \textit{as} \quad n \to \infty. \tag{2.1}$$

(i) With the notation (1.1), we have

$$\sum_{k=1}^{\infty} c_{n,k} z_k = z_n' - z_n''.$$

But if $|z_k| \leqslant M$ for every k, then

$$\left| \sum_{k=1}^{\infty} c_{n,k} z_k \right| \leqslant M \sum_{k=1}^{\infty} |c_{n,k}|.$$

Hence, if (2.1) is satisfied, then $z_n' - z_n'' \to 0$; so the condition is sufficient.

(ii) Conversely, let $z_n' - z_n'' \to 0$; *i.e.*,

$$\sum_{k=1}^{\infty} c_{n,k} z_k \to 0 \text{ as } n \to \infty \text{ for } \textit{every} \text{ bounded } z_k.$$

Then we require to prove that $S_n \to 0$.

(*a*) Suppose that $c_{n,k}$ is *real*. If S_n does not tend to 0, it must either tend to a positive limit, say α, or else have several limiting-points, including (possibly) zero; for

$$S_n \leqslant \sum_{k=1}^{\infty} |a_{n,k}| + \sum_{k=1}^{\infty} |b_{n,k}| \leqslant K \quad \text{independently of } n. \tag{2.2}$$

First suppose that

$$S_n \to \alpha, \quad i.e. \quad \sum_{k=1}^{\infty} |c_{n,k}| \to \alpha. \tag{2.3}$$

Then we construct a real sequence† z_k such that

$$-1 \leqslant z_k \leqslant 1, \tag{2.4}$$

and such that $z_n' - z_n''$ does not tend to 0.

We can choose n_1 so that, for every $n > n_1$,

$$\sum_{k=1}^{\infty} |c_{n,k}| > \tfrac{3}{4}\alpha, \quad \text{by (2.3)};$$

and then m_1 so that

$$\sum_{k=m_1+1}^{\infty} |c_{n_1,k}| < \tfrac{1}{2\,4}\alpha, \quad \text{by (2.2)}.$$

Then, if $z_k = \operatorname{sgn}(c_{n_1,k})$ for $1 \leqslant k \leqslant m_1$, we have, by (2.4),

$$z_{n_1}^{*} \equiv \sum_{k=1}^{\infty} c_{n_1,k} z_k \geqslant \sum_{k=1}^{\infty} |c_{n_1,k}| - 2 \sum_{k=m_1+1}^{\infty} |c_{n_1,k}| > \tfrac{2}{3}\alpha.$$

Now, since $a_{n,k}$, $b_{n,k}$ are T-matrices,

$$\lim_{n\to\infty} c_{n,k} = 0 \quad \text{for every } k. \tag{2.5}$$

Hence we can choose $n_2 > n_1$ so that

$$\sum_{k=1}^{m_1} |c_{n_2,k}| < \tfrac{1}{6}\alpha,$$

and then $m_2 > m_1$, by (2.2), so that

$$\sum_{k=m_2+1}^{\infty} |c_{n_2,k}| < \tfrac{1}{6}\alpha.$$

Then, if $z_k = 0$ for $m_1 < k \leqslant m_2$, we have

$$z_{n_2}^{*} \equiv \sum_{k=1}^{\infty} c_{n_2,k} z_k \leqslant \sum_{k=1}^{m_1} |c_{n_2,k}| + \sum_{k=m_2+1}^{\infty} |c_{n_2,k}| < \tfrac{1}{3}\alpha.$$

Next choose $n_3 > n_2$, by (2.3), so that

$$\sum_{k=1}^{\infty} |c_{n_3,k}| > \tfrac{3}{4}\alpha,$$

† This construction is on the lines of the theorem of Steinhaus that, for any T-matrix, there is always a bounded sequence which has no generalized T-limit (P. Dienes, *loc. cit.*, 392).

and also, by (2.5), so that

$$\sum_{k=1}^{m_2} |c_{n_3,k}| < \tfrac{1}{48}\alpha;$$

and then choose $m_3 > m_2$, by (2.2), so that

$$\sum_{k=m_3+1}^{\infty} |c_{n_3,k}| < \tfrac{1}{48}\alpha.$$

Then, if $z_k = \operatorname{sgn}(c_{n_3,k})$ for $m_2 < k \leqslant m_3$, we have

$$z^*_{n_3} \equiv \sum_{k=1}^{\infty} c_{n_3,k} z_k \geqslant \sum_{k=1}^{\infty} |c_{n_3,k}| - 2\sum_{k=1}^{m_2} |c_{n_3,k}| - 2\sum_{k=m_3+1}^{\infty} |c_{n_3,k}|$$

$$> \tfrac{2}{3}\alpha.$$

By continuing in this way, we can choose z_k so that all its terms are 0 or ± 1, and $z^*_{n_r}$ $(r = 1, 2, 3, \ldots)$, *i.e.* $z'_{n_r} - z''_{n_r}$, does not tend to a limit.

If S_n has several limiting-points, and S_n does not tend to 0, there is a sequence of suffixes n_r' $(r = 1, 2, 3, \ldots)$ such that $\lim_{r\to\infty} S_{n_r'} = \alpha$; hence, if we replace each n in the above proof by a corresponding n', we obtain the proof in this case. Hence the condition (2.1) is necessary when $c_{n,k}$ is real.

(*b*) If $c_{n,k}$ is complex, write $c_{n,k} = r_{n,k} e^{i\theta_{n,k}}$, and then put $z_k = e^{-i\theta_{n_1,k}}$ for $1 \leqslant k \leqslant m_1$; $z_k = 0$ for $m_1 < k \leqslant m_2$; $z_k = e^{-i\theta_{n_3,k}}$ for $m_2 < k \leqslant m_3$, ...; the proof then proceeds as before.

Hence the condition (2.1) is also necessary when $c_{n,k}$ is complex.

The theorem is therefore proved.

3. A few examples will now be given of the application of Theorem I to well-known cases.

(i) M. Riesz distinguished† between the limits

$$\lim_{n\to\infty} \sum_{k=1}^{n-1} \left(1 - \frac{k}{n}\right)^r z_k \quad \text{and} \quad \lim_{\omega\to\infty} \sum_{k<\omega} \left(1 - \frac{k}{\omega}\right)^r z_k,$$

† M. Riesz, *Comptes rendus*, 152 (1911), 1651–1654, and *Proc. London Math. Soc.* (2), 22 (1923), 412–419. I owe this remark to a Referee, who also pointed out that from my result (given below) on the absolute equivalence of "discontinuous" Riesz means and Cesàro means, and the well-known equivalence of Riesz and Cesàro means, it follows that the existence of one of the limits in question does imply that of the other for *bounded partial sums*. See also R. P. Agnew, *Trans. American Math. Soc.*, 35 (1933), 532–548.

where n is a positive integer and ω is a continuous real positive number; it cannot be assumed that the existence of one limit implies that of the other. The second gives rise to the familiar summation by Riesz means; we shall call the first "summation by *discontinuous Riesz means*."

M. Riesz showed† that in the *general* case (*i.e.*, z_k not being necessarily bounded) the two limits are equivalent when $r < 1$, but not equivalent when $r = 2$ or 3, though equivalence is retained if the first limit is replaced by one of the form $\sum_{k=1}^{n-2}$.

We now prove that

Discontinuous Riesz means and Cesàro means of any order $r \geqslant 0$ are absolutely equivalent for bounded sequences.

Taking the Cesàro means from $k = 0$ to $k = n$, we have

$$S_n = \frac{r}{r+n} + \sum_{k=1}^{n-1} \left| \frac{n!\, r\Gamma(r+n-k)}{(n-k)!\,\Gamma(r+n+1)} - \left(1-\frac{k}{n}\right)^r + \left(1-\frac{k+1}{n}\right)^r \right| + \frac{n!\,\Gamma(r+1)}{\Gamma(r+n+1)}.$$

The first term on the right tends to 0 as n tends to ∞; and so also does the last term on the right, since

$$\lim_{n\to\infty} \frac{(r+1)(r+2)\ldots(r+n)}{n!\, n^r} = \frac{1}{\Gamma(r+1)}.$$

Now

$$\frac{n!\, r\,\Gamma(r+n-k)}{(n-k)!\,\Gamma(r+n+1)} = \frac{r}{n-k}\left(1-\frac{k}{n}\right)^r \{1+o(1)\},$$

where the term $o(1)$ tends to zero as $n-k \to \infty$. Hence, to prove that $S_n \to 0$ as $n \to \infty$, it is sufficient to show that

$$S_n' = \sum_{k=1}^{n-1} \left| \frac{r}{n-k}\left(1-\frac{k}{n}\right)^r - \left(1-\frac{k}{n}\right)^r + \left(1-\frac{k+1}{n}\right)^r \right| \to 0, \qquad (3.1)$$

provided that it has been proved that

$$\sum_{k=1}^{n-1} \frac{r}{n-k}\left(1-\frac{k}{n}\right)^r o(1) \to 0 \quad \text{as} \quad n \to \infty.$$

† M. Riesz, *Proc. London Math. Soc.* (2), 22 (1923), 412–419 (416–418).

But this is the case since the last sum is

$$\frac{r}{n^r}\sum_{k=1}^{n-1} o(n-k)^{r-1} = \frac{r}{n^r}\, o(n^r) = o(1) \quad \text{as} \quad n\to\infty.$$

Now, since

$$\left(1-\frac{k+1}{n}\right)^r = \left(1-\frac{k}{n}\right)^r \left(1-\frac{1}{n-k}\right)^r,$$

we have

$$\begin{aligned}\frac{r}{(n-k)}\left(1-\frac{k}{n}\right)^r - \left(1-\frac{k}{n}\right)^r + \left(1-\frac{k+1}{n}\right)^r & \\ = \left(1-\frac{k}{n}\right)^r \left[\frac{r(r-1)}{2!}\,\frac{1}{(n-k)^2} + O\left(\frac{1}{(n-k)^3}\right)\right] & \\ = \frac{r(r-1)}{2n^r(n-k)^{2-r}} + O\left[\frac{(n-k)^{r-3}}{n^r}\right] & \qquad (3.2) \\ = O(n^{-s}), \quad \text{where} \quad s = \min(r, 2). &\end{aligned}$$

(*a*) If $r > 2$, then

$$S_n' = \sum_{k=1}^{n-1} O(n^{-2}) \to 0 \quad \text{as} \quad n\to\infty.$$

(*b*) If $1 < r \leqslant 2$, then

$$S_n' = \sum_{k=1}^{n-1} O(n^{-r}) \to 0.$$

(*c*) If $r = 1$, the Riesz and Cesàro means are identical when the summations are both for n terms.

(*d*) If $0 < r < 1$, then, by (3.2),

$$S_n' = O\left[\frac{1}{n^r}\sum_{p=1}^{n-1} p^{r-2}\right] \to 0 \quad \text{as} \quad n\to\infty,$$

since the series converges absolutely when $n\to\infty$.

(*e*) If $r = 0$, both reduce to ordinary convergence.

This proves the result.

(ii) $(C, 1)$ *and* $(C, 2)$ *summabilities are not absolutely equivalent for bounded sequences.*

Here

$$S_n = \sum_{k=0}^{n} \left| \frac{2(n-k+1)}{(n+1)(n+2)} - \frac{1}{n+1} \right|$$

$$= \sum_{k=0}^{m} \left\{ \frac{2(n-k+1)}{(n+1)(n+2)} - \frac{1}{n+1} \right\} + \sum_{k=m+1}^{n} \left\{ \frac{1}{n+1} - \frac{2(n-k+1)}{(n+1)(n+2)} \right\},$$

where $m = [\frac{1}{2}n]$, so that $\lim_{n\to\infty} \frac{m}{n} = \frac{1}{2}$.

Thus it is easily seen that

$$\lim_{n\to\infty} S_n = \tfrac{1}{2},$$

and the result follows.

(iii) *The arithmetic and Borel means† are not absolutely equivalent for bounded sequences.*

Here $$S_n = \sum_{k=0}^{n} \left| \frac{1}{n+1} - \frac{e^{-n} n^k}{k!} \right| + \sum_{k=n+1}^{\infty} \frac{e^{-n} n^k}{k!}.$$

Using Stirling's formula for $n!$, we have

$$\Sigma_n = \sum_{k=n+1}^{\infty} \frac{e^{-n} n^k}{k!} > \sum_{k=n+1}^{[n+\sqrt{n}]} \frac{e^{-n} n^k}{k!} > \frac{K\sqrt{n}\, e^{-n} n^{[n+\sqrt{n}]}}{e^{-[n+\sqrt{n}]}(n+\sqrt{n}+1)^{n+\sqrt{n}+\frac{1}{2}}},$$

since $e^{-n} n^k/k!$ steadily decreases as k increases from $n+1$ onwards. Hence

$$\Sigma_n > \frac{K' e^{\sqrt{n}}}{(1+1/\sqrt{n})^{n+\sqrt{n}}}, \quad \text{and} \quad \left(1+\frac{1}{\sqrt{n}}\right)^{n+\sqrt{n}} < K'' e^{\sqrt{n}};$$

so that $\Sigma_n > K'/K''$; K, K', K'' are all positive constants. Thus S_n does not tend to zero, and the result follows.

A classical example‡ of a series summable $(C, 1)$, but not summable (B), is

$$\sum_{n=1}^{\infty} e^{i\sqrt{n}}/\sqrt{n}.$$

From (iii) we are led to the following theorem.

† The Borel means which I employ throughout are really "discontinuous" Borel means, in the same sense as the "discontinuous" Riesz means used above.

‡ G. H. Hardy and J. E. Littlewood, "The relations between Borel's and Cesàro's methods of summation", *Proc. London Math. Soc.* (2), 11 (1911), 1–16 (15).

THEOREM II. *A semi-matrix $b_{n,k}$ and a complete matrix $a_{n,k}$ cannot be absolutely equivalent for bounded sequences unless*

$$\lim_{n\to\infty} \sum_{k=n+1}^{\infty} |a_{n,k}| = 0.$$

For
$$S_n = \sum_{k=1}^{n} |c_{n,k}| + \sum_{k=n+1}^{\infty} |a_{n,k}|,$$

and all the terms are positive or zero.

4. We now turn our attention to unbounded sequences.

We first select an unbounded sequence ϑ_k such that (with the notation of § 1)

$$R_n \equiv \sum_{k=1}^{\infty} |c_{n,k}\vartheta_k| \text{ exists}\dagger \text{ for each } n. \tag{4.1}$$

Then the class of unbounded sequences z_k to be considered is that for which

$$|z_k| \leqslant |\vartheta_k|. \tag{4.2}$$

THEOREM III. *A necessary and sufficient condition that the T-matrices $a_{n,k}$, $b_{n,k}$ are absolutely equivalent for all unbounded sequences satisfying the conditions* (4.1) *and* (4.2) *is that $R_n \to 0$ as $n \to \infty$.*

(i) The condition is sufficient; for

$$R_n' \equiv \sum_{k=1}^{\infty} |c_{n,k} z_k| \leqslant \sum_{k=1}^{\infty} |c_{n,k}\vartheta_k| = R_n,$$

so that $R_n' \to 0$ as $n \to \infty$, since $R_n \to 0$.

(ii) Conversely, let $z_n' - z_n'' \to 0$, *i.e.*, $\sum_{k=1}^{\infty} c_{n,k} z_k \to 0$ for all z_k satisfying (4.2) together with (4.1). Then we have to prove that $R_n \to 0$ (and hence $R_n' \to 0$). Suppose that R_n does not tend to 0; then either $R_n \to \alpha$, a positive limit; or R_n has several limiting-points, including (possibly) zero; or $R_n \to \infty$.

(*a*) If $R_n \to \alpha$, we can, as in Theorem I, construct a sequence v_k satisfying (4.2), and such that the corresponding $z_n' - z_n''$ does not tend to zero.

† This condition is *automatically* satisfied if $a_{n,k}$, $b_{n,k}$ are both *semi*-matrices.

First suppose that $c_{n,k}$ is real. Without loss of generality, we may suppose ϑ_k in (4.1) to be real. Then, with the same notation as in Theorem I, we take

$$\begin{aligned} v_k &= \vartheta_k \operatorname{sgn}(c_{n_1,k}\vartheta_k) && \text{for} \quad 1 \leqslant k \leqslant m_1, \\ v_k &= 0 && \text{for} \quad m_1 < k \leqslant m_2, \\ v_k &= \vartheta_k \operatorname{sgn}(c_{n_3,k}\vartheta_k) && \text{for} \quad m_2 < k \leqslant m_3, \\ v_k &= 0 && \text{for} \quad m_3 < k \leqslant m_4, \end{aligned}$$

and so on. We find, as before, that

$$v_{n_1}^* > \tfrac{2}{3}a, \quad v_{n_2}^* < \tfrac{1}{3}a, \quad v_{n_3}^* > \tfrac{2}{3}a, \quad v_{n_4}^* < \tfrac{1}{3}a, \quad \text{and so on.}$$

Thus all the terms of v_k are 0 or $\pm\vartheta_k$, so that v_k satisfies (4.2), and the result therefore follows when $R_n \to a$, and $c_{n,k}$ is real.

The extension to the case when R_n has several limiting-points is immediate, as in Theorem I.

If $c_{n,k}$ is complex, we write $c_{n,k} = r_{n,k}e^{i\theta_{n,k}}$, and then take

$$\begin{aligned} v_k &= |\vartheta_k| e^{-i\theta_{n_1,k}} && \text{for} \quad 1 \leqslant k \leqslant m_1, \\ v_k &= 0 && \text{for} \quad m_1 < k \leqslant m_2, \\ v_k &= |\vartheta_k| e^{-i\theta_{n_2,k}} && \text{for} \quad m_2 < k \leqslant m_3, \\ v_k &= 0 && \text{for} \quad m_3 < k \leqslant m_4, \quad \ldots; \end{aligned}$$

the proof then proceeds as before.

(*b*) Now suppose that $R_n \to \infty$. We construct a real sequence† $x_k \to 0$, such that $|x_k| \leqslant 1$, and such that

$$x_n' \equiv \sum_{k=1}^{\infty} c_{n,k}\vartheta_k x_k$$

has a subsequence tending to infinity; ϑ_k is still assumed to be real.

Since $a_{n,k}$, $b_{n,k}$ are T-matrices,

$$\lim_{n\to\infty} c_{n,k}\vartheta_k = 0 \quad \text{for every } k. \tag{4.3}$$

Hence, if p is fixed, the sequence

$$\sum_{k=1}^{p} |c_{n,k}\vartheta_k| \quad (n = 1, 2, 3, \ldots)$$

† The proof of case (*b*) is on the lines of that of the theorem of Schur and Kojima; see P. Dienes, *loc. cit.*, 385–8.

has a maximum term (dependent on p); we denote the maximum sum by C_p.

Since $R_n \to \infty$, for conveniently chosen n and p, say n_1 and p_1', we have

$$\sum_{k=1}^{p_1'} |c_{n_1,k}\vartheta_k| > \beta^2, \quad \text{where } \beta > 1 \text{ is fixed.}$$

Having thus fixed n_1, there is a number $p_1 \geqslant p_1'$ such that

$$\sum_{k=p_1+1}^{\infty} |c_{n_1,k}\vartheta_k| \leqslant \epsilon, \quad \text{any given positive number.}$$

Assume $c_{n,k}$ to be real†.

Take $$x_k = \frac{1}{\beta} \operatorname{sgn}(c_{n_1,k}\vartheta_k) \quad \text{for} \quad 1 \leqslant k \leqslant p_1;$$

then $$x'_{n_1} = \frac{1}{\beta} \sum_{k=1}^{p_1} |c_{n_1,k}\vartheta_k| + \sum_{k=p_1+1}^{\infty} c_{n_1,k}\vartheta_k x_k;$$

i.e., $$|x'_{n_1}| > \beta - \epsilon, \quad \text{since} \quad |x_k| \leqslant 1.$$

Now determine $n_2 > n_1$, and $p_2' > p_1$, such that

$$\frac{1}{\beta^2} \sum_{k=p_1+1}^{p_2'} |c_{n_2,k}\vartheta_k| > C_{p_1} + \beta^2, \tag{4.4}$$

and n_2 being thus fixed, determine $p_2 \geqslant p_2'$ so that

$$\sum_{k=p_2+1}^{\infty} |c_{n_2,k}\vartheta_k| \leqslant \epsilon. \tag{4.5}$$

Put $x_k = \frac{1}{\beta^2} \operatorname{sgn}(c_{n_2,k}\vartheta_k)$ for $p_1 < k \leqslant p_2$; then

$$x'_{n_2} = \frac{1}{\beta} \sum_{k=1}^{p_1} c_{n_2,k}\vartheta_k \operatorname{sgn}(c_{n_1,k}\vartheta_k) + \frac{1}{\beta^2} \sum_{k=p_1+1}^{p_2} |c_{n_2,k}\vartheta_k| + \sum_{k=p_2+1}^{\infty} c_{n_2,k}\vartheta_k x_k;$$

hence, by (4.4) and (4.5), $|x'_{n_2}| > \beta^2 - \epsilon$.

Continuing in this way, we determine a sequence $x_k \to 0$ such that, for certain values of n, $x_n' \to \infty$; *i.e.*, we determine a sequence $\vartheta_k x_k$ [which satisfies (4.2)] for which the corresponding $z_n' - z_n''$ contains a subsequence tending to infinity.

Cases (a) and (b) together prove that the condition $R_n \to 0$ is necessary.

This proves the theorem.

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† If $c_{n,k}$ is complex, the procedure is as before; write $c_{n,k}\vartheta_k = r_{n,k}e^{i\vartheta_{n,k}}$, and take $x_k = \beta^{-1}e^{-i\vartheta_{n_1,k}}$ for $1 \leqslant k \leqslant p_1$, ...

THEOREM IV. *A semi-matrix $b_{n,k}$ and a complete matrix $a_{n,k}$ cannot be absolutely equivalent for unbounded sequences satisfying* (4.2) *together with* (4.1) *unless*

$$\lim_{n\to\infty} \sum_{k=n+1}^{\infty} |a_{n,k}\vartheta_k| = 0.$$

For
$$R_n = \sum_{k=1}^{n} |c_{n,k}\vartheta_k| + \sum_{k=n+1}^{\infty} |a_{n,k}\vartheta_k|,$$

and all the terms are positive or zero.

5. We finally consider *regular T-limits*†.

A T-limit is said to be *regular* if the existence of the T-limits of $z_1, z_2, z_3, \ldots$ and of $z_2, z_3, z_4, \ldots$ imply one another, and lead to the same limit.

Thus $\sum_{k=1}^{\infty} a_{n,k}z_k$ is a regular T-limit if

$$\sum_{k=1}^{\infty} (a_{n,k} - a_{n,k+1})z_k \to 0 \quad \text{as} \quad n\to\infty.$$

Now, since $a_{n,k}$ is a T-matrix, so is $a_{n,k+1}$; hence the proofs of Theorems I and III apply with $a_{n,k+1}$ in place of $b_{n,k}$.

We are therefore led to theorems which imply rather more than regularity of the T-limit as defined above, namely that the T-limit is *absolutely regular*; *i.e.*, if

$$z_n' = \sum_{k=1}^{\infty} a_{n,k}z_k, \quad z_n'' = \sum_{k=1}^{\infty} a_{n,k+1}z_k,$$

then either z_n', z_n'' both tend to the same limit, or else neither of them tends to a limit, but their difference tends to zero.

These theorems are :

THEOREM V. *A necessary and sufficient condition that* $\sum_{k=1}^{\infty} a_{n,k}z_k$ *should be an absolutely regular T-limit for all bounded* z_k *is that*

$$\sigma_n \equiv \sum_{k=1}^{\infty} |a_{n,k} - a_{n,k+1}| \to 0 \quad \text{as} \quad n\to\infty.$$

† P. Dienes, *loc. cit.*, 418.

For unbounded sequences, we consider the class z_k satisfying (4.2), where

$$\rho_n \equiv \sum_{k=1}^{\infty} |(a_{n,k}-a_{n,k+1})\vartheta_k| \text{ exists for each } n, \qquad (5.1)$$

i.e., the result of replacing $b_{n,k}$ by $a_{n,k+1}$ in (4.1).

THEOREM VI. *A necessary and sufficient condition that* $\sum_{k=1}^{\infty} a_{n,k}z_k$ *should be an absolutely regular T-limit for all unbounded* z_k *satisfying* (4.2) *together with* (5.1) *is that* $\rho_n \to 0$ *as* $n \to \infty$.

As illustrations of Theorem V, it can very easily be deduced that *the arithmetic and Borel means are absolutely regular T-limits for all bounded sequences.*

(i) With $a_{n,k} = \frac{1}{n}$ $(k \leqslant n)$, $= 0$ $(k > n)$, we have $\sigma_n = \frac{1}{n} \to 0$.

(ii) With $a_{n,k} = e^{-n}n^k/k!$, we have

$$\sigma_n = e^{-n}\sum_{k=0}^{\infty}\left|\frac{n^k}{k!}-\frac{n^{k+1}}{(k+1)!}\right|$$

$$= e^{-n}\sum_{k=0}^{n-1}\left(\frac{n^{k+1}}{(k+1)!}-\frac{n^k}{k!}\right)+e^{-n}\sum_{k=n}^{\infty}\left(\frac{n^k}{k!}-\frac{n^{k+1}}{(k+1)!}\right)$$

$$= e^{-n}\sum_{k=0}^{\infty}\left(\frac{n^k}{k!}-\frac{n^{k+1}}{(k+1)!}\right)+2e^{-n}\sum_{k=0}^{n-2}\left(\frac{n^{k+1}}{(k+1)!}-\frac{n^k}{k!}\right)$$

$$= e^{-n}\left\{\frac{2n^{n-1}}{(n-1)!}-1\right\} \to 0, \quad \text{by Stirling's formula.}$$

In illustration of Theorem VI, the last result may be extended to a class of unbounded sequences†:

The Borel matrix is an absolutely regular T-limit for all unbounded sequences z_k *such that* $z_k = o(\sqrt{k})$.

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† *Cf.* P. Dienes, *loc. cit.*, 419–421. It was pointed out to me by a Referee that, as regards *ordinary* regularity, this last result follows from the theorem of Hardy and Littlewood that, *if* z_n *is convergent* (B), *and* $z_n = o(\sqrt{n})$, *then* z_n *is convergent* $(C, 1)$; the most general form of this theorem is given by R. D. Lord, *Proc. London Math. Soc.* (2), 38 (1935), 241–256 (250). I am also indebted to this Referee for giving me an arrangement of the proof of the last result (which I reproduce below) which is a little simpler than my original proof.

We have

$$\rho_n = e^{-n} \sum_{k=0}^{\infty} \left| \frac{n^k}{k!} - \frac{n^{k+1}}{(k+1)!} \right| o(\sqrt{k})$$

$$= e^{-n} \sum_{k=0}^{n-2} \left\{ \frac{n^{k+1}}{(k+1)!} - \frac{n^k}{k!} \right\} o(\sqrt{k}) + e^{-n} \sum_{k=n}^{\infty} \left\{ \frac{n^k}{k!} - \frac{n^{k+1}}{(k+1)!} \right\} o(\sqrt{k})$$

$$= e^{-n} o(\sqrt{n}) \sum_{k=0}^{n-2} \left\{ \frac{n^{k+1}}{(k+1)!} - \frac{n^k}{k!} \right\} + e^{-n} o\left(\frac{1}{\sqrt{n}}\right) \sum_{k=n}^{\infty} \left\{ \frac{n^k}{k!} - \frac{n^{k+1}}{(k+1)!} \right\} k$$

$$= e^{-n} o(\sqrt{n}) \left(\frac{n^n}{n!} - 1 \right) + e^{-n} o\left(\frac{1}{\sqrt{n}}\right) \sum_{k=n}^{\infty} \left\{ \frac{n^k}{(k-1)!} - \frac{n^{k+1}}{k!} \right\}$$

$$+ e^{-n} o\left(\frac{1}{\sqrt{n}}\right) \sum_{k=n}^{\infty} \frac{n^{k+1}}{(k+1)!}$$

$$= e^{-n} o(\sqrt{n}) \left(\frac{n^n}{n!} - 1 \right) + e^{-n} o\left(\frac{1}{\sqrt{n}}\right) \left\{ \frac{n^n}{(n-1)!} + O(e^n) \right\}$$

$$= e^{-n} o(\sqrt{n})\, O\left(\frac{1}{e^{-n}\sqrt{n}} \right) + e^{-n} o\left(\frac{1}{\sqrt{n}}\right) \left\{ O\left(\frac{\sqrt{n}}{e^{-n}} \right) + O(e^n) \right\}$$

$$= o(1) + o(1),$$

by Stirling's formula.

This proves the result.

I have pleasure in expressing my thanks to Dr. P. Dienes for kindly reading the manuscript, and making some useful comments.

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PROOF THAT EVERY LARGE INTEGER IS A SUM OF SEVENTEEN BIQUADRATES

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Introduction.

The method used in this paper is the well-known Hardy-Littlewood method with modifications and refinements due to Landau and especially to Winogradoff. The new idea—an extension of one of Winogradoff's—is to seek an asymptotic formula not for the number of *all* solutions of the equation

$$\sum_{v=1}^{17} r_v^4 = n$$

in integers $r_1, r_2, \ldots, r_{17}$, but for the number of those which satisfy the inequalities (3.02) to (3.06) below.

It was proved by Hardy and Littlewood* that every sufficiently large integer is a sum of nineteen biquadrates.

I am indebted to Dr. H. Heilbronn for his lecture on Winogradoff's new methods at the Bristol Mathematical Colloquium in June, 1935.

With $G(k)$ defined as by Hardy and Littlewood (**1**, section 3), the result of this paper may be summed up in the inequality

$$G(4) \leqslant 17.$$

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* Hardy-Littlewood, **4**, Theorem 4.

It can probably be proved in the same way that

$$G(k) \leqslant 2m+7+[2^{k-1}(k-2)(1-k^{-1})^{m+1}] \quad (k>4),$$

where

$$m = \left[\frac{(k-2)\log 2+\log(k-2)-\log k}{\log k-\log(k-1)}\right].$$

This would imply $G(5) \leqslant 29$.

1. *Notation.*

h, j, m, and r are integers, and l, n, q, and v are positive integers.

t, x, y, θ, and ϑ are real numbers, and u and w are numbers greater than 1.

ϵ is a positive number less than or equal to 2^{-11}.

$B_1, B_2, \ldots, C_1, C_2, \ldots$ are suitable (sufficiently large) positive numbers, the B's being absolute constants, the C's depending at most on ϵ.

(1.1) $$\xi_q = e^{2\pi i/q}.$$

(1.2) $$f(\theta, u) = \sum_{v<u} e^{2\pi i\theta v^4}.$$

(1.3) $$g(\vartheta, u) = \int_0^u e^{2\pi i\vartheta y^4}\,dy.$$

(1.4) $$S_{h,q} = \sum_{v=1}^{q} \xi_q^{hv^4}.$$

(1.5) $$A_q = q^{-17} \sum_{\substack{0<h\leqslant q\\(h,q)=1}} S_{h,q}^{17}\, \xi_q^{-nh}.$$

(1.6) $$S = \sum_{q=1}^{\infty} A_q.$$

2. *Preliminary theorems.*

2.1. If ρ is a primitive q-th root of unity, then

$$\left|\sum_{r=m}^{m+v-1} \rho^{r^4}\right|^8 \leqslant C_1 q^{\epsilon} v^{\epsilon}(v^7+v^4 q+v^8 q^{-1}).$$

This is a particular case of Weyl's inequality (Landau, **1**, Theorem 267).

2.2. If $(h, q) = 1$, then

$$|S_{h,q}| \leqslant B_1 q^{\frac{3}{4}}.$$

This is a particular case of Lemma 3, Hardy-Littlewood, **4** (Landau, **1**, Theorem 315).

2.3. *Euler's sum formula.*

Bernoulli's functions* are characterized by the following properties:

$$b_1(x) = x - [x] - \tfrac{1}{2}, \tag{2.31}$$

where $[x]$ is the greatest integer less than or equal to x,

$$b_l(x+1) = b_l(x), \tag{2.32}$$

and

$$\int_0^x b_l(y)\,dy = b_{l+1}(x) - b_{l+1}(0) \tag{2.33}$$

for every x and every l.

Let $b > a$, and let $f(x)$ and its derivatives (as far as they occur below) be continuous for $a \leqslant x \leqslant b$. Then, for any t,

$$\sum_{a \leqslant m+t < b} f(m+t) = \int_a^b f(x)\,dx + \sum_{r=0}^{l-1} \{f^{(r)}(b)\,b_{r+1}(t-b) - f^{(r)}(a)\,b_{r+1}(t-a)\} - \int_a^b f^{(l)}(x)\,b_l(t-x)\,dx. \tag{2.34}$$

Proof (by induction). By (2.33),

$$-\int_a^b f^{(l)}(x)\,b_l(t-x)\,dx = f^{(l)}(b)\,b_{l+1}(t-b) - f^{(l)}(a)\,b_{l+1}(t-a) - \int_a^b f^{(l+1)}(x)\,b_{l+1}(t-x)\,dx.$$

From this it follows that the formula obtained from (2.34) by substituting $l+1$ for l is true for any l for which (2.34) is true. Hence it is sufficient to prove (2.34) for $l = 1$, *i.e.*

$$\sum_{m=m_0}^{m_1} f(m+t) = \int_a^b f(x)\,dx + f(b)\,b_1(t-b) - f(a)\,b_1(t-a) - \int_a^b f'(x)\,b_1(t-x)\,dx, \tag{2.35}$$

where m_0 is the least and m_1 the greatest m such that $a \leqslant m+t < b$. (The

* See, *e.g.*, Bessel-Hagen, *Proc. London Math. Soc.* (2), 29 (1929), 387.

trivial case when there is no such m may be left to the reader.) Now, by (2.31),

$$\begin{aligned}
&\int_a^b \{f(x)-f'(x)\,b_1(t-x)\}\,dx \\
&\quad=\int_a^{m_0+t}\{f(x)-f'(x)(m_0+t-x-\tfrac12)\}\,dx \\
&\qquad\qquad+\sum_{m=m_0}^{m_1-1}\int_{m+t}^{m+1+t}\{f(x)-f'(x)(m+t-x+\tfrac12)\}\,dx \\
&\qquad\qquad+\int_{m_1+t}^{b}\{f(x)-f'(x)(m_1+t-x+\tfrac12)\}\,dx \\
&\quad=\Big[f(x)(x+\tfrac12-m_0-t)\Big]_a^{m_0+t}+\sum_{m=m_0}^{m_1-1}\Big[f(x)(x-\tfrac12-m-t)\Big]_{m+t}^{m+1+t} \\
&\qquad\qquad+\Big[f(x)(x-\tfrac12-m_1-t)\Big]_{m_1+t}^{b} \\
&\quad=f(a)\,b_1(t-a)+\tfrac12 f(m_0+t) \\
&\qquad\qquad+\sum_{m=m_0}^{m_1-1}\tfrac12\{f(m+t)+f(m+t+1)\}+\tfrac12 f(m_1+t)-f(b)\,b_1(t-b) \\
&\quad=f(a)\,b_1(t-a)+\sum_{m=m_0}^{m_1} f(m+t)-f(b)\,b_1(t-b),
\end{aligned}$$

which proves (2.35).

2.4. *Application of Euler's sum formula to $f(\theta, u)$.*

Let

$$\theta=\frac{h}{q}+\vartheta. \tag{2.41}$$

Then, by (1.2) and (1.1),

$$1+f(\theta,u)=\sum_{v=1}^{q}\sum_{\substack{0\leqslant r<u\\ r\equiv v \pmod q}}\xi_q^{hr^4}e^{2\pi i\vartheta r^4}=\sum_{v=1}^{q}\xi_q^{hv^4}d_v, \tag{2.42}$$

where

$$d_v=\sum_{\substack{j\\ 0\leqslant qj+v<u}}e^{2\pi i\vartheta(qj+v)^4}=\sum_{\substack{j\\ 0\leqslant j+v/q<u/q}}\phi\Big(j+\frac{v}{q}\Big) \tag{2.43}$$

and

$$\phi(x)=\phi_0\{(2\pi i\vartheta)^{\frac14}qx\},\quad \phi_0(z)=e^{z^4}. \tag{2.44}$$

Let

$$l=\left[\frac{1}{\epsilon}\right]+1. \tag{2.45}$$

By (2.43) and (2.34),

$$d_v=\int_0^{u/q}\phi(x)\,dx+\sum_{r=0}^{l-1}\left\{\phi^{(r)}\left(\frac{u}{q}\right)b_{r+1}\left(\frac{v}{q}-\frac{u}{q}\right)-\phi^{(r)}(0)\,b_{r+1}\left(\frac{v}{q}\right)\right\} \tag{2.46}$$
$$-\int_0^{u/q}\phi^{(l)}(x)\,b_l\left(\frac{v}{q}-x\right)dx.$$

Now, by (2.44) and (1.3),

$$\int_0^{u/q}\phi(x)\,dx=q^{-1}\int_0^{u}\phi(yq^{-1})\,dy=q^{-1}\int_0^{u}e^{2\pi i\vartheta y^4}\,dy=q^{-1}g(\vartheta,u).$$

Hence, by (2.42), (2.46), and (1.4),

$$1+f(\theta,u) \tag{2.47}$$
$$=S_{h,q}q^{-1}g(\vartheta,u)+\sum_{r=0}^{l-1}\left\{\phi^{(r)}\left(\frac{u}{q}\right)a_r\left(\frac{u}{q}\right)-\phi^{(r)}(0)\,a_r(0)\right\}-R,$$

where

$$a_r(t)=\sum_{v=1}^{q}\xi_q^{hv^4}b_{r+1}\left(\frac{v}{q}-t\right) \tag{2.48}$$

and

$$R=\sum_{v=1}^{q}\xi_q^{hv^4}\int_0^{u/q}\phi^{(l)}(x)\,b_l\left(\frac{v}{q}-x\right)dx. \tag{2.49}$$

2.5. *Inequalities for $\phi^{(r)}(x)$ and R.*

It follows from (2.44) that

$$\phi^{(r)}(x)=(2\pi i\vartheta)^{\frac{1}{4}r}q^r\phi_0^{(r)}\{(2\pi i\vartheta)^{\frac{1}{4}}qx\}\quad(r\geqslant 0) \tag{2.51}$$

and that

$$\phi_0^{(r)}(z)=P_r(z)\,e^{z^4}, \tag{2.52}$$

where P_r is a polynomial of degree $3r$, so that

$$|P_r(z)|\leqslant c_r(1+|z|^{3r}),$$

where $c_0, c_1, c_2, \ldots$ are constants. Hence, by (2.45) and (2.52),

$$|\phi_0^{(r)}\{(2\pi i\vartheta)^{\frac{1}{4}}qx\}|\leqslant C_2(1+|\vartheta|^{\frac{3}{4}r}q^{3r}x^{3r})\quad(x\geqslant 0,\ 0\leqslant r\leqslant l),$$

and hence, by (2.51),

$$|\phi^{(r)}(x)| \leqslant C_3(|\vartheta|^{\frac{1}{4}r} q^r + |\vartheta|^r q^{4r} x^{3r}) \quad (x \geqslant 0,\ 0 \leqslant r \leqslant l). \tag{2.53}$$

Now let

$$q \leqslant u^{1-\epsilon},\quad |\vartheta| \leqslant q^{-1} u^{-3-\epsilon}. \tag{2.54}$$

Then, by (2.53),

$$|\phi^{(r)}(x)| \leqslant 2C_3 u^{-r\epsilon} \quad (0 \leqslant x \leqslant u/q,\ 0 \leqslant r \leqslant l). \tag{2.55}$$

In particular, by (2.55) and (2.45),

$$|\phi^{(l)}(x)| \leqslant 2C_3 u^{-1} \quad (0 \leqslant x \leqslant u/q). \tag{2.56}$$

Now $b_r(x)$ is a bounded function of x, so that, by (2.45),

$$|b_r(x)| \leqslant C_4 \quad (0 \leqslant r \leqslant l). \tag{2.57}$$

By (2.49), (2.56), and (2.57),

$$|R| \leqslant 2C_3 C_4. \tag{2.58}$$

2.6. *An inequality for* $a_r(t)$.

Let

$$(h, q) = 1, \tag{2.61}$$

and let

$$s_v = \sum_{m=1}^{v} \xi_q^{hm^4}. \tag{2.62}$$

Then, by (2.48),

$$\begin{aligned} a_r(t) &= s_1 b_{r+1}\left(\frac{1}{q} - t\right) + \sum_{v=2}^{q} (s_v - s_{v-1}) b_{r+1}\left(\frac{v}{q} - t\right) \\ &= \sum_{m=1}^{q-1} s_m \left\{ b_{r+1}\left(\frac{m}{q} - t\right) - b_{r+1}\left(\frac{m+1}{q} - t\right)\right\} + s_q b_{r+1}(1-t). \end{aligned} \tag{2.63}$$

Now, by 2.1,

$$|s_v| \leqslant C_5 q^{\frac{1}{2}+\epsilon} \quad (v \leqslant q),$$

and hence, by (2.63) and (2.57),

$$|a_r(t)| \leqslant C_5 q^{\frac{1}{2}+\epsilon} \left\{ \sum_{m=1}^{q-1} \left| b_{r+1}\left(\frac{m}{q} - t\right) - b_{r+1}\left(\frac{m+1}{q} - t\right)\right| + C_4 \right\} \tag{2.64}$$

$$(0 \leqslant r \leqslant l-1).$$

Let V_r be the total variation of $b_{r+1}(x)$ in the interval $0 \leqslant x \leqslant 1$. Then, by (2.32), V_r is the total variation of $b_{r+1}(x)$ in any closed interval of length 1, and in particular in the interval $-t \leqslant x \leqslant 1-t$, so that

$$\sum_{m=1}^{q-1} \left| b_{r+1}\left(\frac{m}{q}-t\right) - b_{r+1}\left(\frac{m+1}{q}-t\right) \right| \leqslant V_r \leqslant C_6 \qquad (0 \leqslant r \leqslant l-1).$$

From this and (2.64) we obtain

$$|a_r(t)| \leqslant C_7 q^{\frac{7}{8}+\epsilon} \quad (0 \leqslant r \leqslant l-1). \tag{2.65}$$

2.7. *Inequalities for $f(\theta, u)$ and $g(\vartheta, u)$.*

By (2.55) and (2.65),

$$\left| \phi^{(r)}\left(\frac{u}{q}\right) a_r\left(\frac{u}{q}\right) - \phi^{(r)}(0)\, a_r(0) \right| \leqslant C_8 q^{\frac{7}{8}+\epsilon} \quad (0 \leqslant r \leqslant l-1).$$

Hence, by (2.47), (2.45), and (2.58),

$$|f(\theta, u) - S_{h,q} q^{-1} g(\vartheta, u)| \leqslant C_9 q^{\frac{7}{8}+\epsilon}, \tag{2.71}$$

provided that (2.41), (2.54), and (2.61) are satisfied.

It easily follows from (1.3) that

$$|g(\vartheta, u)| \leqslant \min(u,\ B_2 |\vartheta|^{-\frac{1}{4}}) \leqslant B_3 (u^{-4} + |\vartheta|)^{-\frac{1}{4}}. \tag{2.72}$$

2.8. We can now establish the following result:

Let u and θ be such that the conditions (2.41), (2.61), and

$$q \leqslant u^{\frac{1}{2}}, \quad |\vartheta| \leqslant q^{-1} u^{-\frac{7}{2}} \tag{2.81}$$

are not simultaneously satisfied by any h, q, ϑ. Then

$$|f(\theta, u)| \leqslant C_{10} u^{\frac{7}{8}+\epsilon}. \tag{2.82}$$

Proof. It is known that, corresponding to any $y \geqslant 1$, there exist numbers h, q, and ϑ such that

$$\theta = h/q + \vartheta, \quad (h, q) = 1, \quad q \leqslant y, \quad \text{and} \quad |\vartheta| \leqslant (qy)^{-1}.$$

In particular, we can choose h, q, and ϑ so that (2.41) and (2.61) hold, and

$$q \leqslant u^{3+\epsilon}, \quad |\vartheta| \leqslant q^{-1} u^{-3-\epsilon}. \tag{2.83}$$

Then, since (2.81) does not hold, at least one of the following three condi-

tions is satisfied:

(i) $$u^{1-\epsilon} < q \leqslant u^{3+\epsilon},$$

(ii) $$u^{\frac{1}{2}} < q \leqslant u^{1-\epsilon},$$

(iii) $$q \leqslant u^{1-\epsilon}, \quad |\vartheta| > q^{-1}u^{-\frac{7}{2}}.$$

Suppose, first, that (i) holds. Then, with s_v defined by (2.62), and λ defined as the greatest integer less than u, we have, by (1.2), (2.41), and (1.1),

$$f(\theta, u) = \sum_{r=1}^{\lambda} \xi_q^{hr^4} e^{2\pi i\vartheta r^4} = s_1 e^{2\pi i\vartheta} + \sum_{r=2}^{\lambda} (s_r - s_{r-1}) e^{2\pi i\vartheta r^4}$$

$$= \sum_{v=1}^{\lambda-1} s_v (e^{2\pi i\vartheta v^4} - e^{2\pi i\vartheta (v+1)^4}) + s_\lambda e^{2\pi i\vartheta \lambda^4},$$

so that

(2.84) $$|f(\theta, u)| \leqslant \sum_{v=1}^{\lambda-1} |s_v| 2\pi |\vartheta| \{(v+1)^4 - v^4\} + |s_\lambda|.$$

Also, by (2.61), (2.62), 2.1, and (i),

$$|s_v| \leqslant C_{11} u^{\frac{7}{8}+\epsilon} \quad (v \leqslant u),$$

and, by (2.83) and (i), $|\vartheta| < u^{-4}$. Hence, by (2.84) and the definition of λ,

$$|f(\theta, u)| \leqslant C_{11} u^{\frac{7}{8}+\epsilon} \left\{ 2\pi u^{-4} \sum_{v=1}^{\lambda-1} [(v+1)^4 - v^4] + 1 \right\} < C_{11} u^{\frac{7}{8}+\epsilon} (2\pi + 1),$$

which implies (2.82).

Next suppose that (ii) holds. Then, by (2.71), 2.2, and (2.72),

$$|f(\theta, u)| \leqslant B_1 q^{-\frac{1}{2}} u + C_q q^{\frac{7}{8}+\epsilon},$$

which, together with (ii), proves (2.82).

Finally suppose that (iii) holds. Then, by (2.71), 2.2, and (2.72),

$$|f(\theta, u)| \leqslant B_1 q^{-\frac{1}{2}} B_2 |\vartheta|^{-\frac{1}{2}} + C_q q^{\frac{7}{8}+\epsilon},$$

which, together with (iii), again proves (2.82).

3. *The main part of the proof.*

3.0. For abbreviation, put

(3.00) $$\alpha = 2\tfrac{21}{64}, \quad \delta = \tfrac{1}{512}.$$

I assume that n is large, say greater than 1000, and introduce the following functions:

$$f_1 = f_1(\theta) = f(\theta, n^{\frac{1}{4}}),$$
$$f_2 = f_2(\theta) = f_1 - f(\theta, n^{\frac{1}{4}-\delta}),$$
$$f_3 = f_3(\theta) = f(\theta, n^{\frac{3}{16}}) - f(\theta, \tfrac{1}{2}n^{\frac{3}{16}}),$$
$$f_4 = f_4(\theta) = f(\theta, n^{\frac{9}{64}}) - f(\theta, \tfrac{1}{2}n^{\frac{9}{64}}),$$
$$f_5 = f_5(\theta) = f(\theta, n^{\frac{37}{256}}),$$
$$f_6 = f_6(\theta) = f_3 f_4 f_5^2,$$
$$f_7 = f_7(\theta) = f_2 f_6,$$
$$f_8 = f_8(\theta) = f_1^7 f_7^2.$$

It easily follows from these definitions that

$$f_8 = \sum_{r=1}^{r_0} a_r e^{2\pi i\theta r}, \tag{3.01}$$

where r_0 is a sufficiently large number, *e.g.* $17n$, and a_r is the number of solutions of the equation

$$\sum_{v=1}^{17} r_v^4 = r$$

in integers $r_1, r_2, \ldots, r_{17}$ satisfying the inequalities

$$0 < r_v < n^{\frac{1}{4}} \quad (v = 1, 2, \ldots, 7), \tag{3.02}$$

$$n^{\frac{1}{4}-\delta} \leqslant r_v < n^{\frac{1}{4}} \quad (v = 8, 9), \tag{3.03}$$

$$\tfrac{1}{2}n^{\frac{3}{16}} \leqslant r_v < n^{\frac{3}{16}} \quad (v = 10, 11), \tag{3.04}$$

$$\tfrac{1}{2}n^{\frac{9}{64}} \leqslant r_v < n^{\frac{9}{64}} \quad (v = 12, 13), \tag{3.05}$$

and

$$0 < r_v < n^{\frac{37}{256}} \quad (v = 14, 15, 16, 17). \tag{3.06}$$

Hence it is sufficient to prove that $a_n \neq 0$ if n is sufficiently large.

3.1. Similarly,

$$f_5^2 = \sum_{v < 2n^{\frac{37}{64}}} h_1(v)\, e^{2\pi i\theta v},$$

where $h_1(v)$ is the number of certain representations of v as a sum of two biquadrates, so that

$$0 \leqslant h_1(v) \leqslant C_{12} n^{\epsilon}. \tag{3.10}$$

Hence

$$f_4 f_5^2 = \sum_{\frac{1}{2}n^{\frac{9}{14}} \leqslant j < n^{\frac{9}{14}}} e^{2\pi i\theta j^4} \sum_{v<2n^{\frac{27}{14}}} h_1(v)\, e^{2\pi i\theta v} = \sum_{q<3n^{\frac{9}{14}}} h_2(q)\, e^{2\pi i\theta q},$$

where

$$h_2(q) = \sum_j h_1(q-j^4),$$

the summation being taken over those values of j which satisfy the conditions

$$\tfrac{1}{2}n^{\frac{9}{14}} \leqslant j < n^{\frac{9}{14}} \quad \text{and} \quad 0 < q-j^4 < 2n^{\frac{27}{14}}.$$

If j_1 is the least and j_2 the greatest j satisfying these conditions, then

$$\tfrac{1}{2}n^{\frac{9}{14}} \leqslant j_1 \quad \text{and} \quad {j_2}^4-{j_1}^4 < 2n^{\frac{27}{14}},$$

so that

$$j_2-j_1 \leqslant ({j_2}^4-{j_1}^4)/(4{j_1}^3) < 4.$$

It follows that the sum has at most four terms. Hence, by (3.10),

$$0 \leqslant h_2(q) \leqslant 4C_{12}n^{\epsilon}.$$

Similarly,

$$f_6 = f_3 f_4 f_5^2 = \sum_{\frac{1}{2}n^{\frac{3}{14}} \leqslant j < n^{\frac{3}{14}}} e^{2\pi i\theta j^4} \sum_{q<3n^{\frac{9}{14}}} h_2(q)\, e^{2\pi i\theta q} = \sum_{v<4n^{\frac{3}{4}}} h_3(v)\, e^{2\pi i\theta v},$$

and

$$0 \leqslant h_3(v) \leqslant 24C_{12}n^{\epsilon}. \tag{3.11}$$

Finally,

$$f_7(\theta) = f_2 f_6 = \sum_{n^{\frac{1}{4}-\delta} \leqslant j < n^{\frac{1}{4}}} e^{2\pi i\theta j^4} \sum_{v<4n^{\frac{3}{4}}} h_3(v)\, e^{2\pi i\theta v} = \sum_{m=1}^{r_0} b_m\, e^{2\pi i\theta m}, \tag{3.12}$$

where

$$b_m = \sum_j h_3(m-j^4), \tag{3.13}$$

the summation being taken over those values of j which satisfy the conditions

$$n^{\frac{1}{4}-\delta} \leqslant j < n^{\frac{1}{4}} \quad \text{and} \quad 0 < m-j^4 < 4n^{\frac{3}{4}}.$$

If j_3 is the least and j_4 the greatest of these values of j, then

$$j_3 \geqslant n^{\frac{1}{4}-\delta} \quad \text{and} \quad {j_4}^4-{j_3}^4 < 4n^{\frac{3}{4}},$$

so that

$$j_4-j_3 \leqslant ({j_4}^4-{j_3}^4)/(4{j_3}^3) < n^{3\delta}.$$

Hence, by (3.13) and (3.11),

$$0 \leqslant b_m \leqslant C_{13}n^{3\delta+\epsilon}. \tag{3.14}$$

Let θ_0 be any real number. Then, by (3.12) and (3.14),

$$(3.15)\qquad \int_{\theta_0}^{\theta_0+1} |f_7(\theta)|^2 d\theta = \sum_{m=1}^{r_0} b_m{}^2 \leqq C_{13} n^{3\delta+\epsilon} \sum_{m=1}^{r_0} b_m = C_{13} n^{3\delta+\epsilon} f_7(0).$$

Now it easily follows from 3.0 that

$$f_7(0) < n^{\frac{1}{4}+\frac{3}{16}+\frac{9}{64}+\frac{27}{128}} = n^{\frac{101}{128}}.$$

Hence, by (3.15),

$$(3.16)\qquad \int_{\theta_0}^{\theta_0+1} |f_7(\theta)|^2 d\theta \leqq C_{13} n^{\frac{101}{128}+3\delta+\epsilon}.$$

3.2. By (3.01),

$$(3.20)\qquad a_n = \int_{\theta_0}^{\theta_0+1} f_8(\theta)\, e^{-2\pi i \theta n} d\theta$$

for any real number θ_0. It will be convenient to put

$$(3.21)\qquad u = n^{\frac{1}{4}},$$

and to choose θ_0 so that

$$(3.22)\qquad u^{-\frac{7}{2}} < \theta_0 < u^{-\frac{3}{2}} - u^{-4}.$$

For any h and q satisfying (2.61), let $I_{h,q}$ denote the interval

$$(3.23)\qquad \frac{h}{q} - q^{-1} u^{-\frac{7}{2}} \leqq \theta \leqq \frac{h}{q} + q^{-1} u^{-\frac{7}{2}}.$$

Then, by (3.20) and (3.22),

$$(3.24)\qquad a_n = \sum_{q \leqq u^{\frac{1}{2}}} \sum_{\substack{0<h\leqq q \\ (h,q)=1}} \int_{I_{h,q}} f_8(\theta)\, e^{-2\pi i \theta n} d\theta + \int_E f_8(\theta)\, e^{-2\pi i \theta n} d\theta,$$

where E is the set of those points of the interval $\theta_0 \leqq \theta \leqq \theta_0 + 1$ which do not lie in any $I_{h,q}$ occurring in (3.24). It is easily seen that, if θ lies in E, then (2.41), (2.61), and (2.81) are not simultaneously satisfied by any h, q, ϑ. Hence, by 2.8, 3.0, and (3.21),

$$(3.25)\qquad |f_1(\theta)| \leqq C_{10} n^{\frac{7}{32}+\frac{1}{4}\epsilon} \quad (\theta \text{ in } E).$$

From 3.0, (3.25), and (3.16) we obtain

$$(3.26)\qquad \left| \int_E f_8(\theta)\, e^{-2\pi i \theta n} d\theta \right| \leqq \int_E |f_1{}^7 f_7{}^2| d\theta$$

$$\leqq C_{10}^7 n^{\frac{49}{32}+\frac{7}{4}\epsilon} \int_{\theta_0}^{\theta_0+1} |f_7(\theta)|^2 d\theta \leqq C_{14} n^{\alpha-\delta+3\epsilon}.$$

3.3. Until further notice we assume (2.41), (2.61), and (2.81).

Since (2.81) and (3.21) imply (2.54) and the formula obtained by substituting $n^{\frac{1}{4}-\delta}$ for u in (2.54), we have (2.71) and the formula obtained by substituting $n^{\frac{1}{4}-\delta}$ for u in (2.71), *i.e.*

$$|f(\theta, n^{\frac{1}{4}-\delta})-S_{h,q}q^{-1}g(\vartheta, n^{\frac{1}{4}-\delta})| \leqslant C_9 q^{\frac{7}{8}+\epsilon}. \tag{3.30}$$

Now, by 2.2 and (2.72),

$$|S_{h,q}q^{-1}g(\vartheta, n^{\frac{1}{4}})| \leqslant B_1 B_3 q^{-\frac{1}{4}}(n^{-1}+|\vartheta|)^{-\frac{1}{4}} \tag{3.31}$$

and

$$|S_{h,q}q^{-1}g(\vartheta, n^{\frac{1}{4}-\delta})| \leqslant B_1 B_3 q^{-\frac{1}{4}}(n^{-1+4\delta}+|\vartheta|)^{-\frac{1}{4}}. \tag{3.32}$$

Put

$$g_1 = g_1(\vartheta) = S_{h,q}q^{-1}g(\vartheta, n^{\frac{1}{4}}), \quad g_2 = g_2(\vartheta) = g_1 - S_{h,q}q^{-1}g(\vartheta, n^{\frac{1}{4}-\delta}). \tag{3.33}$$

Then, by 3.0, (2.71), (3.21), and (3.30),

$$|f_v-g_v| \leqslant 2C_9 q^{\frac{7}{8}+\epsilon} \quad (v=1, 2),$$

which implies

$$|f_v-g_v| \leqslant B_4 q \quad (v=1, 2).$$

From this and (2.81) and (3.21) we obtain

$$|f_v-g_v| \leqslant B_4 n^{\frac{1}{8}} \leqslant B_5 q^{-\frac{1}{4}}(n^{-1}+|\vartheta|)^{-\frac{1}{4}} \quad (v=1, 2). \tag{3.34}$$

Also, by (3.31), (3.32), and (3.33),

$$|g_v| \leqslant 2B_1 B_3 q^{-\frac{1}{4}}(n^{-1}+|\vartheta|)^{-\frac{1}{4}} \quad (v=1, 2). \tag{3.35}$$

From this and (3.34) it follows that

$$|f_v| \leqslant B_6 q^{-\frac{1}{4}}(n^{-1}+|\vartheta|)^{-\frac{1}{4}} \quad (v=1, 2). \tag{3.36}$$

By (3.34), (3.35), and (3.36),

$$|f_1^7-g_1^7| = |f_1-g_1|\left|\sum_{m=0}^{6} f_1^{6-m}g_1^m\right| \leqslant B_4 n^{\frac{1}{8}} 7 \max(|f_1^6|, |g_1^6|)$$

$$\leqslant B_7 n^{\frac{1}{8}} q^{-\frac{3}{2}}(n^{-1}+|\vartheta|)^{-\frac{3}{2}}$$

and, similarly,

$$|f_2^2-g_2^2| \leqslant B_8 n^{\frac{1}{8}} q^{-\frac{1}{4}}(n^{-1}+|\vartheta|)^{-\frac{1}{4}},$$

so that

$$|f_1^7 f_2^2-g_1^7 g_2^2| \leqslant |f_1^7-g_1^7||f_2^2|+|g_1^7||f_2^2-g_2^2| \tag{3.37}$$

$$\leqslant B_9 n^{\frac{1}{8}} q^{-2}(n^{-1}+|\vartheta|)^{-2}.$$

Now, by 3.0,

$$(3.38)\qquad |f_3|<n^{\frac{5}{16}},\quad |f_4|<n^{\frac{5}{64}},\quad |f_5|<n^{\frac{27}{256}},\quad |f_6|<n^{\frac{53}{128}}.$$

Hence, by (3.37),

$$(3.39)\quad |f_8-g_1^{\,7}g_2^{\,2}f_6^{\,2}|=|f_6^{\,2}||f_1^{\,7}f_2^{\,2}-g_1^{\,7}g_2^{\,2}|\leqq B_9 n^{\frac{77}{64}}q^{-2}(n^{-1}+|\vartheta|)^{-2}.$$

3.4. It follows from (1.2) and (2.41) that

$$\left|f(\theta,w)-f\left(\frac{h}{q},w\right)\right|\leqq\sum_{v<w}\left|e^{2\pi iv^4\theta}-e^{2\pi iv^4h/q}\right|$$

$$\leqq 2\pi\left|\theta-\frac{h}{q}\right|\sum_{v<w}v^4<2\pi|\vartheta|w^5.$$

Hence, by 3.0,

$$(3.40)\quad \left|f_3(\theta)-f_3\left(\frac{h}{q}\right)\right|<8|\vartheta|n^{\frac{5}{16}+\frac{3}{4}},\quad \left|f_4(\theta)-f_4\left(\frac{h}{q}\right)\right|<8|\vartheta|n^{\frac{3}{64}+\frac{5}{16}},$$

$$\left|f_5(\theta)-f_5\left(\frac{h}{q}\right)\right|<8|\vartheta|n^{\frac{27}{256}+\frac{27}{64}}.$$

With the abbreviations

$$f_{v,1}=f_v(\theta),\quad f_{v,2}=f_v\left(\frac{h}{q}\right),\quad \Delta_v=f_{v,1}-f_{v,2}\quad (v=3,\,4,\,5,\,6),$$

we have, by 3.0,

$$\Delta_6=f_{3,1}f_{4,1}f_{5,1}^2-f_{3,2}f_{4,2}f_{5,2}^2$$

$$=f_{3,1}f_{4,1}(f_{5,1}+f_{5,2})\Delta_5+f_{3,1}\Delta_4 f_{5,2}^2+\Delta_3 f_{4,2}f_{5,2}^2.$$

From this and (3.38) and (3.40) we obtain

$$|\Delta_6|<8|\vartheta|n^{\frac{53}{128}}(2n^{\frac{27}{64}}+n^{\frac{5}{16}}+n^{\frac{3}{4}})<32|\vartheta|n^{\frac{53}{128}+\frac{3}{4}},$$

and hence, by (2.81) and (3.21),

$$\left|f_6(\theta)-f_6\left(\frac{h}{q}\right)\right|<32n^{\frac{53}{128}}.$$

From this and (3.38) it follows that

$$\left|f_6^{\,2}(\theta)-f_6^{\,2}\left(\frac{h}{q}\right)\right|<64n^{\frac{53+69}{128}}=64n^{\frac{61}{64}},$$

and from this and (3.35) that

$$\left|g_1^{\,7}g_2^{\,2}\left\{f_6^{\,2}-f_6^{\,2}\left(\frac{h}{q}\right)\right\}\right|\leqq B_{10}n^{\frac{61}{64}}q^{-\frac{3}{4}}(n^{-1}+|\vartheta|)^{-\frac{3}{4}}\leqq B_{10}n^{\frac{77}{64}}q^{-2}(n^{-1}+|\vartheta|)^{-2}.$$

Hence, by (3.39),

$$\left| f_8 - g_1^7 g_2^2 f_6^2\left(\frac{h}{q}\right)\right| \leqslant B_{11} n^{\frac{77}{84}} q^{-2}(n^{-1}+|\vartheta|)^{-2}. \tag{3.41}$$

Now, by (3.33) and (3.32),

$$|g_1 - g_2| \leqslant B_1 B_3 q^{-\frac{1}{2}} n^{\frac{1}{4}-\delta},$$

so that, by (3.35),

$$|g_1^7(g_1^2 - g_2^2)| \leqslant (2B_1 B_3)^9 q^{-\frac{9}{4}} n^{\frac{1}{4}-\delta}(n^{-1}+|\vartheta|)^{-2},$$

and hence, by (3.38),

$$\left|(g_1^7 g_2^2 - g_1^9) f_6^2\left(\frac{h}{q}\right)\right| \leqslant B_{12} q^{-\frac{9}{4}} n^{\frac{35}{84}-\delta}(n^{-1}+|\vartheta|)^{-2}. \tag{3.42}$$

Now it follows from (2.81) and (3.21) that $n^{\frac{77}{84}} q^{\frac{1}{4}} \leqslant n^{\frac{79}{84}}$. Hence, by (3.41), (3.42), and (3.00),

$$\left| f_8(\theta) - g_1^9(\vartheta) f_6^2\left(\frac{h}{q}\right)\right| \leqslant B_{13} q^{-\frac{9}{4}} n^{\alpha-1-\delta}(n^{-1}+|\vartheta|)^{-2}. \tag{3.43}$$

Putting

$$J_1(h, q) = \int_{I_{h,q}} f_8(\theta) e^{-2\pi i \theta n} d\theta, \quad J_2(h, q) = \int_{-\vartheta_0}^{\vartheta_0} g_1^9(\vartheta) e^{-2\pi i \vartheta n} d\vartheta, \tag{3.44}$$

where

$$\vartheta_0 = q^{-1} u^{-\frac{7}{2}}, \tag{3.45}$$

we have, by (3.23) and (1.1),

$$J_1(h, q) = \xi_q^{-hn} \int_{-\vartheta_0}^{\vartheta_0} f_8\left(\frac{h}{q}+\vartheta\right) e^{-2\pi i \vartheta n} d\vartheta,$$

and hence, by (2.41) and (3.43),

$$\left| J_1(h, q) - \xi_q^{-hn} f_6^2\left(\frac{h}{q}\right) J_2(h, q)\right| \leqslant 2B_{13} q^{-\frac{9}{4}} n^{\alpha-\delta}. \tag{3.46}$$

3.5. We now drop the assumptions (2.41), (2.61), and (2.81), but note that (3.35) remains valid.

Let

$$J_3(h, q) = \int_{-\infty}^{\infty} g_1^9(\vartheta)\, e^{-2\pi i \vartheta n}\, d\vartheta. \tag{3.50}$$

Then, by (3.44), (3.45), (3.35), and (3.21),

$$|J_3(h, q) - J_2(h, q)| \leqslant B_{14} \int_{\vartheta_0}^{\infty} (q\vartheta)^{-\frac{9}{4}}\, d\vartheta \leqslant B_{14}\, n^{\frac{35}{112}} q^{-1}. \tag{3.51}$$

Now, by (3.50) and (3.33),

$$J_3(h, q) = S_{h,q}^9\, q^{-9}\, n^{-1} \int_{-\infty}^{\infty} g^9(xn^{-1}, n^{\frac{1}{4}})\, e^{-2\pi i x}\, dx. \tag{3.52}$$

Also, by (1.3),

$$g(xn^{-1}, n^{\frac{1}{4}}) = \int_0^{n^{\frac{1}{4}}} e^{2\pi i x n^{-1} y^4}\, dy = n^{\frac{1}{4}} \psi(x), \tag{3.53}$$

where

$$\psi(x) = \int_0^1 e^{2\pi i x t^4}\, dt. \tag{3.54}$$

By (3.52) and (3.53),

$$J_3(h, q) = L n^{\frac{5}{4}} S_{h,q}^9\, q^{-9}, \tag{3.55}$$

where

$$L = \int_{-\infty}^{\infty} \psi^9(x)\, e^{-2\pi i x}\, dx. \tag{3.56}$$

Put

$$T_{h,q} = \xi_q^{-hn} f_6^2 \left(\frac{h}{q}\right) S_{h,q}^9\, q^{-9}, \quad D_q = \sum_{\substack{0<h\leqslant q \\ (h,q)=1}} T_{h,q}. \tag{3.57}$$

Then, by (3.51), (3.55), and (3.38),

$$\left| \xi_q^{-hn} f_6^2 \left(\frac{h}{q}\right) J_2(h, q) - L n^{\frac{5}{4}} T_{h,q} \right| \leqslant B_{14}\, n^{2\frac{11}{84}} q^{-1}.$$

Hence, by (3.46),

$$|J_1(h, q) - L n^{\frac{5}{4}} T_{h,q}| \leqslant B_{15}(n^{\alpha-\delta} q^{-\frac{9}{4}} + n^{2\frac{11}{84}} q^{-1}),$$

and hence, putting

$$N = \sum_{q\leqslant n^{\frac{1}{8}}} \sum_{\substack{0<h\leqslant q \\ (h,q)=1}} J_1(h, q), \tag{3.58}$$

we have, by (3.57) and (3.00),

$$\left| N - L n^{\frac{5}{4}} \sum_{q\leqslant n^{\frac{1}{8}}} D_q \right| \leqslant B_{16}\, n^{\alpha-\delta}. \tag{3.59}$$

3.6. It follows from (3.57), (3.38), and 2.2 that $|D_q| \leqslant B_1{}^9 n^{\frac{69}{64}} q^{-\frac{5}{4}}$. Hence

$$\left| \sum_{n^{4\delta} < q \leqslant n^4} D_q \right| \leqslant B_{17} n^{\frac{69}{64}-\delta},$$

and hence, by (3.59) and (3.00),

$$\left| N - L n^{\frac{5}{4}} \sum_{q \leqslant n^{4\delta}} D_q \right| \leqslant B_{18} n^{\alpha-\delta}. \tag{3.60}$$

Now, by (1.2), (1.1), and (1.4),

$$|f(h/q, w) - w S_{h,q} q^{-1}| \leqslant 2q.$$

Hence, by 3.0,

$$|f_3(h/q) - \tfrac{1}{2} n^{\frac{3}{16}} S_{h,q} q^{-1}| \leqslant 4q, \quad |f_4(h/q) - \tfrac{1}{2} n^{\frac{2}{64}} S_{h,q} q^{-1}| \leqslant 4q,$$

and

$$|f_5(h/q) - n^{\frac{37}{256}} S_{h,q} q^{-1}| \leqslant 2q.$$

It easily follows that, if $q \leqslant n^{4\delta}$, then

$$\left| f_6{}^2 \left(\frac{h}{q}\right) - \tfrac{1}{16} n^{\frac{69}{64}} S_{h,q}^8 q^{-8} \right| \leqslant B_{19} n^{\frac{69}{64} - \frac{37}{256} + 4\delta} \leqslant B_{19} n^{\frac{69}{64}-\delta}.$$

Hence, by (3.57), 2.2, and (1.5),

$$|T_{h,q} - \tfrac{1}{16} n^{\frac{69}{64}} \xi_q^{-nh} S_{h,q}^{17} q^{-17}| \leqslant B_{20} n^{\frac{69}{64}-\delta} q^{-\frac{9}{4}} \quad (q \leqslant n^{4\delta})$$

and

$$|D_q - \tfrac{1}{16} n^{\frac{69}{64}} A_q| \leqslant B_{20} n^{\frac{69}{64}-\delta} q^{-\frac{5}{4}} \quad (q \leqslant n^{4\delta}).$$

From this and (3.60) and (3.00) we obtain

$$\left| N - \tfrac{1}{16} L n^{\alpha} \sum_{q \leqslant n^{4\delta}} A_q \right| \leqslant B_{21} n^{\alpha-\delta}. \tag{3.61}$$

Now, by (1.5) and 2.2, $|A_q| \leqslant B_1^{17} q^{-\frac{13}{4}}$. Hence, by (1.6),

$$\left| S - \sum_{q \leqslant n^{4\delta}} A_q \right| \leqslant B_{22} n^{-9\delta},$$

and hence, by (3.61),

$$|N - \tfrac{1}{16} L n^{\alpha} S| \leqslant B_{23} n^{\alpha-\delta}. \tag{3.62}$$

Now, by (3.24), (3.21), (3.44), and (3.58),

$$a_n = N + \int_E f_8(\theta)\, e^{-2\pi i \theta n}\, d\theta,$$

and hence, by (3.26) and (3.62),

$$|a_n - \tfrac{1}{16} L n^{\alpha} S| \leqslant C_{15} n^{\alpha-\delta+3\epsilon}. \tag{3.63}$$

Taking, in particular, $\epsilon = 2^{-11}$, we obtain

$$|a_n - \tfrac{1}{16} L n^{\alpha} S| \leqslant B_{24} n^{\alpha - 2^{-11}}. \tag{3.64}$$

Now it was proved by Hardy and Littlewood (**3**, Theorems 12 and 15) that $S \geqslant B_{25}^{-1}$, and by Landau (**2**, p. 338) that $L = \Gamma^9(\frac{5}{4})/\Gamma(\frac{9}{4}) > 0$. Hence, by (3.64), $a_n \neq 0$ for any sufficiently large n.

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ON WARING'S PROBLEM FOR FOURTH POWERS

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Introduction.

The object of this paper is to give a proof that every large positive integer is representable as the sum of seventeen fourth powers. This result has also been proved independently by Dr. Estermann†.

The fact that every large number is representable as the sum of nineteen fourth powers was established by Hardy and Littlewood‡. We make in this paper a straightforward application of the Hardy-Littlewood method as modified recently by Vinogradov§.

It is well known that there are an infinity of numbers not representable as the sum of less than sixteen fourth powers.

Notation.

Let N be any sufficiently large integer, and let $P = \frac{1}{2}N^{\frac{1}{4}}$. Let

$$T(\alpha, P) = \sum_{P<n<2P} e(n^4\alpha) \quad [e(x) = e^{2\pi i x}],$$

$$T_i(\alpha) = T(\alpha, 2^{-i}P^{(\frac{1}{4})^i}) \quad (i = 0, 1, 2, 3),$$

$$Q(\alpha) = T_1(\alpha)\,T_2(\alpha)\,T_3^2(\alpha) = \sum_n r_4'(n)\,e(n\alpha),$$

$$R(\alpha) = T_0(\alpha)\,Q(\alpha) = \sum_n r_5'(n)\,e(n\alpha),$$

$$T_0^9(\alpha)\,Q^2(\alpha) = \sum_n r_{17}'(n)\,e(n\alpha).$$

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† See *Proc. London Math. Soc.* (2), 41 (1936), 126–142. The two proofs were obtained at about the same time.

‡ "Further researches in Waring's problem", *Math. Zeitschrift*, 23 (1925), 1–37.

§ See, for example, *Annals of Math.*, 36 (1935), 395–405. For a simplified account see Heilbronn, *Acta Arithmetica*, 1 (1935), Part 2.

We divide the interval $0 \leqslant \alpha \leqslant 1$ in the usual way into Farey arcs belonging to all rational points a/q with $1 \leqslant q \leqslant P^3$, $0 \leqslant a < q$, $(a, q) = 1$. We divide these arcs into major arcs $\mathfrak{M}$ $(1 \leqslant q \leqslant P)$ and minor arcs $\mathfrak{m}$ $(P < q \leqslant P^3)$. In either case the arc has the form

$$\alpha = \frac{a}{q} + \beta, \quad -\vartheta_1 q^{-1} P^{-3} \leqslant \beta \leqslant \vartheta_2 q^{-1} P^{-3},$$

where $$\tfrac{1}{2} \leqslant \vartheta_i \leqslant 1.$$

It is convenient to divide each major arc into three parts, as follows:

(I) $$|\beta| \leqslant \tfrac{1}{2} P^{-4},$$

(II) $$\tfrac{1}{2} P^{-4} \leqslant |\beta| \leqslant \tfrac{1}{2} q^{-\frac{4}{5}} P^{-\frac{16}{5}},$$

(III) $$|\beta| \geqslant \tfrac{1}{2} q^{-\frac{4}{5}} P^{-\frac{16}{5}}.$$

This is possible, since

$$0 < \tfrac{1}{2} P^{-4} \leqslant \tfrac{1}{2} q^{-\frac{4}{5}} P^{-\frac{16}{5}} \leqslant \vartheta_i q^{-1} P^{-3}.$$

We use the classical notation:

$$S_{a,q} = \sum_{v=1}^{q} e\left(v^4 \frac{a}{q}\right),$$

$$\mathfrak{S}_9(n) = \sum_{q=1}^{\infty} q^{-9} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\left(-n \frac{a}{q}\right) S_{a,q}^9.$$

Let $$\psi_{a,q}(\alpha) = \tfrac{1}{4} \frac{S_{a,q}}{q} \sum_{P^4 < j < (2P)^4} \frac{\Gamma(j + \frac{1}{4})}{j!} e(j\beta).$$

We use ϵ to denote an arbitrarily small positive number, and $c_1, c_2, \ldots$ to denote positive absolute constants.

In order to avoid the repetition of many well-known arguments, we make frequent references to Gelbcke† and Landau‡.

Minor arcs.

LEMMA 1. $$r_5'(n) = O(P^\epsilon).$$

Proof. If $n = x_0^4 + x_1^4 + x_2^4 + x_3^4 + x_3'^4$, where

$$2^{-v} P^{(\frac{3}{4})^v} < x_v < 2^{1-v} P^{(\frac{3}{4})^v},$$

† *Math. Annalen*, 105 (1931), 637–652. Our definitions are not identical with those of Gelbcke, but results for which a reference to Gelbcke is given follow by the same arguments as those used there.

‡ *Vorlesungen über Zahlentheorie* (1927), 1.

the values of x_0, x_1, x_2 are unique. For, if v is the first of the suffixes 0, 1, 2 for which $x_v \neq x_v^*$, then

$$|x_v^4 - x_v^{*4}| > 4 \cdot 16^{-v} P^{3(\frac{3}{4})^v} = 4(2^{1-(v+1)} P^{(\frac{3}{4})^{v+1}})^4 \geqslant x_{v+1}^4 + \ldots + x_3'^4.$$

Also, the number of representations of $n - x_0^4 - x_1^4 - x_2^4$ as the sum of two fourth powers is $O(P^\epsilon)$.

LEMMA 2. $\sum_{\mathfrak{m}} \int_{\mathfrak{m}} |T_0^7(\alpha) R^2(\alpha)| d\alpha = O\{P^{5-\frac{1}{32}+\epsilon} Q^2(0)\}$.

Proof. By Gelbcke, Satz 10,

$$T_0(\alpha) = O(P^{\frac{7}{8}+\epsilon}).$$

Hence
$$\sum_{\mathfrak{m}} \int_{\mathfrak{m}} |T_0^7(\alpha) R^2(\alpha)| d\alpha = O\left(P^{7 \cdot \frac{7}{8}+7\epsilon} \int_0^1 |R(\alpha)|^2 d\alpha\right)$$

$$= O\left(P^{6+\frac{1}{8}+7\epsilon} \sum_n [r_5'(n)]^2\right)$$

$$= O\left(P^{6+\frac{1}{8}+8\epsilon} \sum_n r_5'(n)\right)$$

$$= O\{P^{7+\frac{1}{8}+8\epsilon} Q(0)\}.$$

Also
$$Q(0) > c_1 P^{\frac{3}{4}+(\frac{3}{4})^2+2(\frac{3}{4})^3} = c_1 P^{2+\frac{1}{8}+\frac{1}{32}};$$

hence the result.

Major arcs.

LEMMA 3. *For all a, q, and β, provided that $|\beta| \leqslant \frac{1}{2}$, we have*

$$\psi_{a,q}(\alpha) = O\{q^{-\frac{1}{4}} \min(P, |\beta|^{-\frac{1}{4}})\}.$$

Proof. Gelbcke, Satz 8, with the classical result

$$S_{a,q} = O(q^{\frac{3}{4}})$$

(Landau, Satz 315), in place of Gelbcke's Satz 5.

LEMMA 4. *For all a, q, and β, provided that $|\beta| \leqslant \frac{1}{2}$, we have*

$$T(a, P) - \psi_{a,q}(\alpha) = O\{q^{1-\frac{1}{4}+\epsilon}(1+|\beta| P^4)\}.$$

Proof. Gelbcke, Satz 12, with the inequality

$$\sum_{h=0}^{m} e\left(h^4 \frac{a}{q}\right) = O(q^{\frac{3}{4}+\epsilon})$$

for $0 \leqslant m \leqslant q$, in place of Gelbcke's Satz 5. A proof of this inequality, based on well-known arguments, will be published shortly†.

LEMMA 5. *If* $q \leqslant P$ *and* $|\beta| \leqslant P^{-4}$, *then*

$$T(\alpha, P) = O\{q^{-\frac{1}{4}+\epsilon} T(0, P)\}.$$

Proof. By Lemmas 3, 4,

$$T(\alpha, P) = O(q^{-\frac{1}{4}} P) + O(q^{1-\frac{1}{4}+\epsilon}) = O\{q^{-\frac{1}{4}+\epsilon} T(0, P)\}.$$

LEMMA 6. *On* $\mathfrak{M}$, *provided that* $q \leqslant P^{\frac{1}{2}}$, *we have*

$$Q^2(\alpha) = O\{q^{-1+\epsilon} Q^2(0)\}.$$

Proof. Since

$$q^{-1} P^{-3} < (2^{-1} P^{\frac{3}{4}})^{-4} < (2^{-2} P^{(\frac{3}{4})^2})^{-4},$$

and

$$q \leqslant P^{\frac{1}{2}} < P^{(\frac{3}{4})^2} < P^{\frac{3}{4}},$$

we have, by Lemma 5,

$$T_i(\alpha) = O\{q^{-\frac{1}{4}+\epsilon} T_i(0)\}$$

for $i = 1, 2$, which establishes Lemma 6.

LEMMA 7. *On* (I),

$$T_0^9(\alpha) - \psi_{a,q}^9(\alpha) = O(q^{-1-\frac{1}{4}+\epsilon} P^8).$$

Proof. Lemmas 3 and 4, since $q^{1-\frac{1}{4}} \leqslant q^{-\frac{1}{4}} P$.

LEMMA 8. *On* (II),

$$T_0^9(\alpha) - \psi_{a,q}^9(\alpha) = O(q^{-1-\frac{1}{4}+\epsilon} P^4 |\beta|^{-1}).$$

Proof. Lemmas 3 and 4, since on (II)

$$q^{1-\frac{1}{4}} |\beta| P^4 \leqslant q^{-\frac{1}{4}} |\beta|^{-\frac{1}{4}}.$$

LEMMA 9. *On* (III),

$$T_0^9(\alpha) - \psi_{a,q}^9(\alpha) = O(q^{-\frac{2}{5}} P^{4+\frac{1}{2}+\epsilon} |\beta|^{-\frac{2}{5}}).$$

Proof. By Gelbcke, Satz 14,

$$T_0(\alpha) = O(q^{-\frac{1}{5}} P^{\frac{1}{2}+\epsilon} |\beta|^{-\frac{1}{5}}),$$

and, by Lemma 3,

$$\psi_{a,q}(\alpha) = O(q^{-\frac{1}{4}} |\beta|^{-\frac{1}{4}}) = O(q^{-\frac{1}{5}} P^{\frac{1}{2}+\epsilon} |\beta|^{-\frac{1}{5}}).$$

† For primes q, an equivalent inequality was given by Hardy and Littlewood, *Quart. J. of Math.*, 48 (1920), 277, foot-note.

LEMMA 10.

$$\Sigma'_{\mathfrak{M}}\int_{\mathfrak{M}} |T_0^9(\alpha)-\psi^9_{a,q}(\alpha)||Q(\alpha)|^2\,d\alpha = O\{P^{5-\frac{1}{10}+\epsilon}Q^2(0)\},$$

where the summation is over the major arcs with $q \leqslant P^{\frac{1}{2}}$.

Proof. It is sufficient to prove that

$$\int_{\mathfrak{M}} |T_0^9(\alpha)-\psi^9_{a,q}(\alpha)|\,d\alpha = O(q^{-1}P^{5-\frac{1}{10}+\epsilon}),$$

since then, by Lemma 6, the sum in question is

$$O\Big(\Sigma'_{\mathfrak{M}} q^{-1}P^{5-\frac{1}{10}+\epsilon}q^{-1+\epsilon}Q(0)^2\Big) = O\Big(\sum_{q\leqslant P^{\frac{1}{2}}} q.q^{-2+\epsilon}P^{5-\frac{1}{10}+\epsilon}Q(0)^2\Big).$$

By Lemma 7,

$$\int_I |T_0^9-\psi^9|\,d\alpha = O(q^{-1-\frac{1}{4}+\epsilon}P^8P^{-4}) = O(q^{-1}P^{5-\frac{1}{10}}).$$

By Lemma 8,

$$\int_{II} |T_0^9-\psi^9|\,d\alpha = O\Big(q^{-1-\frac{1}{4}+\epsilon}P^4\int_{\frac{1}{2}P^{-1}}^{\frac{1}{2}}\beta^{-1}\,d\beta\Big)$$
$$= O(q^{-1}P^{5-\frac{1}{10}}).$$

By Lemma 9,

$$\int_{III} |T_0^9-\psi^9|\,d\alpha = O\Big(q^{-\frac{9}{8}}P^{4+\frac{1}{2}+\epsilon}\int_{\frac{1}{2}q^{-\frac{4}{5}}P^{-\frac{16}{5}}}^{\infty}\beta^{-\frac{9}{8}}\,d\beta\Big)$$
$$= O(q^{-\frac{9}{8}}P^{4+\frac{1}{2}+\epsilon}q^{\frac{1}{10}}P^{\frac{9}{8}})$$
$$= O(q^{-1}P^{5-\frac{1}{10}+\epsilon}).$$

LEMMA 11. *If* $\bar{\mathfrak{M}}_{a,q}$ *denotes the part of the interval* (0, 1) *not belonging to* $\mathfrak{M}_{a,q}$, *then*

$$\sum_{q\leqslant P^{\frac{1}{2}}}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{\bar{\mathfrak{M}}_{a,q}} |\psi_{a,q}(\alpha)|^9|\,Q(\alpha)|^2\,d\alpha = O\{P^{5-\frac{3}{4}}Q^2(0)\}.$$

Proof. By Lemma 3,

$$\int_{\bar{\mathfrak{M}}_{a,q}} |\psi_{a,q}(\alpha)|^9\,d\alpha = O\Big(\int_{\frac{1}{2}q^{-1}P^{-3}}^{\infty} q^{-\frac{3}{2}}\beta^{-\frac{3}{2}}\,d\beta\Big)$$
$$= O(q^{-\frac{3}{4}+\frac{5}{4}}P^{\frac{15}{4}})$$
$$= O(q^{-1}P^{4-\frac{1}{4}}).$$

Hence

$$\sum_{q\leqslant P^{\frac{1}{2}}}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{\mathfrak{M}_{a,q}}|\psi_{a,q}(\alpha)|^9\,|Q(\alpha)|^2\,d\alpha = O\Big(\sum_{q\leqslant P^{\frac{1}{2}}} q.q^{-1}P^{4-\frac{1}{4}}Q^2(0)\Big),$$

giving the result.

LEMMA 12.

$$\sum_{\mathfrak{M}}{}''\int_{\mathfrak{M}}|T_0{}^7(\alpha)\,R^2(\alpha)|\,d\alpha = O\{P^{5-\frac{1}{32}+\epsilon}Q^2(0)\},$$

where the summation is over the major arcs with $q > P^{\frac{1}{2}}$.

Proof. By Lemmas 3 and 4 we have, for $P^{\frac{1}{2}} < q \leqslant P$,

$$\begin{aligned}T_0(\alpha) &= O(q^{-\frac{1}{4}}P)+O\{q^{\frac{3}{4}+\epsilon}(1+q^{-1}P)\}\\ &= O(q^{-\frac{1}{4}+\epsilon}P) = O(P^{\frac{7}{8}+\epsilon}).\end{aligned}$$

The result now follows exactly as in the proof of Lemma 2.

The singular series.

LEMMA 13.

$$\int_0^1 e(-n\alpha)\,\psi_{a,q}^9(\alpha)\,d\alpha = \Big(\frac{S_{a,q}}{q}\Big)^9 e\Big(-\frac{na}{q}\Big)\,F(n),$$

where, for $10P^4 < n < 100P^4$,

$$c_2 P^5 < F(n) < c_3 P^5.$$

Proof. The integral in question is

$$(\tfrac{1}{4})^9\Big(\frac{S_{a,q}}{q}\Big)^9 e\Big(-\frac{na}{q}\Big)\sum_{\substack{j_1+\ldots+j_9=n\\P^4<j_i<(2P)^4}}\frac{\Gamma(j_1+\frac{1}{4})\ldots\Gamma(j_9+\frac{1}{4})}{j_1!\ldots j_9!}.$$

Since $$c_4 j^{-\frac{3}{4}}j! < \Gamma(j+\tfrac{1}{4}) < c_5 j^{-\frac{3}{4}}j!,$$

we have

$$c_6(P^{-3})^9\sum_{\substack{j_1+\ldots+j_9=n\\P^4<j_i<(2P)^4}}1 < F(n) < c_7(P^{-3})^9\sum_{\substack{j_1+\ldots+j_9=n\\P^4<j_i<(2P)^4}}1.$$

Hence the result, since $10P^4 < n < 100P^4$, and so

$$c_8(P^4)^8 < \sum_{\substack{j_1+\ldots+j_9=n\\P^4<j_i<(2P)^4}}1 < c_9(P^4)^8.$$

LEMMA 14. $\mathfrak{S}_9(n) \geqslant 0$ *for all* n, *and* $\mathfrak{S}_9(n) > c_{10}$ *provided that* n *is congruent to one of* 1, 2, ..., 9 (mod 16).

Proof. In Landau's notation,

$$\mathfrak{S}_9(n) = \prod_p \chi_p,$$

where
$$\chi_p = \sum_{\nu=0}^{\infty} p^{-9\nu} \sum_{\substack{a=1 \\ p \nmid a}}^{p^\nu - 1} e\left(-\frac{na}{p^\nu}\right) (S_{a,p^\nu})^9,$$

the product being absolutely convergent. For $p > c_{11}$,

$$\chi_p > 1 - p^{-\frac{5}{4}}$$

(Landau, Satz 324).

For $p > 2$, we have, by (349) of Landau†,

$$\chi_p \geqslant \begin{cases} p^{-8} N(p, 0) & \text{if } p^4 \mid n, \\ p^{-8} N(p, n) & \text{if } p^4 \nmid n, \end{cases}$$

where, by Satz 301 (2), $N(p, n) \geqslant 1$ and $N(p, 0) \geqslant 1$ since

$$9 \geqslant r + 1 = (4, p-1) + 1.$$

For $p = 2$, provided that $16 \nmid n$, we have, by the same equation,

$$\chi_2 = 16^{-8} N(16, n).$$

Now, if n is congruent to one of 1, 2, ..., 9 (mod 16), $N(16, n) \geqslant 1$, and so $\chi_2 \geqslant 16^{-8}$.

By equation (349) of Landau, $\chi_p \geqslant 0$ for all p and n, and if n is congruent to one of 1, 2, ..., 9 (mod 16),

$$\mathfrak{S}_9(n) > 16^{-8} \prod_{p \leqslant c_{11}} p^{-8} \prod_{p > c_{11}} (1 - p^{-\frac{5}{4}}) = c_{10}.$$

Proof of the theorem.

THEOREM. *$r'_{17}(N) > 0$ for $N > c_{12}$.*

Proof. By Lemmas 2, 10, 11, 12

$$r'_{17}(N) = \int_0^1 e(-Na)\, T_0{}^9(a)\, Q^2(a)\, da$$

$$= \sum_{\mathfrak{M}}' \int_0^1 e(-Na)\, \psi_{a,q}^9(a)\, Q^2(a)\, da + O\{P^{5-\frac{1}{2}+\epsilon} Q^2(0)\}.$$

† $N(p^l, m)$ is the number of solutions of the congruence $x_1{}^4 + \ldots + x_9{}^4 \equiv m \pmod{p^l}$ with not all of $x_1, \ldots, x_9$ divisible by p.

The sum on the right is

$$\sum_{n_1}\sum_{n_2} r_4'(n_1)\, r_4'(n_2) \sum_{\mathfrak{M}}' \int_0^1 e\{-(N-n_1-n_2)\,\alpha\}\, \psi_{a,q}^9(\alpha)\, d\alpha$$

$$= \sum_{n_1}\sum_{n_2} r_4'(n_1)\, r_4'(n_2)\, F(N-n_1-n_2) \sum_{q\leqslant P^{\frac{1}{2}}} \sum_{\substack{a=1\\(a,q)=1}}^{q} (q^{-1}S_{a,q})^9\, e\left(-(N-n_1-n_2)\frac{a}{q}\right)$$

by Lemma 13. Now

$$\sum_{q\leqslant P^{\frac{1}{2}}} \sum_{\substack{a=1\\(a,q)=1}}^{q} (q^{-1}S_{a,q})^9\, e\left(-\frac{na}{q}\right) = \mathfrak{S}_9(n) + O\left(\sum_{q>P^{\frac{1}{2}}} q(q^{-\frac{1}{4}})^9\right)$$

$$= \mathfrak{S}_9(n) + O(P^{-\frac{1}{8}}).$$

The condition $10P^4 < N-n_1-n_2 < 100P^4$ of Lemma 13 is satisfied, if $r_4'(n_1) > 0$, $r_4'(n_2) > 0$. Hence we have

$$r_{17}'(N) = \sum_{n_1}\sum_{n_2} r_4'(n_1)\, r_4'(n_2)\, F(N-n_1-n_2)\, \mathfrak{S}_9(N-n_1-n_2) + O\{P^{5-\frac{1}{32}+\epsilon}\, Q^2(0)\}.$$

We define t to be 0 if N is congruent to one of 1, 2, ..., 8 (mod 16), and to be 1 if N is congruent to one of 9, 10, ..., 16 (mod 16). If $n_1 \equiv n_2 \equiv 4t \pmod{16}$, $N-n_1-n_2$ is congruent to one of 1, 2, ..., 8 (mod 16). Also, clearly,

$$\sum_{n_1\equiv 4t\,(16)} r_4'(n_1) > c_{13} \sum_{n_1} r_4'(n_1) = c_{13}\, Q(0).$$

Hence

$$r_{17}'(N) > c_2 P^5 c_{10} \left(\sum_{n_1\equiv 4t} r_4'(n_1)\right)^2 + O\{P^{5-\frac{1}{32}+\epsilon}\, Q^2(0)\}$$

$$> c_2 P^5 c_{10} c_{13}^2\, Q^2(0) + O\{P^{5-\frac{1}{32}+\epsilon}\, Q^2(0)\}$$

$$> 0.$$

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A LINEAR TRANSFORMATION CONNECTED WITH THE RIEMANN ZETA FUNCTION

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Introduction.

In this paper we consider the linear transformation

$$N(f) = \text{l.i.m.}_{n\to\infty} \left(\sum_{\nu=1}^{n} f\{\nu x \sqrt{(2\pi)}\} - \int_0^n f\{ux\sqrt{(2\pi)}\}\,du \right).$$

The domain $\mathfrak{D}_N$ of this transformation consists of all functions $f(x)$ such that $f(x)$ and $N\{f(x)\}$ both belong to $L_2(0, \infty)$.

We define
$$e_X = e_X(x) = \begin{cases} 1 & (0 \leqslant x < X), \\ 0 & (X \leqslant x), \end{cases}$$

then

$$N(e_X) = \text{l.i.m.}_{n\to\infty} \left\{ \min\left(\left[\frac{X}{x\sqrt{(2\pi)}} \right], \ n \right) - \min\left(\frac{X}{x\sqrt{(2\pi)}}, \ n \right) \right\}$$

$$= \left(\frac{X}{x\sqrt{(2\pi)}} \right),$$

i.e. the fractional part of $X/\{x\sqrt{(2\pi)}\}$.

Thus e_X is in $\mathfrak{D}_N$. Hence all step functions $\sum_{\nu=1}^{n} c_\nu e_{X_\nu}(x)$ are in $\mathfrak{D}_N$. Thus $\mathfrak{D}_N$ is everywhere dense in $L_2(0, \infty)$. On the other hand, $\mathfrak{D}_N \subset L_2(0, \infty)$, since $N(f)$ is an unbounded transformation.

A linear transformation is said to be closed if, whenever $\text{l.i.m.}_{n\to\infty} f_n = f$ and $\text{l.i.m.}_{n\to\infty} N(f_n) = g$, then $N(f) = g$. Now

$$\zeta_n(\tfrac{1}{2} - i\lambda) = \sum_{\nu=1}^{n} \frac{1}{\nu^{\frac{1}{2}-i\lambda}} - \int_0^n \frac{du}{u^{\frac{1}{2}-i\lambda}} \to \zeta(\tfrac{1}{2} - i\lambda) \quad \text{as} \quad n \to \infty.$$

A necessary and sufficient condition for the closure of $N(f)$ is the existence of a constant K such that

$$(A) \qquad |\zeta_n(\tfrac{1}{2}-i\lambda)| \leqslant K\{1+|\zeta(\tfrac{1}{2}-i\lambda)|\}$$

for all n and all λ.

We investigate the transformation $N(f)$ by means of the associated set of orthogonal differentials†. A set of functions $f_\lambda(x)$, defined for x in a measurable set E and for $-\infty < \lambda < \infty$, is said to form a set of orthogonal differentials‡

$$\Delta f_\lambda(x) = f_{\lambda_2}(x) - f_{\lambda_1}(x) \quad (\lambda_2 > \lambda_1),$$

if, for all finite intervals Δ_1 and Δ_2, the Lebesgue integral

$$\int_E \Delta_1 f_\lambda(x)\, \overline{\Delta_2 f_\lambda(x)}\, dx = \Delta_{12}\rho(\lambda), \quad \rho(\lambda) \text{ real},$$

where Δ_{12} is the interval common to Δ_1 and Δ_2. The set $\Delta f_\lambda(x)$ is called complete if there is no function belonging to $L_2(0, \infty)$ which is orthogonal to all the functions $\Delta f_\lambda(x)$.

In paragraph I we give the theory of orthogonal differentials and linear transformations as far as it is required here. This theory is well known†; it is included so as to exhibit it in a form suitable to our requirements. The connection between the transformation $N(f)$ and a complete set of orthogonal differentials is given by the equation

$$N(\Delta f_\lambda) = \int_\Delta \zeta(\tfrac{1}{2}-i\lambda)(2\pi)^{-\frac{1}{2}+\frac{1}{2}i\lambda} \frac{df_\lambda(x)}{d\lambda}\, d\lambda, \quad \text{where} \quad f_\lambda(x) = \frac{x^{-\frac{1}{2}+i\lambda}}{\log x}.$$

From this we then deduce that

$$N(f) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \zeta(\tfrac{1}{2}-i\lambda)\{x\sqrt{(2\pi)}\}^{-\frac{1}{2}+i\lambda} \int_0^\infty f(y)\, y^{-\frac{1}{2}-i\lambda}\, dy\, d\lambda,$$

whenever $N(f)$ exists, *i.e.* for all f in $\mathfrak{D}_N$. In the double integral the integrations are to be interpreted as $\underset{H\to\infty}{\text{l.i.m.}} \int_{-H}^{H}$ and $\underset{n\to\infty}{\text{l.i.m.}} \int_{1/n}^{n}$ respectively.

† Hellinger, *Journal für Math.*, 136 (1909), 210–271; Weyl, *Math. Annalen*, 66 (1909), 273–324, and 68 (1910), 220–269. See also Stone, "Linear transformations in Hilbert space", *American Math. Soc. Publications*, 15 (1932).

‡ Δ always refers to an interval and is used only in connection with λ. Henceforth f and g with a suffix always refer to orthogonal differential functions. Suffixes to other letters are defined as they arise, *e.g.* e_λ, h_n, etc. Δ_λ is the interval $(0, \lambda)$, *i.e.* $\Delta_\lambda f_\lambda = f_\lambda - f_0$ for all $\lambda \gtrless 0$. $\Delta_\nu f_\lambda = f_{\lambda_{\nu+1}} - f_{\lambda_\nu}$ $(\lambda_{\nu+1} > \lambda_\nu)$.

If a transformation $N(f)$ has a closed linear extension, then the least such closed linear extension is defined as follows: if $h_n = h_n(x)$ is a sequence of functions and if $\underset{n\to\infty}{\text{l.i.m.}}\ h_n = h(x)$ and if $\underset{n\to\infty}{\text{l.i.m.}}\ N(h_n) = g(x)$, then we define $\tilde{N}(h) = g(x)$. $\tilde{N}(f)$ is a closed linear transformation. We show that

$$\tilde{N}(f) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \zeta(\tfrac{1}{2}-i\lambda)\{x\sqrt{(2\pi)}\}^{-\frac{1}{2}+i\lambda}\int_0^{\rho} f(y)\, y^{-\frac{1}{2}-i\lambda}\, dy\, d\lambda,$$

and that $\mathfrak{D}_{\tilde{N}}$ is the set of all functions of $L_2(0, \infty)$ for which

$$\int_{-\infty}^{\infty} |\zeta(\tfrac{1}{2}-i\lambda)|^2 \cdot \left|\int_0^{\infty} f(y)\, y^{-\frac{1}{2}-i\lambda}\, dy\right|^2 d\lambda$$

exists. If (A) is true, $\mathfrak{D}_N = \mathfrak{D}_{\tilde{N}}$.

In paragraph IV we obtain the equation $N(f) = SNF(f)$ valid when f and Ff are in $\mathfrak{D}_N$; here

$$Sf(x) = \frac{1}{x} f\left(\frac{1}{x}\right), \qquad F\{f(x)\} = \sqrt{\left(\frac{2}{\pi}\right)}\ \underset{n\to\infty}{\text{l.i.m.}} \int_0^n f(y)\cos xy\, dy.$$

We also show that FS (and $SF \neq FS$) are permutable with N, *i.e.* that $FSN(f) = NFS(f)$ whenever f and $FS(f)$ are in $\mathfrak{D}_N$. From this we deduce that $N^k(f) = SN^k(FS)^k S(f)$†, whenever f and $(FS)^k S(f)$ are in $\mathfrak{D}_{N^k}$.

In paragraph V we investigate the connection between the transformation $N(f)$ and the hypothesis of Lindelöf‡. We show that (L) is equivalent to:

(i) $\tilde{N}^k(e_X)$ exists for all positive integers k,

(ii) $\displaystyle\int_{-\infty}^{\infty} \frac{|\zeta(\frac{1}{2}-i\lambda)|^h}{1+\lambda^2}$ exists for all $h > 0$.

We further show that (L) is implied by:

(iii) e_X is in $\mathfrak{D}_{N^k}$ for all positive integers k,

(iv) the sequence of functions $\zeta_n{}^k(\frac{1}{2}-i\lambda)$ is strongly convergent in $-\infty < \lambda < \infty$, for all positive integers k.

$$\text{(v)}\quad \lim_{n\to\infty} \zeta_{k,n}(s) = \lim_{n\to\infty}\left(\sum_{\nu=1}^{n}\frac{d_k(\nu)}{\nu^s} - \sum_{h=1}^{k}\frac{C_{k,h}}{(h-1)!}\int_0^n \frac{(\log u)^{h-1}}{u^s}\, du\right) = \zeta^k(s)$$

† $N^1 = N$, $N^k = N.N^{k-1}$.

‡ This hypothesis we denote by L. The Riemann hypothesis we denote by R.

for $\Re(s) > \frac{1}{2}$ and all positive integers k, where

$$\zeta^k(1-s) = \sum_{h=1}^{k} \frac{(-)^h}{s^h} C_{h,k} + Z_k(s),$$

$Z_k(s)$ being an integral function of s†.

We next show that, if

$$\lim_{n\to\infty} \zeta_{k,n}(\tfrac{1}{2}-i\lambda) = \zeta^k(\tfrac{1}{2}-i\lambda) \tag{B}$$

and

$$N^{(k)}(f) = \text{l.i.m.}_{n\to\infty}\left(\sum_{\nu=1}^{n} d_k(\nu) f\{\nu x(2\pi)^{\frac{1}{2}k}\} - \sum_{h=1}^{k} \frac{C_{k,h}}{(h-1)!} \int_0^n (\log u)^{h-1} f\{ux(2\pi)^{\frac{1}{2}k}\}\, du \right),$$

then $\tilde{N}^k(f) = N^{(k)}(f)$ in $\mathfrak{D}_{N^k}$. The equation $N^{(k)}(f) = SN^{(k)}(FS)^k S(f)$ then gives formulae of the type given by Voronoi‡ for $f(x) = e_X(x)$.

In paragraph VI we investigate the connection between the transformations

$$N_\alpha(f) = \text{l.i.m.}_{n\to\infty}\left(\sum_{\nu=1}^{n} \frac{f(\nu x)}{\nu^\alpha} - \int_0^n \frac{f(ux)}{u^\alpha}\, du \right) \quad (-\tfrac{1}{2} < \alpha < \tfrac{1}{2}).$$

$$M_\alpha(f) = \text{l.i.m.}_{n\to\infty}\left(\sum_{\nu=1}^{n} \frac{\mu(\nu)}{\nu^\alpha} f(\nu x) \right) \qquad (\alpha > 0),$$

and the Riemann hypothesis; in the last series, $\mu(\nu)$ denotes Möbius' function. We obtain the equation

$$M_\alpha N_\alpha(f) = N_\alpha M_\alpha(f)$$

for a restricted set of functions $f(x)$.

We further show that (R) is equivalent to:

(i) $\dfrac{\mathcal{M}_\alpha(x)}{x} = \dfrac{1}{x} \sum_{\nu=1}^{x} \dfrac{\mu(\nu)}{\nu^\alpha}$ belongs to $L_2(0, \infty)$ for all $\alpha > 0$,

(ii) $e_X(x)$ belongs to the range of $\tilde{N}_\alpha(f)$ for all $\alpha > 0$.

Further, if $\mathcal{M}_\alpha(x)/x$ belongs to $L_2(0, \infty)$, for $\alpha > 0$, then so does $\mathcal{M}_{\alpha+\delta}(x)/x$ for all $\delta > 0$, and this implies the non-existence of zeros of $\zeta(s)$ in $\Re(s) \geqslant \frac{1}{2}+\alpha$.

Finally we show that, if $\alpha > 0$, then

$$\tilde{N}(f) = \mathbf{N}(f) = \frac{d}{dx}\left[\lim_{n\to\infty} \int_a^x \left(\sum_{\nu=1}^{n} f\{y\nu\sqrt{(2\pi)}\} - \int_0^n f\{yu\sqrt{(2\pi)}\}\, du \right) dy \right].$$

† See Hardy and Littlewood, *Proc. Royal Soc.* (A), 103 (1923), 403–412, for a list of known equivalents of (L).

‡ *Journal für Math.*, 126 (1903), 241–282.

Nothing is known about the truth, falsity, etc., of the inequalities A, A', A_a, B, C, D, E; and, with the exception of (B) and (C), nothing is known about their relationships, if any, with L and/or R†. No attempt has been made to discuss the possibility of certain classes of functions existing in $\mathfrak{D}_N$ or $\mathfrak{R}_N$ (the range of N), *e.g.* continuous functions, discontinuous functions, etc., except in the case of step functions. No attempt has been made to define $\tilde{N}(f)$ by means of the theory of summability. Lastly, the metric $\int f\bar{g}$ has been used, and no other metric, *e.g.* $\lim \frac{1}{T}\int^T f\bar{g}$ in the closure B_2 of almost periodic functions, has been considered.

I. *Orthogonal differentials.*

1. Suppose that $\Delta f_\lambda(x)$ is a complete set of orthogonal differentials for x in a measurable set E. Suppose also that:

(i) $\rho(\lambda) = \lambda$,

(ii) there is a sequence of measurable sets E_n, each of finite measure, such that $E_n \subset E_{n+1} \to E$ as $n \to \infty$; and (a) $g_\lambda(x) = df_\lambda(x)/d\lambda$ exists for all x of E and belongs to $L_2(E_n)$ for every n, (b) $\delta f_\lambda(x)/\delta\lambda \to g_\lambda(x)$ uniformly in each E_n,

(iii) $g_\lambda(x)$, *qua* function of λ, belongs to $L_2(-n, n)$ for every n and for each x of E.

Actual examples of such sets and functions are $E_n = (1/n, n)$ and $g_\lambda(x) = x^{-\frac{1}{2}+i\lambda}(2\pi)^{-\frac{1}{2}}$.

2. The following notation will be used: if $f(x)$ and $g(x)$ are $L_2(E)$ and $\phi(\lambda)$, $\psi(\lambda)$ are $L_2(-\infty, \infty)$, then we write

$$\int_E f(x)\overline{g(x)}\,dx = (f, g)_1, \qquad \int_{-\infty}^{\infty} \phi(\lambda)\overline{\psi(\lambda)}\,d\lambda = (\phi, \psi)_2,$$

$$\sqrt{(f, f)_1} = \|f\|_1, \qquad \sqrt{(\phi, \phi)_2} = \|\phi\|_2;$$

if $f(x)$ and $g(x)$ are $L_2(E_n)$ and $\phi(\lambda)$, $\psi(\lambda)$ are $L_2(-n, n)$, then we write

$$\int_{E_n} f(x)\overline{g(x)}\,dx = [f, g]_n, \qquad \int_{-n}^{n} \phi(\lambda)\overline{\psi(\lambda)}\,d\lambda = \{\phi, \psi\}_n;$$

† A, A', A_a, D, and E are connected with the closure of certain transformations; L and R are related to the existence or non-existence of particular functions in the domains of these transformations; it would seem then that the two sets of hypotheses are not intimately connected.

if $f(x)$ is $L_2(E)$ and $\phi(\lambda)$ is $L_2(-\infty, \infty)$, then we show that $[f, g_\lambda]_n$, which is a function of λ, tends in the mean to a function which we call $[f, g_\lambda]$. Similarly we write l.i.m. $\{\phi, \bar{g}_\lambda\}_n = \{\phi, \bar{g}_\lambda\}$.

We also write

$$\phi(\lambda) = [f, g_\lambda], \quad \psi(\lambda) = [g, g_\lambda], \quad \chi(\lambda) = [h, g_\lambda], \quad \xi(\lambda) = [k, g_\lambda],$$

$$e_\mu(\lambda) = 0 \text{ for } \lambda > \mu, \quad e_\mu(\lambda) = 1 \text{ for } \lambda \leqslant \mu.$$

Equalities between functions are to be interpreted as holding almost everywhere.

3. *If $f(x)$ and $g(x)$ belong to $L_2(E)$, then $\phi(\lambda)$ and $\psi(\lambda)$ belong to $L_2(-\infty, \infty)$ and $(f, g)_1 = (\phi, \psi)_2$.*

We first prove

$$(f, g)_1 = \left(\frac{d}{d\lambda}(f, \Delta_\lambda f_\lambda)_1, \frac{d}{d\lambda}(g, \Delta_\lambda f_\lambda)_1\right)_2;$$

this is easily verified if $f(x)$ and $g(x)$ are of the form $\sum_{\nu=1}^{n} c_\nu \Delta_\nu f_\lambda$. Let $h_n(x)$ be a sequence of functions of this form such that $\|h_n - h\|_1 \to 0$ as $n \to \infty$; write $\alpha_n(\lambda) = d(h_n, \Delta_\lambda f_\lambda)_1/d\lambda$, then $\|h_n - h_m\|_1 = \|\alpha_n - \alpha_m\|_2$. Thus there is an $\alpha(\lambda)$ of $L_2(-\infty, \infty)$ such that $\|\alpha_n - \alpha\|_2 \to 0$ as $n \to \infty$. Further,

$$(\alpha, \Delta e_\lambda)_2 = \lim_{n\to\infty}(\alpha_n, \Delta e_\lambda)_2 = \lim_{n\to\infty}(h_n, \Delta f_\lambda)_1 = (h, \Delta f_\lambda)_1;$$

thus $\alpha(\lambda) = d(h, \Delta_\lambda f_\lambda)_1/d\lambda$. Now let $k_n(x)$ be a sequence of functions of the form $\sum_{\nu=1}^{n} c_\nu \Delta_\nu f_\lambda$ such that $\|k_n - k\|_1 \to 0$ as $n \to \infty$. Then

$$(h, k)_1 = \lim_{n\to\infty}(h_n, k_n)_1 = \lim_{n\to\infty}(\alpha_n, \beta_n)_2 = (\alpha, \beta)_2,$$

where

$$\beta_n(\lambda) = \frac{d}{d\lambda}(k_n, \Delta_\lambda f_\lambda)_1, \quad \|\beta_n - \beta\|_2 \to 0 \text{ as } n \to \infty, \quad \text{and} \quad \beta(\lambda) = \frac{d}{d\lambda}(k, \Delta_\lambda f_\lambda)_1.$$

We next show that $\phi_n(\lambda) = [f, g_\lambda]_n$ converges in the mean over $(-\infty, \infty)$.

We have

$$[f, g]_n = \left(\frac{d}{d\lambda}[f, \Delta_\lambda f_\lambda]_n, \frac{d}{d\lambda}[g, \Delta_\lambda f_\lambda]_n\right)_2 = ([f, g_\lambda]_n, [g, g_\lambda]_n)_2;$$

on taking $f(x) = g(x)$ we obtain, if $m > n$,

$$\int_{E_m - E_n} |f(x)|^2 dx = [f, f]_m - [f, f]_n = \|[f, g_\lambda]_m - [f, g_\lambda]_n\|_2,$$

so that $\|\phi_m - \phi_n\|_2 \to 0$ as $n, m \to \infty$. Hence there exists a $\phi(\lambda)$ of $L_2(-\infty, \infty)$ such that $\|\phi_n - \phi\|_2 \to 0$ as $n \to \infty$.

Lastly, we have

$$(f, g)_1 = \lim_{n\to\infty} [f, g]_n = \lim_{n\to\infty} (\phi_n, \psi_n)_2 = (\phi, \psi)_2,$$

where ψ_n is related to g in the same way as ϕ_n is related to f.

4. *There is a* (1, 1) *correspondence between* $f(x)$ *of* $L_2(E)$ *and* $\phi(\lambda)$ *of* $L_2(-\infty, \infty)$ *set up by either of the equations*

$$\phi(\lambda) = [f, g_\lambda], \quad f(x) = \{\phi, \bar{g}_\lambda\}.$$

We have already shown that, if $f(x)$ belongs to $L_2(E)$, then $\phi(\lambda) = [f, g_\lambda]$ belongs to $L_2(-\infty, \infty)$, and that $\|f\|_1 = \|\phi\|_2$; hence different $f(x)$ of $L_2(E)$ correspond to different $\phi(\lambda)$ of $L_2(-\infty, \infty)$. Let $\mathfrak{F} \subseteq L_2(-\infty, \infty)$ be the set of functions $\phi(\lambda)$ for which $\phi(\lambda) = [f, g_\lambda]$ for some $f(x)$ of $L_2(E)$. Let $\omega(\lambda)$ be any function of $L_2(-\infty, \infty)$ and consider

$$L(f) = (\phi, \omega)_2 = ([f, g_\lambda], \omega)_2.$$

Then $L(f)$ is a linear form in $f(x)$, and $|L(f)| \leqslant \sqrt{\|f\|_1 . \|\omega\|_2}$, and so $L(f) = (f, g)_1 = (\phi, \psi)_2$ for all $\phi(\lambda)$ of $\mathfrak{F}$ and for some unique $g(x)$. Now take $f(x) = \Delta f_\lambda(x)$, then $\phi(\lambda) = 1$ when λ is in Δ and $\phi(\lambda) = 0$ elsewhere. Then $\int_\Delta (\psi - \omega)\, d\lambda = 0$ for all Δ, *i.e.* $\psi(\lambda) = \omega(\lambda) = [g, g_\lambda]$; thus $\omega(\lambda)$ is in $\mathfrak{F}$, *i.e.* $\mathfrak{F} = L_2(-\infty, \infty)$. To show that $f(x) = \{\phi, \bar{g}_\lambda\}$, we observe that the functions $\int_{\delta E} \overline{g_\lambda(x)}\, dx$, where δ is the interval (x_1, x_2) on the x-axis, form a complete set of orthogonal differentials for λ in $(-\infty, \infty)$. In the equation $(f, g)_1 = (\phi, \psi)_2$ take $f(x)$ and $g(x)$ as the characteristic functions of the sets $\delta_1 E$ and $\delta_2 E$ respectively. Then

$$\delta_{12}\rho(x) = m(\delta_{12} E) = \left(\int_{\delta_1 E} \overline{g_\lambda(x)}\, dx, \ \int_{\delta_2 E} \overline{g_\lambda(x)}\, dx \right)_2$$

where δ_{12} is the interval common to δ_1 and δ_2. Further, if

$$\left(\phi, \int_{\delta E} \overline{g_\lambda(x)}\, dx \right)_2 = 0$$

for all δ, then $\int_{\delta E} f(x)\, dx = 0$, *i.e.* $f(x) = 0$, so that $\phi(\lambda) = 0$. By arguments similar to those used above, we see that $v(x) = \{\phi, \bar{g}_\lambda\}$ exists for all $\phi(\lambda)$ of

$L_2(-\infty, \infty)$ and $v(x)$ belongs to $L_2(E)$. Further, this equation sets up a $(1, 1)$ correspondence between $\phi(\lambda)$ of $L_2(-\infty, \infty)$ and $v(x)$ of $L_2(E)$. Again, $(\phi, \psi)_2 = (\{\phi, \bar{g}_\lambda\}, \{\psi, \bar{g}_\lambda\})_1$. Finally, there is a $(1, 1)$ correspondence between $v(x)$ and $f(x)$ and $\|v\|_1 = \|\phi\|_2 = \|f\|_1$; this correspondence is a bounded linear transformation, and it is the identical transformation when $f(x) = \Delta f_\lambda(x)$; it is thus the identical transformation for all $f(x)$.

5. Let $F(\lambda)$ be a measurable function of λ in $(-\infty, \infty)$ and let it be bounded in any finite interval†. When $F(\lambda)\phi(\lambda)$ belongs to $L_2(-\infty, \infty)$, then $Af = \{F\phi, \bar{g}_\lambda\}$ belongs to $L_2(E)$ and is a linear transformation of $f(x)$. Also $(Af, g)_1 = (F\phi, \psi)_2$, $\|Af\|_1 = \|F\phi\|_2$. $A(\Delta f_\lambda)$ exists for all Δ; thus Af is defined in $\mathfrak{D}_F \subseteq L_2(E)$, and $\mathfrak{D}_F$ is everywhere dense in $L_2(E)$. $\mathfrak{D}_F$ is the set of functions $f(x)$ of $L_2(E)$ for which $F(\lambda)\phi(\lambda)$ belongs to $L_2(-\infty, \infty)$.

Af is a closed linear transformation.

It suffices to show that, if $h_n(x)$ is a sequence of functions of $\mathfrak{D}_F$ and if $\chi_n(\lambda) = [h_n, g_\lambda]$ and if $\|(1+|F|^2)^{\frac{1}{2}}(\chi_n - \chi_m)\|_2 \to 0$ as $n, m \to \infty$, then there is a $\chi(\lambda)$ of $L_2(-\infty, \infty)$ such that $\|(1+|F|^2)^{\frac{1}{2}}(\chi - \chi_n)\|_2 \to 0$ as $n \to \infty$. For if this is so, then $\|h_n - h_m\|_1 \to 0$ and $\|Ah_n - Ah_m\|_1 \to 0$ imply the existence of $h(x) = \{\chi, \bar{g}_\lambda\}$ such that $\|h - h_n\|_1 \to 0$ and $\|Ah\|_1 = \|F\chi\|_2$ and $\|Ah - Ah_n\|_1 \to 0$ as $n \to \infty$, *i.e.* the closure of Af. The proof, which is word for word the same as Weyl's proof of the Riesz-Fischer theorem, is omitted.

We note that Af is bounded if and only if $|F(\lambda)|$ is bounded in $(-\infty, \infty)$.

6. Now suppose that Tf is a linear transformation of $f(x)$ defined, *i.e.* Tf belongs to $L_2(E)$, when $f(x)$ is in $\mathfrak{D}_T \subseteq L_2(E)$. Suppose also that Tf has a closed linear extension, and that $F(\lambda)$ is as before, and that, for all Δ, $T\Delta f_\lambda = \int_\Delta F(\lambda) g_\lambda(x)\, d\lambda$. Lastly, suppose that $P_\Delta Tf = TP_\Delta f$ for all $f(x)$ of $\mathfrak{D}_T$, where $P_\Delta f$ is the projection of f on to the closed linear manifold determined by all $\Delta_\nu f_\lambda$, where $\Delta_\nu \subseteq \Delta$. Clearly

$$P_\Delta f = \int_\Delta \phi(\lambda)\, g_\lambda(x)\, d\lambda.$$

If Tf is defined, then $Tf = \{F\phi, \bar{g}_\lambda\}$, $\mathfrak{D}_T \subseteq \mathfrak{D}_F$.

The transformation $Bf = \int_\Delta F\phi\, g_\lambda\, d\lambda$ is a bounded linear transformation defined for all $f(x)$ of $L_2(E)$. If $TP_\Delta f$ exists, then $TP_\Delta f = Bf$, since this equa-

† The theory is similar if we allow $F(\lambda)$ to be unbounded in the neighbourhood of isolated points.

tion is true when $f(x)=\Delta_\nu f_\lambda(x)$. If $TP_\Delta f$ is defined, then let $\sum_{\nu=1}^{n} c_\nu \Delta_\nu f_\lambda \to f(x)$ in the mean; since T has a closed linear extension, then $TP_\Delta \Sigma c_\nu \Delta_\nu f_\lambda \to TP_\Delta f$; also Bf is bounded and continuous, and the result follows. Now suppose that Tf exists; then $P_\Delta Tf = TP_\Delta f = \int_\Delta F\phi\, g_\lambda\, d\lambda$. Take $\Delta=(-n,n)$; then, since P_Δ tends to the identical transformation as $n\to\infty$, we obtain

$$Tf = \text{l.i.m. } P_\Delta Tf = \text{l.i.m. } \{F\phi, \bar{g}_\lambda\}_n = \{F\phi, \bar{g}_\lambda\}.$$

It remains to show that $F(\lambda)\phi(\lambda)$ belongs to $L_2(-\infty, \infty)$. We have

$$\|Tf\|_1 = \lim_{\Delta\to\infty} \|P_\Delta Tf\|_1 = \lim_{\Delta\to\infty} \int_\Delta |F\phi|^2 d\lambda;$$

this proves the result; consequently $\mathfrak{D}_T \subseteq \mathfrak{D}_F$.

If Tf is defined everywhere, we may omit the hypothesis $P_\Delta T = TP_\Delta$.

7. *If T_1 and T_2 are two linear transformations having all the properties of T above, but corresponding to functions $F_1(\lambda)$ and $F_2(\lambda)$ respectively, then, whenever T_1T_2f or T_2T_1f or both exist, they are equal to $\{F_1F_2\phi, \bar{g}_\lambda\}$; also $\mathfrak{D}_{T_1T_2}$ and $\mathfrak{D}_{T_2T_1}$ are both $\subseteq \mathfrak{D}_{F_1F_2} = \mathfrak{D}_{F_2F_1}$.*

Suppose, first, that $T_1P_\Delta f$ is defined for all Δ and for all f; if it is not, then, since T_1P_Δ has a bounded extension, we work with its closed linear extension. We then have

$$T_1T_2\Delta f_\lambda = \int_\Delta F_1(\lambda)F_2(\lambda)g_\lambda(x)\,d\lambda \quad \text{for} \quad T_1T_2\Delta f_\lambda = \{F_1[T_2\Delta f_\lambda, g_\lambda], \bar{g}_\lambda\}$$

and

$$[T_2\Delta f_\lambda, g_\lambda] = \left[\int_\Delta F_2(\lambda)g_\lambda(x)\,d\lambda, g_\lambda\right] = F_2(\lambda)$$

for λ in Δ and is zero for λ elsewhere. The result now follows if T_1 is closed and hence it follows generally.

We note that, if $T^{-1}f = \{F^{-1}\phi, \bar{g}_\lambda\}$, then $T^{-1}Tf = TT^{-1}f = f$ whenever these are defined. Here we suppose that $|F(\lambda)| > \delta_n > 0$ in $(-n, n)$.

8. *If $Af = \{F\phi, \bar{g}_\lambda\}$, then $A^*f = \{\bar{F}\phi, \bar{g}_\lambda\}$, $\mathfrak{D}_F = \mathfrak{D}_{\bar{F}}$.*

Here A^* is the adjoint of A, *i.e.* if $(Af, g)_1 = (f, g^*)_1$ for all $f(x)$ of $\mathfrak{D}_F$, then we write $g^*(x) = A^*g$. Now A^* is a closed linear transformation. If $A = A^*$, *i.e.* A is self-adjoint, then its inverse is also self-adjoint. Write A_F for A; then $A_{F/(1+|F|)}$ is a bounded linear transformation and its adjoint is

$A_{\bar{F}/(1+|F|)}$. Thus, for all $f(x)$ of $\mathfrak{D}_F$,

$$(A_F f, g)_1 = (f, g^*)_1 = (A_{F/(1+|F|)} \cdot A_{1+|F|} f, g)_1 = (A_{1+|F|} f, A_{\bar{F}/(1+|F|)} g)_1$$
$$= (f, A_{1+|F|} \cdot A_{\bar{F}/(1+|F|)} g)_1 = (f, A_{\bar{F}} g)_1.$$

Thus $(f, g^* - A_{\bar{F}} g)_1 = 0$ for all $f(x)$ of $\mathfrak{D}_F$. But $\mathfrak{D}_F$ is everywhere dense in $L_2(-\infty, \infty)$, so that $g^* = A_{\bar{F}} g$. $A_{1+|F|}$ is self-adjoint, since it is the inverse of the self-adjoint transformation $A_{1/(1+|F|)}$.

II. *The transformations FS and SF.*

We have

$$FS(f) = \underset{n\to\infty}{\text{l.i.m.}} \sqrt{\left(\frac{2}{\pi}\right)} \int_{1/n}^{\infty} f(y) \cos \frac{x}{y} \frac{dy}{y},$$

$$SF(f) = \underset{n\to\infty}{\text{l.i.m.}} \sqrt{\left(\frac{2}{\pi}\right)} \int_{0}^{n} f(y) \cos \frac{y}{x} \frac{dy}{x}.$$

FS and SF are defined everywhere in $L_2(0, \infty)$. Also $S^2 = F^2 = I$, the identical transformation; hence $(FS)^k (SF)^k = I$, *i.e.* $(FS)^k = (SF)^{-k}$. Further, $SS^* = FF^* = I$, hence $SF = (FS)^*$, $FS = (SF)^*$. The orthogonal differentials associated with SF and $FS \neq SF$ are $g_\lambda(x) = x^{-\frac{1}{2}+i\lambda}(2\pi)^{-\frac{1}{2}}$, since†

$$FS\left(\Delta \frac{x^{-\frac{1}{2}+i\lambda}}{\sqrt{(2\pi)}\log x}\right) = \int_\Delta \sqrt{\left(\frac{2}{\pi}\right)} \Gamma(\tfrac{1}{2} - i\lambda) \cos\left(\frac{\pi}{4} - \frac{i\lambda\pi}{2}\right) x^{-\frac{1}{2}+i\lambda} (2\pi)^{-\frac{1}{2}} d\lambda.$$

We write

$$\phi(\lambda) = \underset{n\to\infty}{\text{l.i.m.}} \int_{1/n}^{n} f(y)\, y^{-\frac{1}{2}-i\lambda}\, dy$$

in future. Then

$$FS(f) = \frac{1}{2\pi} \left\{ \sqrt{\left(\frac{2}{\pi}\right)} \Gamma(\tfrac{1}{2} - i\lambda) \cos\left(\frac{\pi}{4} - \frac{i\lambda\pi}{2}\right) \phi(\lambda), \quad x^{-\frac{1}{2}-i\lambda} \right\},$$

$$SF(f) = \frac{1}{2\pi} \left\{ \sqrt{\left(\frac{\pi}{2}\right)} \Gamma(\tfrac{1}{2} + i\lambda) \cos\left(\frac{\pi}{4} + \frac{i\lambda\pi}{2}\right) \phi(\lambda), \quad x^{-\frac{1}{2}-i\lambda} \right\},$$

$$(FS)^k(f) = \frac{1}{2\pi} \left\{ \left[\sqrt{\left(\frac{2}{\pi}\right)} \Gamma(\tfrac{1}{2} - i\lambda) \cos\left(\frac{\pi}{4} - \frac{i\lambda\pi}{2}\right) \right]^k \phi(\lambda), \quad x^{-\frac{1}{2}-i\lambda} \right\},$$

where k is any integer, positive or negative.

† We have $\int_0^c y^{-\frac{1}{2}-i\lambda} \cos xy\, dy = \Gamma(\frac{1}{2} - i\lambda) \cos(\frac{1}{4}\pi - \frac{1}{2}i\lambda\pi)\, x^{-\frac{1}{2}+i\lambda}$, etc.

III. *The transformation* $N(f)$.

We write

$$N_n(f) = \sum_{\nu=1}^{n} f\{\nu x\sqrt{(2\pi)}\} - \int_0^n f\{ux\sqrt{(2\pi)}\}\,du;$$

clearly $N_n(f)$ is a linear transformation defined for all $f(x)$ of $L_2(0, \infty)$. Further,

$$N_n\left(\Delta\,\frac{x^{-\frac{1}{2}+i\lambda}}{\sqrt{(2\pi)}\log x}\right) = \int_\Delta \zeta_n(\tfrac{1}{2}-i\lambda)\{x\sqrt{(2\pi)}\}^{-\frac{1}{2}+i\lambda}(2\pi)^{-\frac{1}{2}}\,d\lambda,$$

where

$$\zeta_n(\tfrac{1}{2}-i\lambda) = \sum_{\nu=1}^{n}\frac{1}{\nu^{\frac{1}{2}-i\lambda}} - \int_0^n \frac{du}{u^{\frac{1}{2}-i\lambda}} \to \zeta(\tfrac{1}{2}-i\lambda) \quad \text{as} \quad n\to\infty.$$

Thus
$$N_n(f) = \frac{1}{2\pi}\{\zeta_n(\tfrac{1}{2}-i\lambda)(2\pi)^{-\frac{1}{2}+\frac{1}{2}(i\lambda)}\phi(\lambda),\ \ x^{-\frac{1}{2}-i\lambda}\}$$

for all $f(x)$ of $L_2(0, \infty)$.

Further, if
$$P_\Delta(f) = \frac{1}{2\pi}\int_\Delta x^{-\frac{1}{2}+i\lambda}\phi(\lambda)\,d\lambda,$$

then $N_n P_\Delta = P_\Delta N_n$.

We now take $E_n = \left(\frac{1}{n}, n\right)$, so that $\int_0^\infty f\bar{g}\,dx = (f, g)_1$, etc., and define:

$$N(f) = \text{l.i.m.}_{n\to\infty} N_n(f)$$

whenever this exists,

$$N''(f) = \lim_{n\to\infty} N_n(f)$$

whenever this is a function in $L_2(0, \infty)$,

$$N'''(f) = \text{weak}\lim_{n\to\infty} N_n(f)$$

whenever this exists, *i.e.* $\left(N'''(f), g\right)_1 = \lim_{n\to\infty}\left(N_n(f), g\right)_1$ for all $g(x)$ of $L_2(0, \infty)$, and

$$\tilde{N}(f) = \text{l.i.m.}_{\Delta\to\infty}\ \text{l.i.m.}_{n\to\infty}\ N_n P_\Delta(f)$$

whenever this exists.

$N(f)$ has a closed linear extension $\tilde{N}(f)$ and

$$\tilde{N}(f) = \frac{1}{2\pi}\{\zeta(\tfrac{1}{2}-i\lambda)(2\pi)^{-\frac{1}{2}+\frac{1}{2}(i\lambda)}\phi(\lambda),\ \ x^{-\frac{1}{2}-i\lambda}\},$$

$\mathfrak{D}_{\tilde{N}}$ is the set of functions of $L_2(0, \infty)$ for which $\|\zeta(\frac{1}{2}-i\lambda)\phi(\lambda)\|_2$ exists.

First, we show that

$$N\left(\Delta \frac{x^{-\frac{1}{2}+i\lambda}}{\sqrt{(2\pi)}\log x}\right) = \int_\Delta \zeta(\tfrac{1}{2}-i\lambda)\{x\sqrt{(2\pi)}\}^{-\frac{1}{2}+i\lambda}(2\pi)^{-\frac{1}{2}}\,d\lambda;$$

we have

$$\frac{1}{2\pi}\int_0^\infty \left|\int_\Delta \{\zeta_n(\tfrac{1}{2}-i\lambda)-\zeta(\tfrac{1}{2}-i\lambda)\}\{x\sqrt{(2\pi)}\}^{-\frac{1}{2}+i\lambda}\,d\lambda\right|^2 dx$$

$$= \frac{1}{\sqrt{(2\pi)}}\int_\Delta |\zeta_n(\tfrac{1}{2}-i\lambda)-\zeta(\tfrac{1}{2}-i\lambda)|^2\,d\lambda \to 0,$$

which is the required result.

Secondly, we show that $P_\Delta N(f) = NP_\Delta(f)$ whenever $N(f)$ is defined; we have, if $N(f)$ is defined,

$$P_\Delta N(f) = \underset{n\to\infty}{\text{l.i.m.}}\, P_\Delta N_n(f) = \underset{n\to\infty}{\text{l.i.m.}}\, N_n P_\Delta(f) = NP_\Delta(f),$$

for

$$N_n P_\Delta(f) = \frac{1}{2\pi}\int_\Delta \zeta_n(\tfrac{1}{2}-i\lambda)\{x\sqrt{(2\pi)}\}^{-\frac{1}{2}+i\lambda}\phi(\lambda)\,d\lambda$$

$$\to \frac{1}{2\pi}\int_\Delta \zeta(\tfrac{1}{2}-i\lambda)\{x\sqrt{(2\pi)}\}^{-\frac{1}{2}+i\lambda}\phi(\lambda)\,d\lambda$$

in the mean, since

$$\int_\Delta |\zeta_n(\tfrac{1}{2}-i\lambda)-\zeta(\tfrac{1}{2}-i\lambda)|^2\,|\phi(\lambda)|^2\,d\lambda \to 0 \quad \text{as} \quad n\to\infty;$$

thus $NP_\Delta(f)$ exists.

Thirdly, we show that $N(f)$ has a closed linear extension. Let $h_n(x)$ be a sequence of functions of $L_2(0,\infty)$ such that $\|h_n-h\|_1 \to 0$ as $n\to\infty$, also suppose that $N(h_n)\to k(x)$, then $P_\Delta N(h_n)\to P_\Delta(k)$; further,

$$\sqrt{(2\pi)}\,\|P_\Delta(k)\|_1 = \frac{1}{2\pi}\int_\Delta |\zeta(\tfrac{1}{2}-i\lambda)\,\phi(\lambda)|^2\,d\lambda,$$

and, since $P_\Delta(k)\to k(x)$ as $\Delta\to\infty$, then $\zeta(\frac{1}{2}-i\lambda)\,\phi(\lambda)$ belongs to $L_2(-\infty,\infty)$, and so $h(x)$ belongs to $\mathfrak{D}_{\tilde{N}}$, where

$$\tilde{N}(f) = \frac{1}{2\pi}\{\zeta(\tfrac{1}{2}-i\lambda)(2\pi)^{-\frac{1}{4}+\frac{1}{2}(i\lambda}\phi(\lambda),\ x^{-\frac{1}{2}-i\lambda}\},$$

and $\tilde{N}(f)$ is a closed linear transformation. We can now apply the results of §I and conclude that

$$N(f) = \tilde{N}(f) = \underset{\Delta\to\infty}{\text{l.i.m.}}\ \underset{n\to\infty}{\text{l.i.m.}}\, N_n P_\Delta(f)$$

if $f(x)$ is in $\mathfrak{D}_N \subseteq \mathfrak{D}_{\tilde{N}}$.

If $N(f)$ is defined, then

$$\int_0^\infty N(f)x^{-\frac{1}{2}-i\lambda}\,dx = \zeta(\tfrac{1}{2}-i\lambda)(2\pi)^{-\frac{1}{2}+\frac{1}{2}(i\lambda)}\phi(\lambda),$$

and the left-hand side vanishes when $\zeta(\frac{1}{2}-i\lambda)=0$ if it is continuous at this point.

This is simply $[N(f), g_\lambda] = F(\lambda)\phi(\lambda)$. Now suppose that $\zeta(\frac{1}{2}-i\lambda_0)=0$ and that $[N(f), x^{-\frac{1}{2}+i\lambda}]$ is continuous at $\lambda=\lambda_0$, it then tends to a limit a as $\lambda\to\lambda_0$. If $a\neq 0$, then $\phi(\lambda) = a(\lambda-\lambda_0)^{-h}\{a'+o(1)\}$, $a'\neq 0$, $h\geqslant 1$, and so $\phi(\lambda)$ is not in $L_2(0,\infty)$.

The inequality (A) *is a necessary and sufficient condition for the closure of the transformation $N(f)$.*

Since $\mathfrak{D}_N \subseteq \mathfrak{D}_{\tilde{N}}$ and since $\mathfrak{D}_N$ is everywhere dense in $L_2(0,\infty)$, then $N(f)$ is closed if and only if $\mathfrak{D}_N = \mathfrak{D}_{\tilde{N}}$, *i.e.* if and only if

$$\lim_{n\to\infty}\frac{1}{2\pi}\|\{\zeta_n(\tfrac{1}{2}-i\lambda)-\zeta(\tfrac{1}{2}-i\lambda)\}\phi(\lambda)\|_2 = 0,$$

for all $f(x)$ in $\mathfrak{D}_{\tilde{N}}$.

First, suppose that (A) is true; then, if $\tilde{N}(f)$ exists, we have

$$\sqrt{(2\pi)}\|(N_n-\tilde{N})f\|_1^2 \leqslant \frac{1}{2\pi}\int_{-H}^{H}|\zeta_n(\tfrac{1}{2}-i\lambda)-\zeta(\tfrac{1}{2}-i\lambda|^2.|\phi(\lambda)|^2\,d\lambda + \int_H^\infty + \int_{-\infty}^{-H} K\{1+|\zeta(\tfrac{1}{2}-i\lambda)|^2\}|\phi(\lambda)|^2\,d\lambda,$$

which can be made arbitrarily small by choice first of H and then of n. Hence l.i.m. $N_n(f) = \tilde{N}(f)$ whenever $\tilde{N}(f)$ exists, *i.e.* $N(f)=\tilde{N}(f)$, and $\mathfrak{D}_N = \mathfrak{D}_{\tilde{N}}$.

Secondly, suppose that (A) is false; then we shall show that $\mathfrak{D}_N \subset \mathfrak{D}_{\tilde{N}}$, so that $N(f)$ cannot be closed. We define a sequence of integers $n_\nu < n_{\nu+1}$ as follows: let n_1 be the least integer for which $|\zeta_{n_1}(\frac{1}{2}-i\lambda)| \geqslant 1.|\zeta(\frac{1}{2}-i\lambda)|$ for some λ. We name the intervals $A_{1,n}$ in which this inequality is satisfied as follows: enumerate the positive rationals $r_1, r_2, \ldots$, let $m = \min(n)$, where r_n lies in a given interval, this interval we call $A_{1,m}$; we now call $A_1 = A_{1,\min(m)}$. Generally let A_m be a closed interval which is on the right of A_{m-1} and in which $|\zeta_{n_m}(\frac{1}{2}-i\lambda)| \geqslant m\{1+|\zeta(\frac{1}{2}-i\lambda)|\}$, n_m being the least integer for which this is possible. We make the choice of A_m unique by using the same process as for A_1, considering only intervals on the right of A_{m-1}. This arrangement is possible, since we can always find $\lambda>\Lambda$, for any Λ, and an integer n such that $|\zeta_n(\frac{1}{2}-i\lambda)| > m\{1+|\zeta(\frac{1}{2}-i\lambda)|\}$ for any given m; for if this was impossible, *i.e.* if λ was bounded, then, since $\zeta_n(\frac{1}{2}-i\lambda)\to\zeta(\frac{1}{2}-i\lambda)$, (A) would be true, contrary to supposition.

Now let $m^2 b_m^2 \int_{A_m} \{1+|\zeta(\frac{1}{2}-i\lambda)|^2\} d\lambda = 1$; also let $\phi(\lambda)$ be equal to b_m in A_m and zero elsewhere. Then $\phi(\lambda)$ belongs to $L_2(-\infty, \infty)$ and so $f(x) = (2\pi)^{-\frac{1}{2}}\{\phi, x^{-\frac{1}{2}-i\lambda}\}$ belongs to $L_2(0, \infty)$, and

$$\|\phi\|_2^2 = \|f\|_1^2 = \sum_{m=1}^{\infty} b_m^2 A_m,$$

A_m denoting the measure of the set A_m.

Further,
$$\|\zeta(\tfrac{1}{2}-i\lambda)\phi(\lambda)\|_2^2 \leqslant \sum_{m=1}^{\infty} \frac{1}{m^2};$$

thus $\tilde{N}(f)$ is defined.

Finally,

$$\sqrt{(2\pi)}\,\|(N_{n_m}-\tilde{N})f\|_1^2 \geqslant \int_{A_m} |\zeta_{n_m}(\tfrac{1}{2}-i\lambda)-\zeta(\tfrac{1}{2}-i\lambda)|^2\,|\phi(\lambda)|^2\,d\lambda$$
$$\geqslant (m-1)^2 b_m^2 \int_{A_m} (1+|\zeta(\tfrac{1}{2}-i\lambda)|^2)\,d\lambda \geqslant \tfrac{1}{2}.$$

Thus $N_{n_m}(f)$ does not tend to $\tilde{N}(f)$, and so $N(f)$ is not defined, *i.e.* $\mathfrak{D}_N \subset \mathfrak{D}_{\tilde{N}}$.

If $N(f)$ is closed, then so is $N^k(f) = \text{l.i.m.}\, N_n\{N^{k-1}(f)\}$.

The proof is by induction. Suppose that $N, \ldots, N^{k-1}$ are closed and $h_n(x) \to h(x)$, $N^k(h_n) \to k(x) = \tilde{N}^k(h)$. Then $N^m(h_n) \to N^m(h)$ for $1 \leqslant m \leqslant k-1$, since $\tilde{N}^m(h_n)$ and $\tilde{N}^m(h)$ exist for $1 \leqslant m \leqslant k-1$, since

$$\tilde{N}^k(f) = \frac{1}{2\pi}\{[\zeta(\tfrac{1}{2}-i\lambda)(2\pi)^{-\frac{1}{2}+\frac{1}{2}(i\lambda)}]^k \phi(\lambda),\ x^{-\frac{1}{2}-i\lambda}\}.$$

Thus $NN^{k-1}(h_n) \to k(x)$ and $N^{k-1}(h_n) \to N^{k-1}(h)$, and the result follows from the closure of N.

A necessary and sufficient condition for the closure of the transformation $N^{\{k\}}(f) = \underset{n\to\infty}{\text{l.i.m.}}\, N_n{}^k(f)$ is the existence of a constant K such that, for all λ,

$$(\text{A}') \qquad |\zeta_n{}^k(\tfrac{1}{2}-i\lambda)| < K\{1+|\zeta^k(\tfrac{1}{2}-i\lambda)|\};$$

further, (A) *and* (A′) *are equivalent.*

The result that (A′) is a necessary and sufficient condition for the closure of $N^{\{k\}}(f)$ is proved exactly as in the case of (A), since $\tilde{N}^{\{k\}}(f) = \tilde{N}^k(f)$. It remains to show that, if one of (A), (A′) is true, then $N^k(f) = N^{\{k\}}(f) = \tilde{N}^k(f)$ for all f in $\mathfrak{D}_{\tilde{N}^k}$. Suppose that (A) is true, then

$$|\zeta_n{}^k(\tfrac{1}{2}-i\lambda)| < K^k\{1+|\zeta(\tfrac{1}{2}-i\lambda)|\}^k < K'\{1+|\zeta^k(\tfrac{1}{2}-i\lambda)|\}.$$

Now suppose that (A′) is true; then

$$|\zeta_n(\tfrac{1}{2}-i\lambda)| < K^{1/k}\{1+|\zeta^k(\tfrac{1}{2}-i\lambda)|\}^{1/k} < K'\{1+|\zeta(\tfrac{1}{2}-i\lambda)|\},$$

i.e. (A) is true.

IV. $N = SNF$.

We have

$$\zeta(\tfrac{1}{2}-i\lambda) = 2(2\pi)^{-\frac{1}{2}-i\lambda}\cos\left(\frac{\pi}{4}+\frac{i\lambda\pi}{2}\right)\Gamma(\tfrac{1}{2}+i\lambda)\,\zeta(\tfrac{1}{2}+i\lambda),$$

hence

$$N(f) = \frac{1}{2\pi}\left\{\zeta(\tfrac{1}{2}+i\lambda)(2\pi)^{-\frac{1}{2}-\frac{1}{2}(i\lambda)}\sqrt{\left(\frac{2}{\pi}\right)}\,\Gamma(\tfrac{1}{2}+i\lambda)\cos\left(\frac{\pi}{4}+\frac{i\lambda\pi}{2}\right)\phi(\lambda), x^{-\frac{1}{2}-i\lambda}\right\}$$

$$= \tilde{N}^* SF(f),$$

where $\tilde{N}^*(f)$ is the adjoint of $\tilde{N}(f)$; *i.e.*

$$\tilde{N}^*(f) = \frac{1}{2\pi}\{\zeta(\tfrac{1}{2}+i\lambda)(2\pi)^{-\frac{1}{2}-\frac{1}{2}(i\lambda)}\phi(\lambda),\ x^{-\frac{1}{2}-i\lambda}\} = S\tilde{N}S(f).$$

We thus obtain the equation $\tilde{N}(f) = S\tilde{N}F(f)$ whenever $f(x)$ is in $\mathfrak{D}_{\tilde{N}}$.

If $\|\{\zeta(\tfrac{1}{2}-i\lambda)\}^k\phi(\lambda)\|_2$ *exists, then* $\tilde{N}^k(f) = \tilde{N}^k(FS)^k S(f)$, *and*

$$N^k(f) = N^k(FS)^k S(f)$$

whenever both sides are defined.

The proof is by induction, using $FS\tilde{N} = \tilde{N}FS$.

V. *Relations with* (*L*).

(*L*) *is equivalent to* "$\zeta^k(\tfrac{1}{2}+i\lambda)/(\tfrac{1}{2}+i\lambda)$ *belongs to* $L_2(-\infty, \infty)$ *for all positive integers* k".

It suffices to prove that (*L*) is equivalent to the existence of

$$I_h = \int_0^\infty \frac{|\zeta(\tfrac{1}{2}+i\lambda)|^h}{1+\lambda^2}\,d\lambda \quad \text{for all } h>0.$$

First suppose that (*L*) is true. Then $|\zeta(\tfrac{1}{2}+i\lambda)| < K|\lambda|^\epsilon$, and so I_h exists for all $h>0$. Secondly, suppose that I_h exists for all $h>0$. Let $\epsilon>0$ and (λ_i, λ_i'), $\lambda_i < \lambda_i' < \lambda_{i+1}$, be the intervals in which $|\zeta(\tfrac{1}{2}+i\lambda)| \geqslant \lambda^{\epsilon/h}$, $\lambda>0$.

Outside these intervals $|\zeta(\frac{1}{2}+i\lambda)| \leqslant \lambda^{\epsilon/h}$. Thus

$$\int_0^T |\zeta(\tfrac{1}{2}+i\lambda)|^h d\lambda \leqslant T^{1+\epsilon} + \sum_{\lambda_\nu \leqslant T} \int_{\lambda_\nu}^{\lambda_\nu'} (\lambda^2+1) \frac{|\zeta(\frac{1}{2}+i\lambda)|^h}{1+\lambda^2} d\lambda$$

$$\leqslant T^{1+\epsilon} + \sum_{\lambda_\nu \leqslant T} \int_{\lambda_\nu}^{\lambda_\nu'} \frac{|\zeta(\frac{1}{2}+i\lambda)|^{h\{1+(2/\epsilon)\}}}{1+\lambda^2} d\lambda + \sum_{\lambda_\nu \leqslant T} \int_{\lambda_\nu}^{\lambda_\nu'} \frac{|\zeta(\frac{1}{2}+i\lambda)|^h}{1+\lambda^2} d\lambda$$

$$\leqslant T^{1+\epsilon} + \int_0^\infty \frac{|\zeta(\frac{1}{2}+i\lambda)|^{h\{1+(2/\epsilon)\}}}{1+\lambda^2} d\lambda + \int_0^\infty \frac{|\zeta(\frac{1}{2}+i\lambda)|^h}{1+\lambda^2} d\lambda$$

$$\leqslant T^{1+\epsilon} + 2K \leqslant T^{1+2\epsilon},$$

if T is sufficiently large, and this inequality implies (L).

(L) is equivalent to "$e_X(x)$ belongs to $\mathfrak{D}_{\tilde{N}^k}$ for all positive integers k".

If (L) is true, then

$$\tilde{N}^k(e_X) = \frac{1}{2\pi} \left\{ \frac{[\zeta(\frac{1}{2}-i\lambda)(2\pi)^{-\frac{1}{2}+i\lambda}]^k X^{\frac{1}{2}-i\lambda}}{\frac{1}{2}-i\lambda},\ x^{-\frac{1}{2}-i\lambda} \right\}$$

exists for all positive integers k. For $X^{\frac{1}{2}-i\lambda}/(\frac{1}{2}-i\lambda) = [e_X, x^{-\frac{1}{2}-i\lambda}]$, and if (L) is true, then $\{\zeta(\frac{1}{2}-i\lambda)\}^k/(\frac{1}{2}-i\lambda)$ belongs to $L_2(-\infty, \infty)$. Secondly, if $e_X(x)$ belongs to $\mathfrak{D}_{\tilde{N}^k}$ for all positive integers k, then

$$\frac{[\zeta(\frac{1}{2}-i\lambda)(2\pi)^{-\frac{1}{2}+i\lambda}]^k X^{\frac{1}{2}-i\lambda}}{\frac{1}{2}-i\lambda}$$

belongs to $L_2(-\infty, \infty)$, and hence (L) follows.

Now let $d_k(\nu)$ be the number of solutions of the equation $\nu = n_1 \ldots n_k$ in positive integers $n_1, \ldots, n_k$. Write

$$\sum_{\nu=1}^{X} d_k(\nu) = \sum_{h=1}^{k} \frac{C_{k,h}}{(h-1)!} \int_1^X (\log u)^{h-1} du + R_k(X) \quad (0! = 1).$$

We have $$\lim_{n\to\infty} N_n{}^k(e_X) = R_k\left(\frac{X}{x(2\pi)^{\frac{1}{2}k}}\right),$$

[*whether* $L_2(0, \infty)$ *or not*].

For, if $c > \frac{1}{2}$,

$$N_n{}^k(e_X) = \frac{1}{2\pi} \int_{ci-\infty}^{ci+\infty} \zeta_n{}^k(\tfrac{1}{2}-i\lambda) \left(\frac{x(2\pi)^{\frac{1}{2}k}}{X}\right)^{-\frac{1}{2}+i\lambda} \frac{d\lambda}{\frac{1}{2}-i\lambda} + 2\pi i \text{ (residue at } \lambda = \tfrac{1}{2}i).$$

The integrals which involve the term $n^{\frac{1}{2}+i\lambda}$ are in absolute value less than

$$n^\epsilon \frac{K}{2\pi} \int_{-\infty}^{\infty} \sum_{h=1}^{k} \sum_{\nu=1}^{n} \frac{n^{(\frac{1}{2}-c)h} \nu^{-(\frac{1}{2}+c)(k-h)} (2\pi)^{-\frac{1}{2}k-\frac{1}{2}(ck)}}{\{(\frac{1}{2}-c)^2+\mu^2\}^{\frac{1}{2}h} \{(\frac{1}{2}+c)^2+\mu^2\}^{\frac{1}{2}}} \left(\frac{x}{X}\right)^{-(\frac{1}{2}+c)} d\mu \to 0 \quad \text{as} \quad n \to \infty.$$

The integrals which do not involve the term $n^{\frac{1}{2}+i\lambda}$ give, if n is sufficiently large,

$$\sum_{\nu=1}^{z} d_k(\nu),$$

if $Xz = x(2\pi)^{\frac{1}{2}k}$. The residue at $\lambda = \frac{1}{2}i$ is

$$\frac{1}{2\pi i}\int_C \left(\sum_{\nu=1}^{n}\frac{1}{\nu^{1-s}} - \int_1^n \frac{du}{u^{1-s}} - \frac{1}{s}\right)^k \frac{z^{s-1}}{1-s}\,ds,$$

where C is a circle with centre $s = 0$ and radius less than unity.

Now $$\zeta_n(1-s) = \sum_{\nu=1}^{n}\frac{1}{\nu^{1-s}} - \int_1^n \frac{du}{u^{1-s}} - \frac{1}{s} \to \zeta(1-s)$$

uniformly in $\Re(s) = \sigma < 1-\kappa$, $\kappa > 0$ and $|s| < H$, $H > 0$. Thus

$$\lim_{n\to\infty}\frac{1}{2\pi i}\int_C \zeta_n{}^k(1-s)\,\frac{z^{s-1}}{1-s}\,ds$$

$$= \frac{1}{2\pi i}\int_C \zeta^k(1-s)\,\frac{z^{s-1}}{1-s}\,ds$$

$$= \sum_{h-1}^{k}\frac{(-)^h C_{k,h}}{z}\sum_{m=0}^{h-1}\frac{(\log z)^m}{m!} = -\sum_{h=1}^{k}\frac{C_{k,h}}{(h-1)!}\int_1^{1/z}(\log u)^{h-1}\,du.$$

The result now follows.

If $N^{(k)}(e_X)$ or if $N^k(e_X)$ exists, then $R_k[X/\{x(2\pi)\}^{\frac{1}{2}k}]$ belongs to $L_2(0, \infty)$.

For if $N^{(k)}(e_X)$ or if $N^k(e_X)$ exists, they must be

$$\lim_{n\to\infty} N_n{}^k(e_X) = R_k\left(\frac{X}{x(2\pi)^{\frac{1}{2}k}}\right),$$

and the result follows.

If $\left(N_n{}^k(e_X), g\right)_1$ tends to a limit for all $g(x)$ of $L_2(0, \infty)$, then (L) follows.

For, if $\left(N_n{}^k(e_X), g\right)_1$ tends to a limit for all $g(x)$ of $L_2(0, \infty)$, then, from the theory of weak convergence, we have $\|N_n{}^k(e_X)\|_1 \leqslant K$ for all n. Hence

$$\|N_n{}^k(e_X)\|_1 = \frac{1}{2\pi}\left\|(2\pi)^{-\frac{1}{2}k}\frac{[\zeta_n(\frac{1}{2}-i\lambda)]^k}{\frac{1}{2}-i\lambda}\right\|_2 \leqslant K$$

for all n, and so

$$\int_{-T}^{T}\frac{|\zeta_n(\frac{1}{2}-i\lambda)|^{2k}}{\frac{1}{4}+\lambda^2}\,d\lambda \leqslant K$$

for all n and all T, and this implies the existence of

$$\int_{-\infty}^{\infty} \frac{|\zeta(\frac{1}{2}-i\lambda)|^{2k}}{\frac{1}{4}+\lambda^2}\, d\lambda,$$

i.e. the truth of (L).

If $\zeta_n{}^k(\frac{1}{2}-i\lambda)/(\frac{1}{2}-i\lambda)$ *is convergent in the mean over* $(-\infty, \infty)$ *for all positive integers* k, *then* (L) *is true.*

For if it was convergent in the mean, it would converge to $\zeta^k(\frac{1}{2}-i\lambda)/(\frac{1}{2}-i\lambda)$, and this would then belong to $L_2(-\infty, \infty)$ for all k.

If (L) *is true, then* $R_k(X/x)$ *belongs to* $L_2(0, \infty)$.

For if (L) is true we obtain, by contour integration,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\zeta^k(\frac{1}{2}-i\lambda)}{\frac{1}{2}-i\lambda}\left(\frac{x(2\pi)^{\frac{1}{2}k}}{X}\right)^{-\frac{1}{2}+i\lambda} d\lambda = R_k\left(\frac{X}{x(2\pi)^{\frac{1}{2}k}}\right),$$

and the result follows since $\zeta^k(\frac{1}{2}-i\lambda)/(\frac{1}{2}-i\lambda)$ belongs to $L^2(-\infty, \infty)$ if (L) is true.

If $R_k(X/x)$ *belongs to* $L_2(0, \infty)$, *then* (L) *is true.*

For

$$\int_1^n R_k{}^2(y)\, dy = \sum_{\nu=2}^{n} (\nu-1)^2 \int_{\nu-1}^{\nu} \frac{R_k{}^2(y)}{y^2}\, dy \leqslant n^2 \int_0^n \frac{R_k{}^2(y)}{y^2}\, dy \leqslant Kn^2,$$

and this inequality implies (L).

It follows immediately that (L) is equivalent to the truth of the equation

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|\zeta(\frac{1}{2}-i\lambda)|^{2k}}{\frac{1}{4}+\lambda^2}\, d\lambda = \int_0^{\infty} R_k{}^2(x)\, \frac{dx}{x^2}.$$

(L) *is equivalent to*

$$\lim_{n\to\infty} \zeta_{k,n}(s) = \lim_{n\to\infty}\left(\sum_{\nu=1}^{n} \frac{d_k(\nu)}{\nu^s} - \sum_{h=1}^{k} \frac{C_{k,h}}{(h-1)!}\int_0^n \frac{(\log u)^{h-1}}{u^s}\, du\right) = \zeta^k(s)$$

for $1 > \sigma = \Re(s) > \frac{1}{2}$, *and for every positive integer* k.

We have

$$\zeta_{k,n}(s) = \sum_{\nu=1}^{n} \frac{d_k(\nu)}{\nu^s} - \sum_{h=1}^{k} \frac{C_{k\,h}}{(h-1)!}\int_0^n \frac{(\log u)^{h-1}}{u^s}\, du - \sum_{h=1}^{k} \frac{(-)^{h-1} C_{k,h}}{(1-s)^h},$$

which is a regular function of s, if $s \neq 1$. If $\sigma > 1$, then $\lim_{n\to\infty} \zeta_{k,n}(s) = \zeta^k(s)$. Hence, by analytic continuation, this equality will hold as long as

$\lim_{n\to\infty} \zeta_{k,n}(s)$ represents a regular function of s. Now

$$\sum_{\nu=1}^{n} \frac{d_k(\nu)}{\nu^s} = 1 + \sum_{\nu=2}^{n} \frac{R_k(\nu) - R_k(\nu-1)}{\nu^s} + \sum_{h=1}^{k} \frac{C_{k,h}}{(h-1)!} \sum_{\nu=2}^{n} \int_{\nu-1}^{\nu} \frac{(\log u)^{h-1}}{\nu^s}\, du;$$

also $$w_h(s) = \lim_{n\to\infty} \left(\int_1^n \frac{(\log u)^{h-1}}{u^s}\, du - \sum_{\nu=2}^{n} \int_{\nu-1}^{\nu} \frac{(\log u)^{h-1}}{\nu^s}\, du \right)$$

is a regular function of s for $\sigma > 0$. Hence

$$v(s) = \sum_{\nu=2}^{\infty} \frac{R_k(\nu) - R_k(\nu-1)}{\nu^s}$$

is a regular function of s for $\sigma > 1$, and can be continued over $\sigma > 0$; for, if $\sigma > 0$, then

$$\zeta^k(s) = 1 + v(s) - \sum_{h=1}^{k} \frac{C_{k,h}}{(h-1)!}\, w_h(s) - \sum_{h=1}^{k} \frac{(-)^{h-1} C_{k,h}}{(1-s)^h}.$$

It remains to show that (L) is equivalent to the convergence for $\sigma > \frac{1}{2}$ of the series which defines $v(s)$. First suppose that this series is convergent for $\sigma > \frac{1}{2}$; then $v(\frac{1}{2}+\epsilon+it) = o(t)$, also $w_h(\frac{1}{2}+\epsilon+it) = O(t)$, whence $|\zeta^k(\frac{1}{2}+\epsilon+it)| < K|t|$ for any $\epsilon > 0$, $K = K(\epsilon, k)$; thus

$$|\zeta(\tfrac{1}{2}+\epsilon+it)| < K'|t|^{\eta}$$

for all $\eta > 0$, $\epsilon > 0$, and this is equivalent to (L). Secondly, suppose that (L) is true; then $R_k(\nu) = O(\nu^{\frac{1}{2}+\epsilon})$, for every $\epsilon > 0$. Hence

$$\sum_{\nu=2}^{n} \frac{R_k(\nu) - R_k(\nu-1)}{\nu^s} = \sum_{\nu=2}^{n-1} R_k(\nu) \left(\frac{1}{\nu^s} - \frac{1}{(\nu+1)^s} \right) + \frac{R_k(n)}{n^s} - \frac{R_k(1)}{2^s}$$

is convergent for $\sigma > \frac{1}{2}$. This proves the result.

Now suppose that, for all λ and all positive integers k,

(B) $$\lim_{n\to\infty} \zeta_{k,n}(\tfrac{1}{2}+i\lambda) = \zeta^k(\tfrac{1}{2}+i\lambda).$$

(B) implies the convergence of the series $v(s)$ for $\sigma = \frac{1}{2}$; hence (B) implies (L). Further, (B) is implied by

(C) $$R_k(\nu) = O(\nu^{\frac{1}{2}-\frac{1}{2}k+\epsilon})$$

for all $\epsilon > 0$ and all k.

If (B) is true, we define

$$N^{(k)}(f) = \underset{n\to\infty}{\text{l.i.m.}} \left(\sum_{\nu=1}^{n} d_k(\nu) f\{\nu x (2\pi)^{\frac{1}{2}k}\} - \sum_{h=1}^{k} \frac{C_{k,h}}{(h-1)!} \times \int_0^n (\log u)^{h-1} f\{ux(2\pi)^{\frac{1}{2}k}\}\, du \right).$$

We can show, as in the case of $N(f)$, that, if $N^{(k)}(f)$ is defined,

$$P_\Delta N^{(k)}(f) = N^{(k)} P_\Delta(f), \quad \mathfrak{D}_{N^{(k)}} \subseteq \mathfrak{D}_{\tilde{N}^k},$$

so that $N^{(k)}(f)$ has $\tilde{N}^k(f)$ as a closed linear extension; further, that

$$N^{(k)}\Big(\frac{\Delta x^{-\frac{1}{2}+i\lambda}}{\sqrt{(2\pi)}\log x}\Big) = \tilde{N}^k\Big(\frac{\Delta x^{-\frac{1}{2}+i\lambda}}{\sqrt{(2\pi)}\log x}\Big).$$

If, in addition to (B), we have, for all λ and some K,

$$(D) \qquad |\zeta_{k,n}(\tfrac{1}{2}-i\lambda)| \leqslant K\{1+|\zeta^k(\tfrac{1}{2}+i\lambda)|\},$$

then $\mathfrak{D}_{N^{(k)}} = \mathfrak{D}_{\tilde{N}^k}$. In fact, (D) is a necessary and sufficient condition for this to be so; the proof is similar to the proof of the corresponding result about (A).

Now suppose that (B) is true, and consider

$$\mathbf{N}^{(k)}(f) = \frac{d}{dx}\Big[\lim_{n\to\infty}\Big(\int_a^x \Big\{\sum_{\nu=1}^{n} d_k(\nu) f\{\nu y(2\pi)^{\frac{1}{2}k}\} - \sum_{h=1}^{k} \frac{C_{k,h}}{(h-1)!}\int_0^n (\log u)^{h-1} f\{uy(2\pi)^{\frac{1}{2}k}\}\,du\Big\}\,dy\Big)\Big],$$

where $a > 0$. Then we can show that, if $\mathbf{N}^{(k)}(f)$ is defined,

$$P_\Delta \mathbf{N}^{(k)}(f) = \mathbf{N}^{(k)} P_\Delta(f), \text{ etc.}, \quad \mathfrak{D}_{\mathbf{N}^{(k)}} \subseteq \mathfrak{D}_{\tilde{N}^k}.$$

If (B) is true, then

$$N^{(k)}(f) = SN^{(k)}(FS)^k S(f) \quad \text{and} \quad \mathbf{N}^{(k)}(f) = S\mathbf{N}^{(k)}(FS)^k S(f)$$

whenever they are significant. These formulae are of the type of the explicit formulae of Voronoi. It would seem that (B) is not sufficient for the closure of $\mathbf{N}^{(k)}(f)$. Similarly, (L) does not seem to be sufficient for the existence of $N^{(k)}(e_X)$ or $N^k(e_X)$.

VI. *The transformations $N_a(f)$ and $M_a(f)$.*

1. We now omit the factor $\sqrt{(2\pi)}$ in $f\{x\nu\sqrt{(2\pi)}\}$ for brevity. This factor was introduced into the definition of $N(f)$ so as to simplify the equation $N = SNF$. If $a > \frac{1}{2}$, we define

$$N_a(f) = \text{l.i.m.}_{n\to\infty} \sum_{\nu=1}^{n} \frac{f(\nu x)}{\nu^a};$$

then, for $a > \frac{1}{2}$, we have

$$\tilde{N}_a(f) = \frac{1}{2\pi}\{\zeta(\tfrac{1}{2}+a-i\lambda)\,\phi(\lambda),\ x^{-\frac{1}{2}-i\lambda}\}.$$

This is a bounded transformation defined for all $f(x)$ of $L_2(0, \infty)$. Since, for $\alpha > \frac{1}{2}$, $|\zeta_n(\frac{1}{2}+\alpha-i\lambda)| < K\{1+|\zeta(\frac{1}{2}+\alpha-i\lambda)|\}$ for all λ, it follows that N_α is closed; hence $N_\alpha(f) = \tilde{N}_\alpha(f)$ for all $f(x)$ of $L_2(0, \infty)$. Similarly,

$$M_\alpha(f) = \underset{n\to\infty}{\text{l.i.m.}} \sum_{\nu=1}^{n} \mu(\nu) \frac{f(\nu x)}{\nu^\alpha} = \frac{1}{2\pi} \left\{ \frac{\phi(\lambda)}{\zeta(\frac{1}{2}+\alpha-i\lambda)}, \quad x^{-\frac{1}{2}-i\lambda} \right\} = \tilde{M}_\alpha(f),$$

if $\alpha < \frac{1}{2}$, for all $f(x)$ of $L_2(0, \infty)$. Further, $M_\alpha(f) = N_\alpha^{-1}(f)$, $\alpha > \frac{1}{2}$, for all $f(x)$ of $L_2(0, \infty)$.

2. If $-\frac{1}{2} < \alpha < \frac{1}{2}$, we define

$$N_\alpha(f) = \underset{n\to\infty}{\text{l.i.m.}} \left(\sum_{\nu=1}^{n} \frac{f(\nu x)}{\nu^\alpha} - \int_0^n \frac{f(ux)}{u^\alpha} du \right).$$

As before, $N_\alpha(f)$ has a closed linear extension

$$\tilde{N}_\alpha(f) = \frac{1}{2\pi} \{\zeta(\tfrac{1}{2}+\alpha-i\lambda)\, \phi(\lambda), \quad x^{-\frac{1}{2}-i\lambda}\},$$

and a necessary and sufficient condition that $\mathfrak{D}_{N^\alpha} = \mathfrak{D}_{\tilde{N}^\alpha}$ is that there exists a constant K such that, for all λ and all n,

$$(A_\alpha) \qquad |\zeta_n(\tfrac{1}{2}+\alpha-i\lambda)| < K\{1+|\zeta(\tfrac{1}{2}+\alpha-i\lambda)|\}.$$

$\tilde{N}_\alpha(f)$ is unbounded because $\zeta(\frac{1}{2}+\alpha-i\lambda)$ is unbounded in $-\frac{1}{2} < \alpha \leqslant \frac{1}{2}$.

If $-\frac{1}{2} < \alpha < \frac{1}{2}$, $\alpha > 0$, *then*

$$N_\alpha'^{**}(f) = \frac{d}{dx} \left[\lim_{n\to\infty} \int_a^x \left(\sum_{\nu=1}^{n} \frac{f(\nu y)}{\nu^\alpha} - \int_0^n \frac{f(uy)}{u^\alpha} du \right) dy \right] = \tilde{N}_\alpha(f)$$

whenever $\| \zeta(\frac{1}{2}+\alpha-i\lambda)\, \phi(\lambda) \|_2$ *exists.*

Consider the linear transformation $N_\alpha'(f)$ defined in $\mathfrak{D}_\alpha'$, where $\mathfrak{D}_\alpha'$ is the linear manifold determined by all functions $e_X(x)$ for all X, which converts $e_X(x)$ into

$$\sum_{x\nu \leqslant X} \frac{1}{\nu^\alpha} - \left(\frac{X}{x}\right)^{1-\alpha} \frac{1}{1-\alpha}.$$

These functions belong to $L_2(0, \infty)$; also $\mathfrak{D}_\alpha'$ is everywhere dense in $L_2(0, \infty)$. Since

$$\underset{n\to\infty}{\text{l.i.m.}} \left[\sum_{\nu=1}^{\min(n,\, X/x)} \frac{1}{\nu^\alpha} - \left\{ \min\left(n, \frac{X}{x}\right) \right\}^{1-\alpha} \frac{1}{1-\alpha} \right] = \sum_{x\nu \leqslant X} \frac{1}{\nu^\alpha} - \left(\frac{X}{x}\right)^{1-\alpha} \frac{1}{1-\alpha},$$

we have $$N_\alpha'(f) = \underset{n\to\infty}{\text{l.i.m.}} \left(\sum_{\nu=1}^{n} \frac{f(\nu x)}{\nu^\alpha} - \int_0^n \frac{f(ux)}{u^\alpha} du \right),$$

if $f(x)$ is in $\mathfrak{D}_\alpha'$.

Thus $N_a'(f) = N_a(f)$ if $f(x)$ is in $\mathfrak{D}_a'$. Since $\mathfrak{D}_a'$ is everywhere dense in $L_2(0, \infty)$, then $N_a'^*$ exists and is a closed linear transformation. If $\left(N_a'(f), g\right)_1 = (f, g^*)$, for all $f(x)$ of $\mathfrak{D}_a'$, then $g^*(x) = N_a'^*(g)$. Now $\left(e_X(\nu x), g(x)\right)_1 = \left(e_X(x), (1/\nu)\, g(x/\nu)\right)_1$; further,

$$\int_0^X \int_0^n \frac{\overline{g}(x/u)}{u^{1+a}}\, du\, dx = \int_0^n \int_0^{X/u} \frac{\overline{g}(x)}{u^a}\, dx\, du = \int_0^\infty \overline{g}(x) \int_0^{\min(n,\, X/x)} \frac{du}{u^a}\, dx,$$

since $L(g) = \int_0^X \int_0^n \frac{g(x/u)}{u^{1+a}}\, du\, dx$ is a linear form in $g(x)$, and

$$|L(g)| \leqslant \left| \int_0^X x^{-a} \int_{x/n}^\infty g(v)\, v^{a-1}\, dv \right| \leqslant \int_0^X x^{-a} \|g\|_1 \sqrt{\left\{ \frac{1}{1-2a} \left(\frac{x}{n}\right)^{2a-1} \right\}}\, dx$$

$$\leqslant K \|g\|_1.$$

Thus $L(g)$ is a bounded linear form defined for all $g(x)$ of $L_2(0, \infty)$; hence $L(g) = (g, h)_1$ for some unique $h(x)$, but when $g(x) = e_X(x)$, then $h(x) = \int_0^{\min(n,\, X/x)} \frac{du}{u^a}$. The result now follows.

Thus if $\left(N_a'(f), g\right)_1 = (f, g^*)_1$, for all $f(x)$ of $\mathfrak{D}_a'$, then

$$(f, g^*)_1 = \lim_{n \to \infty} \int_0^\infty f(x) \left(\sum_{\nu=1}^n \frac{\overline{g}(x/\nu)}{\nu^{1+a}} - \int_0^n \frac{\overline{g}(x/u)}{u^{1+a}}\, du \right) dx.$$

On taking $f(x) = e_X(x) - e_a(x)$, we see that

$$g^*(x) = N_a'^*(g) = \frac{d}{dx} \left[\lim_{n \to \infty} \int_a^x \left(\sum_{\nu=1}^n \frac{g(x/\nu)}{\nu^{1+a}} - \int_0^n \frac{g(x/u)}{u^{1+a}}\, du \right) dx \right] \quad (a > 0).$$

This is then a closed linear transformation. Further, it is easily shown that $P_\Delta N_a'^* = N_a'^* P_\Delta$, etc., whence $N_a'^*(f) = \tilde{N}_a^*(f)$ for all $f(x)$ in $\mathfrak{D}_{N_a'^*} = \mathfrak{D}_{\tilde{N}_a}$. In a similar manner we can show that $N_a'^{**}(f)$, defined above, is equal to $\tilde{N}_a(f) = (N_a'^*)^*(f)$.

Now consider the transformation

$$M_a''(f) = \lim_{n \to \infty} \sum_{\nu=1}^n \frac{\mu(\nu)}{\nu^a} f(\nu x)$$

whenever this is a function in $L_2(0, \infty)$. If $M_a''(e_X)$ exists, it is equal to

$$\mathcal{M}_a\left(\frac{X}{x}\right) = \sum_{x\nu \leqslant X} \frac{\mu(\nu)}{\nu^a}.$$

If $f(x) = e_X(x)\, g(x)$ belongs to $L_2(0, \infty)$, and if $N_a'^{**}(f)$ and $M_a''(f)$ both exist, then

$$M_a''\, N_a'^{**}(f) = N_a'^{**}\, M_a''(f) = f(x).$$

We have

$$N_\alpha'^{**}(e_X g) = N_\alpha''(e_X g) = \sum_{\nu x \leqslant X} \frac{g(\nu x)}{\nu^\alpha} - \int_0^{X/x} \frac{g(ux)}{u^\alpha}\, du,$$

if this is a function in $L_2(0, \infty)$. Further,

$$\begin{aligned} M_\alpha'' N_\alpha'^{**}(e_X g) &= \lim_{n\to\infty} \left(\sum_{m=1}^{n} \frac{\mu(m)}{m^\alpha} \left\{ \sum_{m\nu x \leqslant X} \frac{g(\nu m x)}{\nu^\alpha} - \int_0^{X/mx} \frac{g(umx)}{u^\alpha}\, du \right\} \right) \\ &= \lim_{n\to\infty} \left(\sum_{xl \leqslant X} \frac{g(lx)}{l^\alpha} \sum_{\substack{m \mid l \\ m<n}} \mu(m) - \int_0^{X/x} \frac{g(ux)}{u^\alpha}\, du \sum_{m=1}^{n} \frac{\mu(m)}{m} \right) \\ &= e_X(x)\, g(x). \end{aligned}$$

Again, since $M_\alpha''(e_X g) = 0$ for $x > X$, we have

$$\begin{aligned} & N_\alpha'^{**}\, M_\alpha''(e_X g) \\ &= N_\alpha''\, M_\alpha''(e_X g) \\ &= \lim_{n\to\infty} \left(\sum_{m=1}^{n} \frac{1}{m^\alpha} \sum_{m\nu x \leqslant X} \frac{\mu(\nu)}{\nu^\alpha}\, g(m\nu x) - \int_0^n \frac{1}{u^\alpha} \sum_{u\nu x \leqslant X} \frac{\mu(\nu)}{\nu^\alpha}\, g(\nu u x)\, du \right) \\ &= \lim_{n\to\infty} \left(\sum_{xl \leqslant X} \frac{g(lx)}{l^\alpha} \sum_{\substack{\nu \mid l \\ n\nu > l}} \mu(\nu) - \sum_{\nu=1}^{\infty} \frac{\mu(\nu)}{\nu} \int_0^{\min(n\nu,\, X/x)} \frac{g(vx)}{v^\alpha}\, dv \right) \\ &= e_X(x)\, g(x), \end{aligned}$$

and the result now follows. We thus have $M_\alpha''(f) = \tilde{N}_\alpha^{-1}(f)$ whenever both sides exist.

If $e_X(x)$ is in the range of $\tilde{N}_\alpha$ for $\alpha > 0$, then (R) is true, and conversely.

For if $e_X(x)$ is in the domain of $\tilde{N}_\alpha^{-1}$ for $\alpha > 0$, then

$$\| \{\zeta(\tfrac{1}{2}+\alpha-i\lambda)(\tfrac{1}{2}-i\lambda)\}^{-1} \|_2$$

exists, but this implies $\zeta(\frac{1}{2}+\alpha-i\lambda) \neq 0$ for $\alpha > 0$ and any λ, *i.e.* it implies (R). Conversely, if (R) is true, then $\zeta(s) \sum_{\nu=1}^{\infty} \frac{\mu(\nu)}{\nu^s} = 1$†, for $\sigma > \frac{1}{2}$, and thus $|\mathcal{M}_0(X)| < KX^{\frac{1}{2}+\epsilon}$, so that $|\mathcal{M}_\alpha(X)| < KX^{\frac{1}{2}-\alpha+\epsilon}$, for $0 < \alpha \leqslant \frac{1}{2}$; this implies the existence of $\mathcal{M}_\alpha(X/x)$ in $L_2(0, \infty)$ and hence $M_\alpha''(e_X)$ is defined. But $\tilde{N}_\alpha M_\alpha''(e_X) = N_\alpha'^{**}\, M_\alpha''(e_X) = e_X(x)$ when $M_\alpha''(e_X)$ is defined. The result now follows. Thus $(R) \equiv$ "$\mathcal{M}_\alpha(x)/x$ is in $L_2(0, \infty)$ for $\alpha > 0$".

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† Littlewood, *Comptes rendus*, 154 (1912), 263–266.

If $\alpha > 0$ and if $M_\alpha(x)/x$ belongs to $L_2(0, \infty)$, then so does $M_{\alpha+\delta}(x)/x$ for $0 < \delta \leqslant \frac{1}{2}$, and $\zeta(s) \neq 0$ for $\sigma \geqslant \alpha$.

Suppose that $M_\alpha(x)/x$ belongs to $L_2(0, \infty)$ and that $0 < \delta \leqslant \frac{1}{2}$; then

$$M_{\alpha+\delta}(x) = \sum_{\nu=1}^{x} \frac{\mu(\nu)}{\nu^\alpha \nu^\delta} = 1 + \sum_{\nu=2}^{x} \frac{M_\alpha(\nu) - M_\alpha(\nu-1)}{\nu^\delta}$$

$$= 1 + \sum_{\nu=2}^{x} M_\alpha(\nu)\left(\frac{1}{\nu^\delta} - \frac{1}{(\nu+1)^\delta}\right) + \frac{M_\alpha([x])}{([x]+1)^\delta}.$$

Now $|M_\alpha(\nu)| < K\nu^{1-\alpha}$ for all ν if $\alpha > 0$, and hence

$$M_{\alpha+\delta}(x) = 1 + \sum_{\nu=2}^{x} \frac{M_\alpha(\nu)}{\nu^{1+\delta}} + O(x^{-(\alpha+\delta)}+1) + \theta \frac{M_\alpha(x)}{x^\delta} \quad (0 < \theta < 1).$$

Clearly $$\left[1 + O(x^{-(\alpha+\delta)}+1) + \theta \frac{M_\alpha(x)}{x^\delta}\right] \frac{1}{x}$$

belongs to $L_2(1, \infty)$. It suffices to consider $L_2(1, \infty)$, since $M_\alpha(x) = 0$ for $0 \leqslant x < 1$. Thus we need only show that

$$\frac{1}{x} \sum_{\nu=2}^{x} \frac{M_\alpha(\nu)}{\nu^{1+\delta}}$$

belongs to $L_2(1, \infty)$. Now

$$\int_1^H \left(\sum_{\nu=2}^{x} \frac{M_\alpha(\nu)}{\nu^{1+\delta}}\right)^2 \frac{dx}{x^2} = \sum_{\nu,\,\mu=2}^{H} \frac{M_\alpha(\nu) M_\alpha(\mu)}{(\nu\mu)^{1+\delta}} \int_{\max(\nu,\,\mu)}^{H} \frac{dx}{x^2}$$

$$= 2 \sum_{\mu=2}^{H} \frac{M_\alpha(\mu)}{\mu^{2+\delta}} \sum_{\nu<\mu} \frac{M_\alpha(\nu)}{\nu^{1+\delta}} - \frac{1}{H} \sum_{\nu,\,\mu=2}^{H} \frac{M_\alpha(\nu) M_\alpha(\mu)}{(\nu\mu)^{1+\delta}}.$$

Now $$2 \int_1^\infty \frac{\{M_\alpha(x)\}^2}{x^2}\, dx > \sum_{\nu=1}^{\infty} \frac{\{M_\alpha(\nu)\}^2}{\nu^2};$$

thus

$$\sum_{\nu=1}^{n} \frac{|M_\alpha(\nu)|}{\nu^{1+\delta}} \leqslant \sqrt{\left\{\sum_{\nu=1}^{n} \frac{1}{\nu^{2\delta}} \sum_{\nu=1}^{n} \frac{|M_\alpha(\nu)|^2}{\nu^2}\right\}} \leqslant K(n^{\frac{1}{2}-\delta}+1),$$

$$\left| \sum_{\mu=2}^{n} \frac{M_\alpha(\mu)}{\mu^{2+\delta}} \sum_{\nu<\mu} \frac{M_\alpha(\nu)}{\nu^{1+\delta}} \right| \leqslant \sum_{\mu=2}^{n} \frac{|M_\alpha(\mu)|}{\mu^{2+\delta}} K\mu^{\frac{1}{2}-\delta} \leqslant K'.$$

The result now follows.

If (R) is true and if

$$M_\alpha(f) = \text{l.i.m.}_{n\to\infty} \sum_{\nu=1}^{n} \frac{\mu(\nu)}{\nu^\alpha} f(\nu x) \quad (\tfrac{1}{2} > \alpha > 0),$$

then $M_\alpha(f) = \tilde{N}_\alpha^{-1}(f)$, *when* $M_\alpha(f)$ *is defined; further, a necessary and sufficient condition that* $M_\alpha(f) = \tilde{N}_\alpha^{-1}(f)$, *whenever* $\| \{\zeta(\frac{1}{2}+\alpha-i\lambda)\phi(\lambda)\}^{-1} \|_2$ *exists, is the existence of a constant* K *such that, for all* λ *and all* n,

$$\text{(E)} \qquad \left| \sum_{\nu=1}^{n} \frac{\mu(\nu)}{\nu^{\frac{1}{2}+\alpha-i\lambda}} \right| \leqslant K\left(1+\frac{1}{|\zeta(\frac{1}{2}+\alpha-i\lambda)|}\right).$$

The proof is similar to the proof of the corresponding results about $N(f)$. A further way of defining $M_\alpha(f)$ is

$$\mathbf{M}_\alpha(f) = \frac{d}{dx}\left[\lim_{n\to\infty} \int_a^x \sum_{\nu=1}^{n} \frac{\mu(\nu)}{\nu^\alpha} f(\nu y)\, dy\right] \quad (\alpha > 0),$$

but (R) does not seem to be sufficient for $\mathbf{M}_\alpha(f) = \tilde{N}_\alpha^{-1}(f)$ whenever

$$\| \{\zeta(\tfrac{1}{2}+\alpha-i\lambda)\phi(\lambda)\}^{-1} \|_2$$

exists.

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ON THE SUMMABILITY OF SCHLÖMILCH SERIES

By Richard G. Cooke.

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1. The Schlömilch series of a function $f(x)$, integrable in the sense of Lebesgue over the closed interval $(-\pi, \pi)$, is*

$$\Sigma A_m \equiv \sum_{m=1}^{\infty} \frac{a_m J_\nu(mx)+b_m \mathbf{H}_\nu(mx)}{(\frac{1}{2}mx)^\nu},$$

where

$$a_m = \frac{m}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}-\nu)} \int_{-\pi}^{\pi} t\phi(t) \sin mt\, dt,$$

$$b_m = -\frac{m}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}-\nu)} \int_{-\pi}^{\pi} t\phi(t) \cos mt\, dt,$$

$$\phi(t) = \int_0^{\frac{1}{2}\pi} f(t \sin\theta) \sin^{2\nu+1}\theta \cos^{-2\nu}\theta\, d\theta, \quad \text{and} \quad |\nu| < \tfrac{1}{2}.$$

We denote the half-open intervals $\epsilon \leqslant x < \pi$, $-\pi < x \leqslant -\epsilon$ $(\epsilon > 0)$ by e_1, e_2.

It is the object of this paper to consider the $(C, 1)$ summability of the Schlömilch series.

It will be shown that theorems of as general a character as the corresponding theorems in the theory of Fourier series can be deduced from the work of Young on *ORF* series† for all points in the intervals e_1 and e_2.

A general necessary and sufficient condition that the Schlömilch series should be summable $(C, 1)$ at any point x of $-\pi < x < \pi$ to the sum s is given, and this is then applied to show that the series is summable

* R. G. Cooke, "On the theory of Schlömilch series", *Proc. London Math. Soc.* (2), 28 (1928), 207–241. See *ibid.*, 209 (1.14), (1.15), (1.16).

† W. H. Young, "On the convergence of the derived series of Fourier series", *Proc. London Math. Soc.* (2), 17 (1918), 195–236. For the definition of *ORF* series, see R. G. Cooke, *loc. cit.*, 216.

$(C, 1)$ at the origin (where the series is, in general, either divergent or oscillatory*) to the sum $\frac{1}{2}\{f(0+0)+f(0-0)\}$, provided only that these limits exist, and that $f(x)$ satisfies Lipschitz conditions of any order $p>0$ in arbitrarily short intervals on either side of the origin.

It remains only to consider the end-points $\pm\pi$, where the series is, in general, divergent†. Here more stringent conditions are required, namely that $f(x)$ is continuous at $x=\pm\pi$, that $\phi(\pi)+\phi(-\pi)=0$, and that $\frac{d}{dt}\{t\phi(t)\}$ is integrable over $(-\pi, \pi)$, and bounded in the neighbourhoods of $\pm\pi$; the series is then summable $(C, 1)$ at $x=\pi$ to the sum $f(\pi)$, and at $x=-\pi$ to the sum $f(-\pi)$.

2. In this section, the intervals e_1 and e_2 are considered.

THEOREM I. *The Schlömilch series is summable* $(C, 1)$ *almost everywhere in* e_1 *and* e_2 *to the sum* $\frac{1}{2}\{f(x+0)+f(x-0)\}$.

It is known‡ that, at any point x of e_1 or e_2,

$$\Sigma A_m = x^{-\nu-\frac{1}{2}}\{S_1+T(x)\}, \qquad (2.1)$$

where S_1 is an ORF series and $T(x)$ is an integral. Also, if $G(x)$ is the function associated with the ORF series S_1, then§

$$G(x) = x^{\nu+\frac{1}{2}}f(x)-T(x); \qquad (2.2)$$

so that $G(x)$ is integrable over $(-\pi, \pi)$, since the same is true of $f(x)$ and therefore of $x^{\nu+\frac{1}{2}}f(x)$, since $|\nu|<\frac{1}{2}$.

Write $$F(x)-F(a) \equiv \int_a^x G(t)\,dt;$$

then $F(x)$ is of bounded variation in e_1 and e_2, and the first derived series of the Fourier series of $F(x)$ is the ORF series S_1.

But we have the theorem of Young||:

If $f(x)$ *is, throughout a certain interval* (a, b), *a function of bounded variation, then the derived series of the Fourier series of* $f(x)$ *converges* $(C, 1)$,

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* R. G. Cooke, *loc. cit.*, 230, Theorem VI.
† *Ibid.*, 235, Theorem VIII.
‡ *Ibid.*, 223 (3.81).
§ *Ibid.*, 223 (3.82).
|| W. H. Young, *loc. cit.*, 218.

almost everywhere in the interval in question, to the differential coefficient of the function.

Hence the series S_1 is summable $(C, 1)$ almost everywhere in e_1 and e_2 to the sum $G(x)$, so that, by (2.1) and (2.2), the theorem follows.

Theorem II. *The Schlömilch series of $f(x)$ is summable $(C, 1)$ at any point x of e_1 or e_2 under precisely the same conditions which ensure that the Fourier series of $f(x)$ should be summable $(C, 1)$ at the point x.*

For it has been proved by Young* that

The condition that a restricted Fourier series of the p-th class should converge (C, p), at the point x of the interval to which it is restricted, is precisely of the same form as the condition that a Fourier series should converge (C, p) at that point, the function associated with the restricted Fourier series taking the place of that associated with the Fourier series.

If we take $p = 1$, the theorem then follows as in Theorem I. The theorem evidently includes analogues for the Schlömilch series of the theorems of Fejér and Fejér-Lebesgue in e_1 or e_2. Thus the Schlömilch series is, in particular, summable $(C, 1)$ to the sum $f(x)$ at all points of continuity of $f(x)$ in e_1 or e_2; it is summable $(C, 1)$ to the sum

$$\tfrac{1}{2}\{f(x+0)+f(x-0)\}$$

in e_1 or e_2 whenever these limits exist, and to this same sum for all points x of e_1 or e_2 for which

$$\int_0^t |f(x+u)-f(x)|\,du = o(t). \qquad (2.3)$$

Since (2.3) is true almost everywhere, Theorem I is a particular case of Theorem II.

3. We now find a general necessary and sufficient condition for the series to be summable $(C, 1)$ at any point of the open interval $-\pi < x < \pi$, so that the origin will be included in this condition.

Theorem III. *A necessary and sufficient condition that the Schlömilch series should be summable $(C, 1)$ to the sum s at any point x of the open interval*

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* W. H. Young, *loc. cit.*, 225.

$-\pi < x < \pi$ *is that*

$$\lim_{n\to\infty} \frac{1}{n} \int_0^\delta \frac{d}{du} \left(\frac{\sin^2 \frac{1}{2}nu}{\sin^2 \frac{1}{2}u}\right) du$$

$$\times \int_0^1 (1-v^2)^{\nu-\frac{1}{2}} \{(vx+u)\psi(vx+u) - (vx-u)\psi(vx-u)\} dv = 0, \quad (3.1)$$

where $0 < \delta < \pi$, *and*

$$\psi(t) \equiv \frac{\Gamma(\nu+1)\,\Gamma(\frac{1}{2}-\nu)}{\Gamma(\frac{1}{2})}\, s - \phi(t).$$

From § 1 we see that the n-th partial sum of the series is given by

$$s_n = \frac{1}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}-\nu)} \sum_{m=1}^{n} \left[\frac{mJ_\nu(mx)}{(\frac{1}{2}mx)^\nu} \int_{-\pi}^{\pi} t\phi(t) \sin mt\, dt \right.$$

$$\left. - \frac{m\mathbf{H}_\nu(mx)}{(\frac{1}{2}mx)^\nu} \int_{-\pi}^{\pi} t\phi(t) \cos mt\, dt\right]$$

$$= \frac{1}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}-\nu)} \int_{-\pi}^{\pi} t\phi(t) \left[\sum_{m=1}^{n} \frac{m\{J_\nu(mx)\sin mt - \mathbf{H}_\nu(mx)\cos mt\}}{(\frac{1}{2}mx)^\nu}\right] dt$$

$$= -\frac{1}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}-\nu)} \int_{-\pi}^{\pi} t\phi(t) \frac{d}{dt} \left[\sum_{m=1}^{n} \frac{J_\nu(mx)\cos mt + \mathbf{H}_\nu(mx)\sin mt}{(\frac{1}{2}mx)^\nu}\right] dt$$

$$= -\frac{2}{\pi\Gamma(\frac{1}{2}+\nu)\,\Gamma(\frac{1}{2}-\nu)} \int_{-\pi}^{\pi} t\phi(t) \frac{d}{dt} \int_0^{\frac{1}{2}\pi} \sin^{2\nu}\theta \sum_{m=1}^{n} \cos\{m(t-x\cos\theta)\}\, d\theta\, dt^*$$

$$= -\frac{1}{\pi\Gamma(\frac{1}{2}+\nu)\,\Gamma(\frac{1}{2}-\nu)}$$

$$\times \int_{-\pi}^{\pi} t\phi(t) \frac{d}{dt} \int_0^{\frac{1}{2}\pi} \sin^{2\nu}\theta \left[\frac{\sin\{(n+\frac{1}{2})(t-x\cos\theta)\}}{\sin\frac{1}{2}(t-x\cos\theta)} - 1\right] d\theta\, dt$$

$$= -\frac{1}{\pi\Gamma(\frac{1}{2}+\nu)\,\Gamma(\frac{1}{2}-\nu)}$$

$$\times \int_{-\pi}^{\pi} t\phi(t) \frac{d}{dt} \int_0^{\frac{1}{2}\pi} \sin^{2\nu}\theta \frac{\sin\{(n+\frac{1}{2})(t-x\cos\theta)\}}{\sin\frac{1}{2}(t-x\cos\theta)}\, d\theta\, dt. \quad (3.2)$$

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* G. N. Watson, *Theory of Bessel functions* (Cambridge, 1922), 48 (5), and 328 (1).

Hence, if $\sigma_n = \frac{1}{n}\sum_{k=1}^{n} s_k$, we have

$$\sigma_n = -\frac{1}{n\pi\,\Gamma(\frac{1}{2}+\nu)\,\Gamma(\frac{1}{2}-\nu)} \times\int_{-\pi}^{\pi} t\,\phi(t)\,\frac{d}{dt}\int_0^{\frac{1}{2}\pi}\sin^{2\nu}\theta\,\frac{\sin^2\{\frac{1}{2}(n+1)(t-x\cos\theta)\}}{\sin^2\frac{1}{2}(t-x\cos\theta)}\,d\theta\,dt. \quad (3.3)$$

Now, when $f(t)=1$, then $\phi(t)=\Gamma(\nu+1)\,\Gamma(\frac{1}{2}-\nu)/\Gamma(\frac{1}{2})$, and

$$\sigma_n = -\frac{1}{n\pi\,\Gamma(\frac{1}{2}+\nu)\,\Gamma(\frac{1}{2}-\nu)} \times\int_{-\pi}^{\pi} t\,\frac{\Gamma(\nu+1)\,\Gamma(\frac{1}{2}-\nu)}{\Gamma(\frac{1}{2})}\,\frac{d}{dt}\int_0^{\frac{1}{2}\pi}\sin^{2\nu}\theta\,\frac{\sin^2\{\frac{1}{2}(n+1)(t-x\cos\theta)\}}{\sin^2\frac{1}{2}(t-x\cos\theta)}\,d\theta\,dt. \quad (3.4)$$

But in this particular case

$$s_n = \frac{\Gamma(\nu+1)}{\pi}\sum_{m=1}^{n}\left[\frac{mJ_\nu(mx)}{(\frac{1}{2}mx)^\nu}\int_{-\pi}^{\pi} t\sin mt\,dt - \frac{m\mathbf{H}_\nu(mx)}{(\frac{1}{2}mx)^\nu}\int_{-\pi}^{\pi} t\cos mt\,dt\right]$$

$$= -2\Gamma(\nu+1)\sum_{m=1}^{n}\frac{(-1)^m J_\nu(mx)}{(\frac{1}{2}mx)^\nu}$$

$$= 2\Gamma(\nu+1)\left[\frac{1}{2\Gamma(\nu+1)} - \frac{(-1)^n}{x^{2\nu}\,\Gamma(\frac{1}{2})\,\Gamma(\nu+\frac{1}{2})}\int_0^x \frac{\cos(n+\frac{1}{2})u}{\cos\frac{1}{2}u}\,(x^2-u^2)^{\nu-\frac{1}{2}}\,du\right]^*$$

$$= 1 - \frac{2\Gamma(\nu+1)}{\Gamma(\frac{1}{2})\,\Gamma(\nu+\frac{1}{2})}\int_0^{\frac{1}{2}\pi}\frac{\sin\{(n+\frac{1}{2})(\pi-x\cos\theta)\}}{\sin\frac{1}{2}(\pi-x\cos\theta)}\sin^{2\nu}\theta\,d\theta,$$

where $u = x\cos\theta$.

Hence, if $-\pi+\eta \leqslant x \leqslant \pi-\eta$ $(\eta>0)$, then†

$$s_n = 1+o(1), \quad \text{as} \quad n\to\infty,$$

by the Riemann-Lebesgue theorem; and hence also

$$\sigma_n = 1+o(1). \quad (3.5)$$

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* R. G. Cooke, *loc. cit.*, 233.

† I owe a slight simplification in the proof at this stage to the Referee, who pointed out that it is only necessary to use $\sigma_n = 1+o(1)$; I had previously shown that

$$\sigma_n = 1+O(1/n) \quad \text{for} \quad -\pi+\eta \leqslant x \leqslant \pi-\eta.$$

Consequently, from (3.4) and (3.5),

$$1+o(1) = -\frac{1}{n\pi\,\Gamma(\frac{1}{2}+\nu)\,\Gamma(\frac{1}{2}-\nu)}$$

$$\times\int_{-\pi}^{\pi} t\,\frac{\Gamma(\nu+1)\,\Gamma(\frac{1}{2}-\nu)}{\Gamma(\frac{1}{2})}\,\frac{d}{dt}\int_0^{\frac{1}{2}\pi}\sin^{2\nu}\theta\,\frac{\sin^2\{\frac{1}{2}(n+1)(t-x\cos\theta)\}}{\sin^2\frac{1}{2}(t-x\cos\theta)}\,d\theta\,dt. \quad (3.6)$$

Multiply (3.6) by s, and subtract from (3.3); then

$$\sigma_n - s + o(1) = \frac{1}{n\pi\,\Gamma(\frac{1}{2}+\nu)\,\Gamma(\frac{1}{2}-\nu)}$$

$$\times\int_{-\pi}^{\pi} t\psi(t)\,\frac{d}{dt}\int_0^{\frac{1}{2}\pi}\sin^{2\nu}\theta\,\frac{\sin^2\{\frac{1}{2}(n+1)(t-x\cos\theta)\}}{\sin^2\frac{1}{2}(t-x\cos\theta)}\,d\theta\,dt, \quad (3.7)$$

where
$$\psi(t) = \frac{\Gamma(\nu+1)\,\Gamma(\frac{1}{2}-\nu)}{\Gamma(\frac{1}{2})}\,s - \phi(t).$$

Hence, by (3.7), a necessary and sufficient condition that the series should be summable $(C, 1)$ to the sum s at any point x of the open interval $-\pi < x < \pi$ is that

$$\lim_{n\to\infty}\frac{1}{n}\int_{-\pi}^{\pi} t\psi(t)\,\frac{d}{dt}\int_0^{\frac{1}{2}\pi}\frac{\sin^2\{\frac{1}{2}(n+1)(t-x\cos\theta)\}}{\sin^2\frac{1}{2}(t-x\cos\theta)}\,\sin^{2\nu}\theta\,d\theta\,dt = 0. \quad (3.8)$$

This condition can be simplified.

In the inner integral, $n+1$ may clearly be replaced by n, and, writing $u = t - x\cos\theta$, we have

$$\frac{d}{dt}\left[\frac{\sin^2\{\frac{1}{2}n(t-x\cos\theta)\}}{\sin^2\frac{1}{2}(t-x\cos\theta)}\right] = \frac{d}{du}\,\frac{\sin^2\frac{1}{2}nu}{\sin^2\frac{1}{2}u}$$

$$= \frac{n\sin nu}{2\sin^2\frac{1}{2}u} - \frac{\cos\frac{1}{2}u\,\sin^2\frac{1}{2}nu}{\sin^3\frac{1}{2}u}. \quad (3.9)$$

Now (3.8) may be written

$$\lim_{n\to\infty}\frac{1}{n}\int_0^{\frac{1}{2}\pi}\sin^{2\nu}\theta\,d\theta\int_{-\pi}^{\pi} t\psi(t)\,\frac{d}{du}\,\frac{\sin^2\frac{1}{2}nu}{\sin^2\frac{1}{2}u}\,dt = 0,$$

and we proceed to prove that this can be replaced by

$$\lim_{n\to\infty}\frac{1}{n}\int_0^{\frac{1}{2}\pi}\sin^{2\nu}\theta\,d\theta\int_{x\cos\theta-\delta}^{x\cos\theta+\delta} t\psi(t)\,\frac{d}{du}\,\frac{\sin^2\frac{1}{2}nu}{\sin^2\frac{1}{2}u}\,dt = 0 \quad (0<\delta<\pi). \quad (3.10)$$

From (3.9) we have

$$\int_{x\cos\theta+\delta}^{\pi} t\psi(t)\,\frac{d}{du}\,\frac{\sin^2\frac{1}{2}nu}{\sin^2\frac{1}{2}u}\,dt = I_1 - I_2,$$

where
$$I_1 = \tfrac{1}{2}n\int_{x\cos\theta+\delta}^{\pi} t\psi(t)\,\frac{\sin n(t-x\cos\theta)}{\sin^2\frac{1}{2}(t-x\cos\theta)}\,dt = o(n) \qquad (3.11)$$

by the Riemann-Lebesgue theorem, since

$$\frac{t\psi(t)}{\sin^2\frac{1}{2}(t-x\cos\theta)}$$

is integrable over $(x\cos\theta+\delta,\ \pi)$; and

$$I_2 = \int_{x\cos\theta+\delta}^{\pi} t\psi(t)\,\frac{\cos\frac{1}{2}(t-x\cos\theta)\sin^2\frac{1}{2}n(t-x\cos\theta)}{\sin^3\frac{1}{2}(t-x\cos\theta)}\,dt,$$

so that
$$|I_2| \leqslant \int_{x\cos\theta+\delta}^{\pi} \frac{t|\psi(t)|\,dt}{\sin^3\frac{1}{2}(t-x\cos\theta)} = O(1). \qquad (3.12)$$

From (3.11) and (3.12) it follows that

$$\lim_{n\to\infty}\frac{1}{n}\int_0^{\frac{1}{2}\pi}\sin^{2\nu}\theta\,d\theta\int_{x\cos\theta+\delta}^{\pi} t\psi(t)\,\frac{d}{du}\,\frac{\sin^2\frac{1}{2}nu}{\sin^2\frac{1}{2}u}\,dt = 0. \qquad (3.13)$$

Similarly,

$$\lim_{n\to\infty}\frac{1}{n}\int_0^{\frac{1}{2}\pi}\sin^{2\nu}\theta\,d\theta\int_{-\pi}^{x\cos\theta-\delta} t\psi(t)\,\frac{d}{du}\,\frac{\sin^2\frac{1}{2}nu}{\sin^2\frac{1}{2}u}\,dt = 0. \qquad (3.14)$$

From (3.13) and (3.14), (3.10) follows.
If we put $t-x\cos\theta = u$, $v = \cos\theta$, (3.10) becomes

$$\lim_{n\to\infty}\frac{1}{n}\int_0^1(1-v^2)^{\nu-\frac{1}{2}}dv\int_{-\delta}^{\delta}(u+vx)\,\psi(u+vx)\,\frac{d}{du}\,\frac{\sin^2\frac{1}{2}nu}{\sin^2\frac{1}{2}u}\,du = 0,$$

from which (3.1) at once follows.

4. We now proceed to consider summability at the origin.

THEOREM IV. *Let $f(x)$ satisfy Lipschitz conditions of any order $p>0$ in the intervals $(0,\ \epsilon)$, $(-\epsilon,\ 0)$, ϵ being any arbitrarily small positive number; and let the limits $f(0+0)$ and $f(0-0)$ exist. Then the Schlömilch series is summable $(C,\ 1)$ at the origin to the sum $\frac{1}{2}\{f(0+0)+f(0-0)\}$.*

When $s = \frac{1}{2}\{f(x+0)+f(x-0)\}$, write

$$\chi(u, x) \equiv (vx+u)\psi(vx+u)-(vx-u)\psi(vx-u).$$

Then

$$\chi(u, 0) = u\{\psi(u)+\psi(-u)\}$$

$$= u\left[\frac{\Gamma(\nu+1)\,\Gamma(\frac{1}{2}-\nu)}{\Gamma(\frac{1}{2})}\{f(0+0)+f(0-0)\}-\{\phi(u)+\phi(-u)\}\right]$$

$$= u\int_0^{\frac{1}{2}\pi} \sin^{2\nu+1}\alpha\cos^{-2\nu}\alpha\,\Theta(u, \alpha)\,d\alpha, \qquad (4.1)$$

where

$$\Theta(u, \alpha) \equiv f(0+0)+f(0-0)-f(u\sin\alpha)-f(-u\sin\alpha).$$

Since $f(x)$ satisfies Lipschitz conditions of order $p > 0$ in $(0, \epsilon)$ and $(-\epsilon, 0)$, we have, for $0 \leqslant \alpha \leqslant \frac{1}{2}\pi$,

$$|f(u\sin\alpha)-f(0+0)| < Au^p, \quad |f(-u\sin\alpha)-f(0-0)| < A'u^p,$$

where A and A' are constants.

Hence $|\Theta(u, \alpha)| < (A+A')u^p$, and so, by (4.1),

$$|\chi(u, 0)| < Ku^{p+1},$$

where K is a constant.

Consequently,

$$I \equiv \left|\int_0^\epsilon \chi(u, 0)\,\frac{d}{du}\,\frac{\sin^2\frac{1}{2}nu}{\sin^2\frac{1}{2}u}\,du\right| < K\int_0^\epsilon u^{p+1}\left|\frac{d}{du}\,\frac{\sin^2\frac{1}{2}nu}{\sin^2\frac{1}{2}u}\right|du$$

$$< K'\int_0^\epsilon u^{p+1}\left\{\frac{n|\sin nu|}{2u^2}+\frac{\cos\frac{1}{2}u\sin^2\frac{1}{2}nu}{u^3}\right\}du,$$

by (3.9); but* $\sin^2\frac{1}{2}nu \leqslant \frac{1}{2}nu$ $(0 \leqslant u \leqslant \epsilon)$, so that

$$I < K'\int_0^\epsilon nu^{p-1}\,du = K'n\epsilon^p/p.$$

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* I am indebted to the Referee for the use of the inequality $\sin^2\frac{1}{2}nu \leqslant \frac{1}{2}nu$, which resulted in extending the range of values of p for which the theorem holds from $p > 1$ (which I assumed originally) to $p > 0$.

Hence

$$\left| \frac{1}{n} \int_0^1 (1-v^2)^{\nu-\frac{1}{2}} dv \int_0^\epsilon \chi(u, 0) \frac{d}{du} \frac{\sin^2 \frac{1}{2}nu}{\sin^2 \frac{1}{2}u} du \right| < M\epsilon^p, \tag{4.2}$$

where M is a constant.

Also, if $\epsilon < \delta$, we have, by (3.9),

$$\int_\epsilon^\delta \chi(u, 0) \frac{d}{du} \frac{\sin^2 \frac{1}{2}nu}{\sin^2 \frac{1}{2}u} du = I_1 - I_2,$$

where

$$I_1 = \tfrac{1}{2} \int_\epsilon^\delta \chi(u, 0) \frac{n \sin nu}{\sin^2 \frac{1}{2}u} du = o(n), \tag{4.3}$$

by the Riemann-Lebesgue theorem, since

$$\frac{u}{\sin^2 \frac{1}{2}u} \{\psi(u)+\psi(-u)\}$$

is integrable over (ϵ, δ); and

$$|I_2| = \left| \int_\epsilon^\delta \chi(u, 0) \frac{\cos \frac{1}{2}u \sin^2 \frac{1}{2}nu}{\sin^3 \frac{1}{2}u} du \right|$$

$$\leqslant \int_\epsilon^\delta \frac{u}{\sin^3 \frac{1}{2}u} |\psi(u)+\psi(-u)| du = O(1). \tag{4.4}$$

From (4.3) and (4.4), we have

$$\lim_{n\to\infty} \frac{1}{n} \int_0^1 (1-v^2)^{\nu-\frac{1}{2}} dv \int_\epsilon^\delta \chi(u, 0) \frac{d}{du} \frac{\sin^2 \frac{1}{2}nu}{\sin^2 \frac{1}{2}u} du = 0; \tag{4.5}$$

and, from (4.2) and (4.5), we see that (3.1) is satisfied when $x = 0$ and $s = \frac{1}{2}\{f(0+0)+f(0-0)\}$, since ϵ is arbitrarily small.

This proves the theorem.

5. We finally consider the end-points $\pm\pi$.

THEOREM V. *Let* (i) $f(x)$ *be continuous at* $x = \pm\pi$;

(ii) $\frac{d}{dt}\{t\phi(t)\}$ *be integrable over* $(-\pi, \pi)$, *and bounded in* $(\pi-\delta, \pi)$ *and* $(-\pi, -\pi+\delta)$, *where* δ *is arbitrarily small and positive*;

(iii) $$\phi(\pi)+\phi(-\pi) = 0.$$

Then the Schlömilch series is summable $(C, 1)$ *at* $x = \pi$ *to the sum* $f(\pi)$, *and at* $x = -\pi$ *to the sum* $f(-\pi)$.

When
$$\phi(\pi)+\phi(-\pi) = 0, \tag{5.1}$$

we have*

$$s_n = \frac{1}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}-\nu)} \int_{-\pi}^{\pi} \frac{d}{dt}\{t\phi(t)\} \left\{ \sum_{m=1}^{n} \frac{J_\nu(mx)\cos mt + \mathbf{H}_\nu(mx)\sin mt}{(\frac{1}{2}mx)^\nu} \right\} dt \tag{5.2}$$

$$= \frac{2}{\pi\Gamma(\frac{1}{2}+\nu)\,\Gamma(\frac{1}{2}-\nu)} \int_{-\pi}^{\pi} \frac{d}{dt}\{t\phi(t)\}\,dt \int_0^{\frac{1}{2}\pi} \sin^{2\nu}\theta \sum_{m=1}^{n} \cos\{m(t-x\cos\theta)\}\,d\theta$$

$$= \frac{1}{\pi\Gamma(\frac{1}{2}+\nu)\,\Gamma(\frac{1}{2}-\nu)}$$

$$\times \int_{-\pi}^{\pi} \frac{d}{dt}\{t\phi(t)\}\,dt \int_0^{\frac{1}{2}\pi} \sin^{2\nu}\theta \left[\frac{\sin\{(n+\frac{1}{2})(t-x\cos\theta)\}}{\sin\frac{1}{2}(t-x\cos\theta)} - 1 \right] d\theta$$

$$= \frac{1}{\pi\Gamma(\frac{1}{2}+\nu)\,\Gamma(\frac{1}{2}-\nu)} \int_{-\pi}^{\pi} \frac{d}{dt}\{t\phi(t)\}\,dt \int_0^{\frac{1}{2}\pi} \sin^{2\nu}\theta \frac{\sin\{(n+\frac{1}{2})(t-x\cos\theta)\}}{\sin\frac{1}{2}(t-x\cos\theta)}\,d\theta,$$

since
$$\int_{-\pi}^{\pi} \frac{d}{dt}\{t\phi(t)\}\,dt = 0,$$

by (5.1). Hence

$$\sigma_n = \frac{1}{n\pi\Gamma(\frac{1}{2}+\nu)\,\Gamma(\frac{1}{2}-\nu)}$$

$$\times \int_{-\pi}^{\pi} \frac{d}{dt}\{t\phi(t)\}\,dt \int_0^{\frac{1}{2}\pi} \sin^{2\nu}\theta \frac{\sin^2\{\frac{1}{2}(n+1)(t-x\cos\theta)\}}{\sin^2\frac{1}{2}(t-x\cos\theta)}\,d\theta. \tag{5.3}$$

If (5.1) is not satisfied, s_n [as given by the partial integration leading to (5.2)] tends to ∞ with n when $x = \pm\pi$; the same would therefore† be true of σ_n.

Now, when $f(t) = t$, we have $\phi(t) = \frac{1}{2}t\Gamma(\nu+\frac{3}{2})\,\Gamma(\frac{1}{2}-\nu)$, which satisfies

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* R. G. Cooke, *loc. cit.*, 235.

† P. Dienes, *The Taylor series* (Oxford, 1931), 390, Theorem III.

(5.1); and in that case

$$\sigma_n = \frac{1}{n\pi\Gamma(\frac{1}{2}+\nu)\,\Gamma(\frac{1}{2}-\nu)}$$

$$\times\int_{-\pi}^{\pi}\Gamma(\nu+\tfrac{3}{2})\,\Gamma(\tfrac{1}{2}-\nu)\,t\,dt\int_0^{\frac{1}{2}\pi}\sin^{2\nu}\theta\,\frac{\sin^2\{\frac{1}{2}(n+1)(t-x\cos\theta)\}}{\sin^2\frac{1}{2}(t-x\cos\theta)}\,d\theta. \quad (5.4)$$

But, by (5.2), we have, in this particular case,

$$s_n = \frac{\Gamma(\nu+\frac{3}{2})}{2\Gamma(\frac{1}{2})}\Big[\sum_{m=1}^{n}\frac{J_\nu(mx)}{(\frac{1}{2}mx)^\nu}\int_{-\pi}^{\pi}2t\cos mt\,dt+\sum_{m=1}^{n}\frac{\mathbf{H}_\nu(mx)}{(\frac{1}{2}mx)^\nu}\int_{-\pi}^{\pi}2t\sin mt\,dt\Big]$$

$$= -2\Gamma(\tfrac{1}{2})\,\Gamma(\nu+\tfrac{3}{2})\sum_{m=1}^{n}\frac{(-1)^m\mathbf{H}_\nu(mx)}{m(\frac{1}{2}mx)^\nu},$$

which converges uniformly as $n\to\infty$ in the neighbourhoods of $x=\pm\pi$, being $\Sigma O(m^{-\nu-\frac{3}{2}})$; so that $s_n\to\pm\pi$, and therefore $\sigma_n\to\pm\pi$, when $x=\pm\pi$.

We now suppose that $x=\pi$.

Then, in the above particular case, $\sigma_n=\pi+o(1)$ as $n\to\infty$; hence, by (5.4),

$$\pi+o(1) = \frac{1}{n\pi\Gamma(\frac{1}{2}+\nu)\,\Gamma(\frac{1}{2}-\nu)}$$

$$\times\int_{-\pi}^{\pi}\Gamma(\nu+\tfrac{3}{2})\,\Gamma(\tfrac{1}{2}-\nu)\,t\,dt\int_0^{\frac{1}{2}\pi}\sin^{2\nu}\theta\,\frac{\sin^2\{\frac{1}{2}(n+1)(t-\pi\cos\theta)\}}{\sin^2\frac{1}{2}(t-\pi\cos\theta)}\,d\theta. \quad (5.5)$$

Multiply (5.5) by s/π, and subtract from (5.3); then

$$\sigma_n-s+o(1)$$

$$= \frac{1}{n\pi\Gamma(\frac{1}{2}+\nu)\,\Gamma(\frac{1}{2}-\nu)}\int_{-\pi}^{\pi}\psi(t)\,dt\int_0^{\frac{1}{2}\pi}\sin^{2\nu}\theta\,\frac{\sin^2\{\frac{1}{2}(n+1)\,(t-\pi\cos\theta)\}}{\sin^2\frac{1}{2}(t-\pi\cos\theta)}\,d\theta,$$

where $$\psi(t)=\frac{d}{dt}\{t\phi(t)\}-\frac{st}{\pi}\,\Gamma(\nu+\tfrac{3}{2})\,\Gamma(\tfrac{1}{2}-\nu).$$

Hence the series will be summable $(C, 1)$ to the sum s at $x=\pi$ if

$$\lim_{n\to\infty}\frac{1}{n}\int_0^{\frac{1}{2}\pi}\sin^{2\nu}\theta\,d\theta\int_{-\pi}^{\pi}\psi(t)\,\frac{\sin^2\{\frac{1}{2}(n+1)\,(t-\pi\cos\theta)\}}{\sin^2\frac{1}{2}(t-\pi\cos\theta)}\,dt=0. \quad (5.6)$$

If $\pi\cos\theta+\delta<\pi$ $(\delta>0)$, *i.e.*, if $\theta>\arccos\{1-(\delta/\pi)\}$, then

$$\left|\int_{\pi\cos\theta+\delta}^{\pi}\psi(t)\,\frac{\sin^2\{\frac{1}{2}(n+1)(t-\pi\cos\theta)\}}{\sin^2\frac{1}{2}(t-\pi\cos\theta)}\,dt\right|$$

$$\leqslant\int_{\pi\cos\theta+\delta}^{\pi}\frac{|\psi(t)|\,dt}{\sin^2\frac{1}{2}(t-\pi\cos\theta)}=O(1);$$

and

$$\left|\int_{-\pi}^{\pi\cos\theta-\delta}\psi(t)\,\frac{\sin^2\{\frac{1}{2}(n+1)(t-\pi\cos\theta)\}}{\sin^2\frac{1}{2}(t-\pi\cos\theta)}\,dt\right|=O(1).$$

Hence (5.6) may be replaced by

$$\lim_{n\to\infty}\left[\frac{1}{n}\int_{\arccos\{1-(\delta/\pi)\}}^{\frac{1}{2}\pi}\sin^{2\nu}\theta\,d\theta\int_{\pi\cos\theta-\delta}^{\pi\cos\theta+\delta}\psi(t)\,\frac{\sin^2\{\frac{1}{2}(n+1)(t-\pi\cos\theta)\}}{\sin^2\frac{1}{2}(t-\pi\cos\theta)}\,dt\right.$$

$$\left.+\frac{1}{n}\int_{0}^{\arccos\{1-(\delta/\pi)\}}\sin^{2\nu}\theta\,d\theta\int_{\pi\cos\theta-\delta}^{\pi}\psi(t)\,\frac{\sin^2\{\frac{1}{2}(n+1)(t-\pi\cos\theta)\}}{\sin^2\frac{1}{2}(t-\pi\cos\theta)}\,dt\right]=0,$$

and, when we write $v=\cos\theta$, $t=u+\pi v$, this becomes

$$\lim_{n\to\infty}\left[\frac{1}{n}\int_{0}^{1-(\delta/\pi)}(1-v^2)^{\nu-\frac{1}{2}}\,dv\int_{-\delta}^{\delta}\psi(u+\pi v)\,\frac{\sin^2\frac{1}{2}nu}{\sin^2\frac{1}{2}u}\,du\right.$$

$$\left.+\frac{1}{n}\int_{1-(\delta/\pi)}^{1}(1-v^2)^{\nu-\frac{1}{2}}\,dv\int_{-\delta}^{\pi(1-v)}\psi(u+\pi v)\,\frac{\sin^2\frac{1}{2}nu}{\sin^2\frac{1}{2}u}\,du\right]=0,$$

i.e.,

$$\lim_{n\to\infty}\frac{1}{n}\left[\int_{-\delta}^{\delta}\int_{0}^{1-(\delta/\pi)}+\int_{-\delta}^{0}\int_{1-(\delta/\pi)}^{1}\right.$$

$$\left.+\int_{0}^{\delta}\int_{1-(\delta/\pi)}^{1-(u/\pi)}\right]\frac{\sin^2\frac{1}{2}nu}{\sin^2\frac{1}{2}u}\,\psi(u+\pi v)(1-v^2)^{\nu-\frac{1}{2}}\,du\,dv=0. \quad (5.7)$$

Take $s=f(\pi)$; then we shall prove that, under the conditions of the theorem,

$$\int_{0}^{1}\psi(\pi v)(1-v^2)^{\nu-\frac{1}{2}}\,dv=0. \quad (5.8)$$

Since

$$\int_{0}^{1}(1-v^2)^{\nu-\frac{1}{2}}\,\Gamma(\nu+\tfrac{3}{2})\,\Gamma(\tfrac{1}{2}-\nu)\,\frac{f(\pi)}{\pi}\,\pi v\,dv=\tfrac{1}{2}\Gamma(\tfrac{1}{2}+\nu)\,\Gamma(\tfrac{1}{2}-\nu)f(\pi),$$

(5.8) will follow if we show that

$$\int_{0}^{1}(1-v^2)^{\nu-\frac{1}{2}}\,\frac{d}{dv}\{v\phi(\pi v)\}\,dv=\tfrac{1}{2}\Gamma(\tfrac{1}{2}+\nu)\,\Gamma(\tfrac{1}{2}-\nu)f(\pi). \quad (5.9)$$

By a partial integration, we have

$$\int_0^1 (1-v^2)^{\nu+\frac{1}{2}} \frac{d}{dv}\{v\phi(\pi v)\}\,dv = (2\nu+1)\int_0^1 (1-v^2)^{\nu-\frac{1}{2}} v^2\phi(\pi v)\,dv. \quad (5.10)$$

But we also have*

$$\frac{2}{\Gamma(\frac{1}{2}+\nu)\Gamma(\frac{1}{2}-\nu)}\int_0^{\frac{1}{2}\pi}\cos^{2\nu}\theta\,d\theta\int_0^{x\sin\theta} t\phi(t)\,dt = \int_0^x z^{-2\nu-1}\,dz\int_0^z t^{2\nu+1}f(t)\,dt.$$

Differentiating with respect to x, we have

$$\frac{2x^{2\nu+2}}{\Gamma(\frac{1}{2}+\nu)\Gamma(\frac{1}{2}-\nu)}\int_0^{\frac{1}{2}\pi}\cos^{2\nu}\theta\sin^2\theta\,\phi(x\sin\theta)\,d\theta = \int_0^x t^{2\nu+1}f(t)\,dt,$$

and, repeating this operation, we obtain, almost everywhere,

$$f(x) = \frac{2}{\Gamma(\frac{1}{2}+\nu)\Gamma(\frac{1}{2}-\nu)} \times\int_0^{\frac{1}{2}\pi}\cos^{2\nu}\theta\sin^2\theta[2(\nu+1)\phi(x\sin\theta)+x\sin\theta\,\phi'(x\sin\theta)]\,d\theta.$$

Since $f(x)$ is continuous at $x=\pi$, and $\frac{d}{dt}\{t\phi(t)\}$ is bounded in $(\pi-\delta,\pi)$†, we may put $x=\pi$, so that, on writing $v=\sin\theta$, we get

$$\tfrac{1}{2}\Gamma(\tfrac{1}{2}+\nu)\Gamma(\tfrac{1}{2}-\nu)f(\pi) = \int_0^1 (1-v^2)^{\nu-\frac{1}{2}}v^2[2(\nu+1)\phi(\pi v)+\pi v\phi'(\pi v)]\,dv. \quad (5.11)$$

Subtract (5.10) from (5.11); then

$$\begin{aligned}\tfrac{1}{2}\Gamma(\tfrac{1}{2}+\nu)\Gamma(\tfrac{1}{2}-\nu)f(\pi) &= \int_0^1 (1-v^2)^{\nu-\frac{1}{2}}v^2\frac{d}{dv}\{v\phi(\pi v)\}\,dv+\int_0^1(1-v^2)^{\nu+\frac{1}{2}}\frac{d}{dv}\{v\phi(\pi v)\}\,dv\\ &= \int_0^1(1-v^2)^{\nu-\frac{1}{2}}\frac{d}{dv}\{v\phi(\pi v)\}\,dv,\end{aligned}$$

which is (5.9).

Hence (5.8) is established.

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* R. G. Cooke, *loc. cit.*, 214–215.

† I am indebted to the Referee for pointing out these as the simplest conditions under which (5.11) is valid.

Consequently, since $\frac{d}{dt}\{t\phi(t)\}$ is bounded in $(\pi-\delta, \pi)$, given a positive ϵ, a positive η can be found so that, when $-\eta \leqslant u \leqslant \eta$,

$$\left| \int_0^{1-(u/\pi)} \psi(u+\pi v)(1-v^2)^{\nu-\frac{1}{2}}\, dv \right| \leqslant \epsilon, \tag{5.12}$$

and so that, when $-\eta \leqslant u \leqslant 0$,

$$\left| \int_0^{1} \psi(u+\pi v)(1-v^2)^{\nu-\frac{1}{2}}\, dv \right| \leqslant \epsilon. \tag{5.13}$$

Denoting the integrals in (5.7) by I_1, I_2, I_3, respectively, we have

$$\begin{aligned} I_1+I_2+I_3 = \Big[&\int_{-\eta}^{\eta}\int_0^{1-(u/\pi)} - \int_{-\eta}^{\eta}\int_{1-(\delta/\pi)}^{1-(u/\pi)} + \int_{\eta}^{\delta}\int_0^{1-(\delta/\pi)} + \int_{-\delta}^{-\eta}\int_0^{1-(\delta/\pi)} \\ &+ \int_{-\delta}^{-\eta}\int_{1-(\delta/\pi)}^{1} + \int_{-\eta}^{0}\int_{1-(\delta/\pi)}^{1-(u/\pi)} + \int_{-\eta}^{0}\int_{1-(u/\pi)}^{1} + \int_{\eta}^{\delta}\int_{1-(\delta/\pi)}^{1-(u/\pi)} \\ &+ \int_0^{\eta}\int_{1-(\delta/\pi)}^{1-(u/\pi)} \Big] \frac{\sin^2 \frac{1}{2}nu}{\sin^2 \frac{1}{2}u}\, \psi(u+\pi v)(1-v^2)^{\nu-\frac{1}{2}}\, du\, dv \\ = \Big[&\int_{-\eta}^{\eta}\int_0^{1-(u/\pi)} + \int_{\eta}^{\delta}\int_0^{1-(u/\pi)} + \int_{-\delta}^{-\eta}\int_0^{1} \\ &+ \int_{-\eta}^{0}\int_{1-(u/\pi)}^{1} \Big] \frac{\sin^2 \frac{1}{2}nu}{\sin^2 \frac{1}{2}u}\, \psi(u+\pi v)(1-v^2)^{\nu-\frac{1}{2}}\, du\, dv \\ = I_4+I_5+I_6+I_7, &\quad \text{say.} \end{aligned}$$

Now

$$I_4+I_7 = \Big[\int_0^{\eta}\int_0^{1-(u/\pi)} + \int_{-\eta}^{0}\int_0^{1} \Big] \frac{\sin^2 \frac{1}{2}nu}{\sin^2 \frac{1}{2}u}\, \psi(u+\pi v)(1-v^2)^{\nu-\frac{1}{2}}\, du\, dv,$$

so that, by (5.12) and (5.13),

$$|I_4+I_7| \leqslant K_1 \epsilon n \int_{-\frac{1}{2}n\eta}^{\frac{1}{2}n\eta} \frac{\sin^2 w}{w^2}\, dw < 2K_1 \epsilon n \int_0^{\infty} \frac{\sin^2 w}{w^2}\, dw = K_2 \epsilon n. \tag{5.14}$$

Also, by the second mean-value theorem,

$$\begin{aligned} |I_5| &= \left| \int_{\eta}^{\delta} \frac{\sin^2 \frac{1}{2}nu}{\sin^2 \frac{1}{2}u} \left(\frac{2u}{\pi} - \frac{u^2}{\pi^2}\right)^{\nu-\frac{1}{2}} du \int_{\xi}^{1-(u/\pi)} \psi(u+\pi v)\, dv \right| \quad \left(0 \leqslant \xi \leqslant 1-\frac{u}{\pi}\right) \\ &\leqslant K_3 \int_{\eta}^{\delta} \frac{du}{\sin^2 \frac{1}{2}u} \left(\frac{2u}{\pi} - \frac{u^2}{\pi^2}\right)^{\nu-\frac{1}{2}} \int_{u+\pi\xi}^{\pi} |\psi(w)|\, dw \\ &< K_3 \int_{\eta}^{\delta} \frac{du}{\sin^2 \frac{1}{2}u} \left(\frac{2u}{\pi} - \frac{u^2}{\pi^2}\right)^{\nu-\frac{1}{2}} \int_{\eta}^{\pi} |\psi(w)|\, dw, \end{aligned}$$

so that

$$|I_5| = O(1). \tag{5.15}$$

Finally,

$$|I_6| = \left| \int_0^1 (1-v^2)^{\nu-\frac{1}{2}} dv \int_{-\delta}^{-\eta} \psi(u+\pi v) \frac{\sin^2 \frac{1}{2}nu}{\sin^2 \frac{1}{2}u} du \right|$$

$$\leqslant \frac{1}{\sin^2 \frac{1}{2}\eta} \int_0^1 (1-v^2)^{\nu-\frac{1}{2}} dv \int_{\pi v-\delta}^{\pi v-\eta} |\psi(w)| dw$$

$$< \frac{1}{\sin^2 \frac{1}{2}\eta} \int_0^1 (1-v^2)^{\nu-\frac{1}{2}} dv \int_{-\delta}^{\pi-\eta} |\psi(w)| dw;$$

hence

$$|I_6| = O(1), \tag{5.16}$$

η being fixed. From (5.14), (5.15), (5.16), we see that (5.7) is satisfied.

Similarly for the point $x = -\pi$.

The theorem therefore follows.

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THE TOPOLOGICAL CHARACTERISATION OF AN OPEN LINEAR INTERVAL

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A great deal of work has been devoted to the study of the Jordan arc, that is, a set of points homeomorphic with a closed linear interval. In particular, necessary and sufficient topological conditions for a set to be a Jordan arc have been known for many years*. Very little attention, however, seems to have been given to the corresponding problem for the open linear interval. The theorem here proved was suggested to me by Mr. M. H. A. Newman, who also pointed out that the essential difficulty would lie in establishing an order among the points of the set considered. Since a homeomorphism is an "intrinsic" property of a set of points, it is convenient to consider the given set of points as forming a space. The theorem is then as follows:

If a metric space Γ *is separable, connected, and locally connected, and such that, on removing any point* P *of* Γ, *the remaining set* $\Gamma - P$ *consists of exactly two components, then it is homeomorphic with the interval* $0 < x < 1$.

1. We prove first:

THEOREM A. *If* A, B, C *are any three distinct points of such a space* Γ, *there is one of them which separates the other two.*

By saying that C separates A and B we mean that A and B lie in different components of $\Gamma - C$.

Let A, B be any two distinct points of Γ; let the components of $\Gamma - A$ be denoted by α, α', and those of $\Gamma - B$ by β, β', in such a way that $A \in \beta'$ and $B \in \alpha'$.

(1) α *and* α' *are open sets.* This is obvious.

(2) $\bar{\alpha} = \alpha + A$; *similarly* $\bar{\alpha}' = \alpha' + A$.

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* Hausdorff, *Mengenlehre* (1917), 2 ed., 219 ff.

For since $\alpha\alpha' = 0$, by (1), $\bar{\alpha}\alpha' = 0$; that is, $\bar{\alpha} \subset \Gamma - \alpha' = \alpha + A$. But if $\bar{\alpha} = \alpha$, α would be a proper sub-set of Γ, both open and closed, which is impossible since Γ is connected. Hence we must have $\bar{\alpha} = \alpha + A$.

(3) $\beta \subset \alpha'$; *that is*, $\alpha\beta = 0$.

Suppose, if possible, that β and $\bar{\alpha}$ have a point in common. Then, since β and $\bar{\alpha}$ are both connected, so also is $\beta + \bar{\alpha}$. Again, this set contains A, which belongs to β'; hence $\beta' + \beta + \bar{\alpha}$ is connected. But since B is in α' and therefore not in $\bar{\alpha}$, we see that $\beta' + \beta + \bar{\alpha} = \beta' + \beta = \Gamma - B$, which is, by hypothesis, not connected.

2. In what follows [(4)–(9)] we suppose given a fixed finite set of points $\mathcal{A} = (A, B, \ldots, H)$, with at least one member*. If, for any point P, one component of $\Gamma - P$ contains none of the points $A, B, \ldots, H$, then we call that component ϖ_P, and the other component ϖ'_P†. If Q is any point of ϖ_P, we write $P > Q$. It is clear that this involves that P, Q are distinct points.

(4) *If* $P_1 > Q$ *and* $P_2 > Q$ (P_1, P_2 *being distinct*), *then* $P_1 > P_2$ *or* $P_2 > P_1$.

Suppose that $P_1 > P_2$ is not true, *i.e.* that ϖ_{P_1} does not contain P_2; then, since ϖ_{P_1} is connected, it lies entirely in one component of $\Gamma - P_2$. But ϖ_{P_1} includes Q lying in ϖ_{P_2}; hence $\varpi_{P_1} \subset \varpi_{P_2}$. Then we have, using (2),

$$P_1 \epsilon\, \bar{\varpi}_{P_1} \subset \bar{\varpi}_{P_2} = \varpi_{P_2} + P_2.$$

That is, $P_1 \epsilon\, \varpi_{P_2}$, and so $P_2 > P_1$.

(5) $P_1 > P_2$ *and* $P_2 > P_1$ *cannot both be true.*

If $P_1 > P_2$, so that P_2 lies in ϖ_{P_1}, then $\bar{\varpi}'_{P_1} = \varpi'_{P_1} + P_1$ is a connected sub-set of $\Gamma - P_2$ which contains $A, B, \ldots, H$ and P_1‡. It follows that ϖ'_{P_2} exists and contains P_1; so that $P_2 > P_1$ is not true.

(6) *If* $P_1 > Q$ *and* $P_2 > P_1$, *then* $P_2 > Q$.

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* The argument of this paragraph may be compared with that used by Sierpinski, *Annali di Mat.* (3), 26 (1917), 131–150, especially 135–139. His $S(p)$ is defined in a way very similar to our $\bar{\varpi}_P$, but his proofs of its properties require the hypothesis that the space is compact.

† ϖ_P, ϖ'_P will not in general be defined for every P. The definition may apply even if P coincides with a point of $\mathcal{A}$, but it breaks down if $\mathcal{A}$ consists of only one point A, and P coincides with A; in this case we do not define ϖ_P.

‡ This statement is true, as is easily seen, even if P_1 coincides with a point of $\mathcal{A}$.

By (5), ϖ_{P_1} does not include P_2. Hence $\bar{\varpi}_{P_1} = \varpi_{P_1} + P_1$ is a connected sub-set of $\Gamma - P_2$, which includes the point P_1 of ϖ_{P_2}. Thus $\bar{\varpi}_{P_1} \subset \varpi_{P_2}$; and so Q, as a point of ϖ_{P_1}, lies in ϖ_{P_2}. That is, $P_2 > Q$.

(7) *If* $P_1 > Q_0$, $P_2 > Q_0$, $P_1 > Q_1$, $P_2 > Q_2$, *then either* $P_1 > Q_2$ *or* $P_2 > Q_1$.

If P_1, P_2 coincide, the statement is obviously true. Suppose then that P_1, P_2 are distinct. Since $P_1 > Q_0$ and $P_2 > Q_0$, by (4), either $P_1 > P_2$ or $P_2 > P_1$. If $P_1 > P_2$, then, by (6), we have $P_1 > Q_2$. Similarly, if $P_2 > P_1$, then $P_2 > Q_1$.

Take now any fixed point Q_0 such that $P > Q_0$ for some P (if such a point Q_0 exists). Q_0 clearly cannot be a point of $\mathcal{A}$. Write

$$\Pi = \Pi_{Q_0} = \sum_{P > Q_0} \varpi_P.$$

As the sum of open connected sets ϖ_P all containing Q_0, Π is an open connected set.

(8) $\bar{\Pi}_{Q_0} - \Pi_{Q_0}$ *contains exactly one point.*

Suppose, if possible, that $\bar{\Pi} = \Pi$; then, since Π is both open and closed, we must have $\Pi = \Gamma$. But this is impossible, since a point A of $\mathcal{A}$ cannot lie in any ϖ_P, and so cannot lie in Π.

Now suppose, if possible, that there exist two distinct points R_1, R_2 of $\bar{\Pi} - \Pi$. Take non-overlapping neighbourhoods U_1 of R_1, U_2 of R_2. Since Γ is locally connected, there exist connected sets $C_1 \subset U_1$, $C_2 \subset U_2$, of which R_1, R_2 respectively are interior points. Then there must exist points Q_1, Q_2 of Π, lying in C_1, C_2 respectively, since R_1, R_2 lie in $\bar{\Pi}$. That is, there exist points P_1, P_2 such that $P_1 > Q_0$ and $P_1 > Q_1$, $P_2 > Q_0$ and $P_2 > Q_2$. By (7), either ϖ_{P_1} or ϖ_{P_2} (let us suppose ϖ_{P_1}) contains Q_0, Q_1, and Q_2. P_1 cannot lie in both U_1 and U_2; suppose that P_1 is not in U_1. Then

$$C_1 \varpi_{P_1} \supset Q_1 \neq 0, \quad C_1 - \varpi_{P_1} \supset C_1 - \Pi \supset R_1 \neq 0.$$

Since P_1 is not in U_1 and *a fortiori* not in C_1, we have $C_1 - \varpi_{P_1} = C_1 \varpi'_{P_1}$. Hence $C_1 \varpi_{P_1}$ and $C_1 - \varpi_{P_1}$ are disjunct*, and so C_1 cannot be connected. If P_1 is in U_1 but not in U_2, we show similarly that C_2 cannot be connected. This contradiction proves the lemma (8).

(9) *If* P_0 *is the single point of* $\bar{\Pi}_{Q_0} - \Pi_{Q_0}$, *then* $P > P_0$ *is false for all* P; *and* $\varpi_{P_0} = \Pi_{Q_0}$, *except in the case when* $P_0 = A$ *and there are no other points of* $\mathcal{A}$.

* Two sets X, Y are said to be disjunct if $\bar{X}Y = X\bar{Y} = 0$.

Π is connected, open, and closed in $\Gamma - P_0$; that is, it is a component of $\Gamma - P_0$. Since Π contains no point of the set $\mathcal{A}$, we have $\Pi = \varpi_{P_0}$, unless $P_0 = A$ and $\mathcal{A}$ contains no other point. If $P_0 = A \in \mathcal{A}$, clearly $P > P_0$ is false for all P. In the other case, we have just shown that $\Pi = \varpi_{P_0}$, and if $P > P_0$, we should have $P > Q_0$, by (6), and so $\Pi_{Q_0} \supset \varpi_P \supset P_0$, which is in contradiction with $P_0 \in \bar{\Pi} - \Pi$.

3. Now suppose, if possible, that there exist three points A, B, C of Γ, none of which separates the other two.

Apply the results of § 2, taking the set $\mathcal{A}$ to consist of the two points A, B. If there exists no point $P > C$, write $C_0 = C$. If, however, there exist such points P, we can apply (9) with C as Q_0; there exists a point $P_0 > C$ such that there is no P with $P > P_0$. P_0 cannot coincide with A, since, by supposition, A does not separate B from C; similarly P_0 does not coincide with B. In this case write $C_0 = P_0$. A, B, and C_0 are thus three distinct points. We say that

(10) (*a*) *None of* A, B, C_0 *separates the other two.*

(*b*) *No point can separate* C_0 *from both* A *and* B.

(10*b*) follows at once from the definition of C_0. We consider (10*a*). If $C_0 = C$, the result is trivial, so we suppose that $C_0 \neq C$. Then $C_0 > C$ with relation to A, B; that is, C lies in one component of $\Gamma - C_0$, and A, B lie in the other. Thus C_0 cannot separate A and B. Again, with the notation of § 2, $\bar{\varpi}_{C_0}$ is a connected sub-set of $\Gamma - A$ which contains C and C_0. Thus C and C_0 belong to the same component of $\Gamma - A$. Since B and C lie in the same component of $\Gamma - A$, so do B and C_0; that is, A does not separate B and C_0. Similarly, B does not separate A and C_0.

Now we can define B_0, in relation to A, C_0, and B, just as we have defined C_0 in relation to A, B, and C.

(11) (*a*) *None of* A, B_0, C_0 *separates the other two.*

(*b*) *No point can separate* B_0 *from both* A *and* C_0.

(*c*) *No point can separate* C_0 *from both* A *and* B_0.

(*d*) *No point can separate* B_0 *from* C_0.

(*a*) and (*b*) are merely repetitions of (10), and (*d*) is an obvious consequence of (*a*), (*b*), and (*c*), since any point $P \neq A$ which separates B_0 from C_0 must separate A from one of these two points. It remains to prove (*c*). Suppose, if possible, that there exists a point D such that

$\Gamma - D$ has components $\delta \supset C_0$ and $\delta' \supset (B_0) + (A)$. Consider the set ϖ_{B_0} relative to A, C_0. (i) If D is not in ϖ_{B_0}, then $\bar{\varpi}_{B_0}$ is a connected sub-set of $\Gamma - D$ which includes B and B_0. Therefore $B \epsilon \delta'$ and so D separates C_0 from both A and B, which is impossible. (ii) If D is in ϖ_{B_0}, then $\bar{\varpi}'_{B_0}$ is a connected sub-set of $\Gamma - D$ which includes A, C_0, and B_0. This is impossible, since $C_0 \epsilon \delta$ and $A \epsilon \delta'$.

4. Now take B_0, C_0 as the set $\mathcal{A}$ of § 2. By (11), (*d*), ϖ_P is defined for any point P. Thus, by (9), the whole space can be divided up into:

(i) points P_0 such that there is no $P > P_0$;

(ii) the sets ϖ_{P_0}, each of which is open and such that $\bar{\varpi}_{P_0} = \varpi_{P_0} + P_0$.

If P_0, P_1 are distinct points of the set (i), then, since P_0 is not in ϖ_{P_1} and P_1 not in ϖ_{P_0}, by (3), $\varpi_{P_0} \varpi_{P_1} = 0$. Hence, the space being separable and each ϖ_{P_0} being open, we have

(12) *There are at most* $\aleph_0$ *points* P_0.

But there are at least two such points, B_0, C_0 themselves. To obtain the desired contradiction it is sufficient to prove

(13) *The set of points* P_0 *is connected.*

Suppose, if possible, that this set can be written as $E_1 + E_2$, where

$$\bar{E}_1 E_2 = E_1 \bar{E}_2 = 0.$$

Write

$$\mathfrak{E}_1 = E_1 + \sum_{P \epsilon E_1} \varpi_P,$$

$$\mathfrak{E}_2 = E_2 + \sum_{P \epsilon E_2} \varpi_P.$$

Clearly $\mathfrak{E}_1 \mathfrak{E}_2 = 0$ and $\mathfrak{E}_1 + \mathfrak{E}_2 = \Gamma$. We say that $\bar{\mathfrak{E}}_1 \mathfrak{E}_2 = \mathfrak{E}_1 \bar{\mathfrak{E}}_2 = 0$. For any P of E_2, ϖ_P is an open set such that $\mathfrak{E}_1 \varpi_P = 0$ and so $\bar{\mathfrak{E}}_1 \varpi_P = 0$. It follows that $\bar{\mathfrak{E}}_1 \mathfrak{E}_2 = \bar{\mathfrak{E}}_1 E_2$. Suppose, if possible, that a point P_0 lies in $\bar{\mathfrak{E}}_1 E_2$ (and therefore not in $\bar{E}_1$). Let U be a neighbourhood of P_0 such that $U E_1 = 0$. There is a connected set C lying in U and having P_0 as an interior point. Hence there is a point Q of $\mathfrak{E}_1$ lying in C; suppose that $Q \epsilon \varpi_{P_1}$, where $P_1 \epsilon E_1$. P_1 is not in U and therefore not in C; hence $C \varpi_{P_1}$ and $C \varpi'_{P_1}$ are disjunct sets which together make up C. Also we have $C \varpi_{P_1} \supset Q \neq 0$, $C \varpi'_{P_1} \supset P_0 \neq 0$, since P_0 is in E_2; this contradicts the connectedness of C. Hence $\bar{\mathfrak{E}}_1 \mathfrak{E}_2 = \bar{\mathfrak{E}}_1 E_2 = 0$; similarly $\mathfrak{E}_1 \bar{\mathfrak{E}}_2 = 0$, and so we have proved (13)*. Since, however, in a metric space any connected

* For, since Γ is connected, this means that one of $\mathfrak{E}_1$, $\mathfrak{E}_2$ must vanish.

set containing two points must contain at least c points, (13) itself contradicts (12). Thus the original supposition of the existence of A, B, C must be false; that is, we have proved Theorem A.

5. It may be pointed out that each of the two conditions, separability and local connectedness, is necessary for the truth of Theorem A. As an example of a separable but not locally connected space, in which Theorem A is not true, consider the set of points in the Euclidean plane given by

$$-1 < x < 0, \quad y = \sin 1/x;$$
$$0 \leqslant x < 1, \quad y = 1;$$
$$0 \leqslant x < 1, \quad y = -1.$$

As an example of a non-separable locally-connected space, consider the space formed by points (r, θ), $1 \leqslant r < 2$, $0 \leqslant \theta < 2\pi$; the distance between the points (r_1, θ_1) and (r_2, θ_2) being defined as

$$(r_1 - 1) + (r_2 - 1) + \min [|\theta_1 - \theta_2|, 2\pi - |\theta_1 - \theta_2|]$$

if $\theta_1 \neq \theta_2$, and as $|r_1 - r_2|$ if $\theta_1 = \theta_2$. (This space may be schematically represented by a circle, to each point of which is attached a straight line in such a way that any two straight lines are in different dimensions, the whole space being c-dimensional.) It may easily be verified that both these spaces have the property of falling into two components if any one point is removed, but neither of them satisfies Theorem A.

6. THEOREM B. *Γ is homeomorphic with an open linear interval.*

Take any point A of Γ, and let α, α' be the components of $\Gamma - A$. It is sufficient to show that $\bar{\alpha}$ is homeomorphic with the interval $0 < x \leqslant \frac{1}{2}$, for then, by the same arguments, $\bar{\alpha}'$ is homeomorphic with $\frac{1}{2} \leqslant x < 1$, and, by (2), $\bar{\alpha}\bar{\alpha}'$ consists only of the point A, which corresponds to $x = \frac{1}{2}$ in both homeomorphisms.

We make the definitions of § 2 with A as the only point of $\mathcal{A}$. By (3), if $P \in \alpha$, $\varpi_P \subset \alpha$. Since Γ is separable, there exists an enumerable set $\mathfrak{R}$ which is (metrically) dense in α.

(14) *For any distinct P, Q of α, either $P > Q$ or $Q > P$.*

For A does not separate P, Q; hence, by Theorem A, either P separates A, Q (that is, $P > Q$) or Q separates A, P.

(15) *Given two points $P_1 > P_2$ of α, there exist points R_1, R_2, R_3 of $\mathfrak{R}$ such that $R_1 > P_1 > R_2 > P_2 > R_3$.*

Since Γ is connected, so that no set except Γ itself is both open and closed, it follows, from the inclusions*

$$\alpha \supset \bar{\varpi}_{P_1}, \quad \varpi_{P_1} \supset \bar{\varpi}_{P_2},$$

that $\alpha - \bar{\varpi}_{P_1}$, $\varpi_{P_1} - \bar{\varpi}_{P_2}$, ϖ_{P_2} are each non-vanishing open sub-sets of α, and so contain points R_1, R_2, R_3, respectively, of $\mathfrak{R}$. $R_2 \epsilon (\varpi_{P_1} - \bar{\varpi}_{P_2})$ and $R_3 \epsilon \varpi_{P_2}$ show at once that $P_1 > R_2$ and $P_2 > R_3$. Again, since R_1 is not in ϖ_{P_1} and R_2 not in ϖ_{P_2}, we have $P_1 \not> R_1$ and $P_2 \not> R_2$; hence, by (14) and (6), we have the result required.

(16) *If* $\alpha = E_1 + E_2$, *where* $E_1 E_2 = 0$, $E_1 \neq 0$, $E_2 \neq 0$, *and* $P_1 > P_2$ *whenever* $P_1 \epsilon E_1$, $P_2 \epsilon E_2$; *then* $\bar{E}_1 E_2 + E_1 \bar{E}_2$ *consists of a single point* P_0, *and* P_0 *is either the "greatest" point of* E_2 *or the "least" point of* E_1 (*in the sense of the relation* $P_1 > P_2$).

(*a*) Suppose that there exists a point P_0 which is the "greatest" point of E_2; then, clearly†, $E_2 = \bar{\varpi}_{P_0}$, $E_1 = \alpha - \bar{\varpi}_{P_0} = \alpha \varpi'_{P_0}$, or, again,

$$E_1 = \varpi'_{P_0} - \bar{\alpha}'.$$

Hence $\bar{E}_1 \supset \bar{\varpi}'_{P_0} - \bar{\alpha}' \supset P_0$, and so $\bar{E}_1 E_2 \supset P_0$. On the other hand,

$$\bar{E}_1 E_2 \subset \bar{\varpi}'_{P_0} . E_2 = P_0,$$

and clearly $E_1 \bar{E}_2 = E_1 E_2 = 0$. Hence $\bar{E}_1 E_2 + E_1 \bar{E}_2 = (P_0)$.

(*b*) Suppose that there exists no "greatest" point of E_2. Let Q_0, Q_1 be any two points of E_2, then there exist points P of E_2 such that $P > Q_0$ and $P > Q_1$. Letting Q_1 vary, we see that we can write‡

$$E_2 = \sum_{\substack{P > Q_0 \\ P \epsilon E_2}} \varpi_P,$$

so that E_2 is an open connected set. Arguing just as in the proof of (8) and (9), we now show that there exists a point P_0 such that $\bar{E}_2 - E_2 = (P_0)$ and E_2 is a component of $\Gamma - P_0$. Since E_2 is a *proper* sub-set of α, this means that $P_0 \epsilon \alpha$ and $\varpi_{P_0} = E_2$; and, since $P_0 \notin E_2$, we have $P_0 \epsilon E_1$. Let P_1 be any other point of E_1; since $\varpi_{P_0} = E_2$ we have $P_0 \not> P_1$, and so by (14) $P_1 > P_0$. That is, P_0 is the "least" point of E_1. Finally, $\bar{E}_1 E_2 = 0$, $E_1 \bar{E}_2 = P_0$ (since E_2 is open and $E_1 E_2 = 0$), so that $E_1 E_2 + E_1 \bar{E}_2 = P_0$.

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* Compare (6).

† ϖ_{P_0} lies in α, and cannot contain any point of E_1; hence $\varpi_{P_0} \subset E_2$.

‡ Using the same argument as in the note above.

From (5), (14), (15), and (16) we now have

(17) *With the ordering relation "$<$", α is ordinarily similar to the interval* $0 < x < \frac{1}{2}$*.

Now choose one of the possible ordinal correlations of α with $0 < x < \frac{1}{2}$, and complete it to one of $\bar{\alpha}$ with the interval $0 < x \leqslant \frac{1}{2}$, by writing $\varpi_A = a$, $A > P$ for all P of α.

(18) *The ordinal correlation is also topological.*

(a) Let P_0 be any point of α, and U a neighbourhood of P_0. Since Γ is locally connected, there exists a connected set C such that P_0 is interior to C and $C \subset U$.

Since $P_0 \epsilon \overline{\varpi}_{P_0}$ and $P_0 \epsilon \alpha \overline{\varpi}'_{P_0})$, there exist points P_1, P_2 lying in $C\alpha$ such that $P_1 < P_0 < P_2$. We say that whenever $P_1 < P < P_2$, then $P \epsilon U$. For if P is outside U, C is a connected sub-set of $\Gamma - P$, so that P_1 and P_2 lie in the same component of $\Gamma - P$, which is inconsistent with $P_1 < P < P_2$.

(b) Let P_0 be any point of α, and let P_1, P_2 of α be given so that $P_1 < P_0 < P_2$. Since P_0 is not in ϖ_{P_1} and so not in $\overline{\varpi}_{P_1}$, and, similarly, not in $\overline{\varpi}'_{P_2}$, there exists a neighbourhood U of P_0 such that, if $P \epsilon U$, P is not in $\overline{\varpi}_{P_1}$ or $\overline{\varpi}'_{P_2}$, that is, $P_1 < P < P_2$.

(a) and (b) together show that *any* ordinal correlation of α with $0 < x < \frac{1}{2}$ is also topological. The special point A can be treated in a similar way, and this shows that $\alpha + A$ is homeomorphic with $0 < x \leqslant \frac{1}{2}$. As already pointed out, this is sufficient to prove the theorem.

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* Hausdorff, *loc. cit.*, 53, 54.

ON THE CONVERGENCE OF TRANSFORM INTEGRALS

By G. W. MORGAN.

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1. In the general theory of orthogonal developments there has hitherto, with one exception, been no point convergence theory which is independent of very special assumptions about the orthogonal system concerned. The theory of point convergence is made to depend on the Dirichlet kernel, and this aspect has been studied at length by Hobson‡, Lebesgue§, Hahn‖, and others. Recently a theory of transform integrals of the Fourier type has been developed by Watson¶ and others, corresponding closely to the theory of orthogonal developments. The Plancherel theorem concerning Fourier transforms in the L^2 domain is true also of any kernel $K(x)$ whose integral

$$K_1(x) = \int_0^x K(u)\,du$$

satisfies the equation (2.2) below, and it is to such functions $K(x)$ that the title "Fourier kernel" is applied.

There is, however, one theorem due to Hardy and Titchmarsh††, concerning transform integrals of the Fourier type, which is not dependent on the Dirichlet kernel. Roughly it is that, if a function is the integral of a function of L^2 and if $K_1(x)$ satisfies certain order conditions, then the transform integrals exist as Cauchy integrals. It is my object to find

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‡ E. W. Hobson, "On a general convergence theorem, etc.", *Proc. London Math. Soc.* (2), 6 (1908), 349–395.

§ H. Lebesgue, "Sur les intégrales singulières", *Ann. de la Fac. des Sci. de Toulouse* (3), 1 (1909).

‖ H. Hahn, *Wiener Denkschrift*, 93 (1916), 585–692; "Über Folgen lineare Operationen", *Monatshefte für Math.*, 32 (1922), 3–88.

¶ G. N. Watson, "General transforms", *Proc. London Math. Soc.* (2), 35 (1933), 156–199.

†† G. H. Hardy and E. C. Titchmarsh, "A class of Fourier kernels", *Proc. London Math. Soc.* (2), 35 (1933), 116–155 (153).

other theorems of this nature, in which the formula

$$\int_0^\infty K(xu) \int_0^\infty K(ut) f(t)\, dt\, du = f(x) \tag{1.1}$$

is true in some sense, and in which the conditions bearing on $f(x)$ are more nearly those familiar in the classical theory of the Fourier integral. In this direction I have proved the following results.

THEOREM 1. *Let $K(x)$ be a Fourier kernel, such that $K(x)$, $K_1(x)$, and $K_2(x)$† are bounded in $(0, \infty)$; and let*

$$f(x) = \frac{1}{x} \int_0^x \phi(y)\, dy,$$

where $\phi(y)$ and $y^{-\frac{1}{2}} \phi(y)$ belong to $L(0, \infty)$. Then (1.1) is true for all positive x, each integral being a Cauchy integral at each limit.

THEOREM 2. *If $K(x)$ satisfies the same conditions as in Theorem 1, and if $f(x)$ is absolutely integrable, of bounded variation, and continuous in $(0, \infty)$, then (1.1) is true, the inner integral being an ordinary Lebesgue integral, and the outer integral being a Cauchy integral at infinity.*

Theorem 1, in statement and proof, is derivative from the theorem of Hardy and Titchmarsh referred to above. In the proof of Theorem 2 appeal is made to the theory of "convergence in the large" (in its final Rademacher-Kaczmarz form‡, which is at the same time the most delicate and the most concise). It does not seem to me either that I have used this powerful weapon with due economy, or that I have reached the final truth in the matter. In particular it is desirable that the restriction of continuity of $f(x)$ should be removed from Theorem 2. This could be done if one could prove the O-Tauberian theorem of which Lemma 2 is the o-version.

This lemma is due to Prof. Hardy, whom I should like to thank for its inclusion here.

2. LEMMA 1. *If $K(x)$ is a Fourier kernel satisfying the conditions of Theorems 1 and 2, then*

$$\int_0^\infty \frac{K_2(xu)\, K(tu)}{u^2}\, du = \begin{cases} 0 & (x \leqslant t), \\ x - t & (x \geqslant t), \end{cases} \tag{2.1}$$

the convergence being bounded and uniform in any finite domain of x and t.

† $K_2(x)$ is the integral of $K_1(x)$.

‡ See S. Kaczmarz, "Sur la convergence et la sommabilité des développements orthogonales", *Studia Math.*, 1 (1929), 87–121.

We take the equation

$$\int_0^\infty \frac{K_1(xu)\,K_1(tu)}{u^2}\,du = \min(x, t), \tag{2.2}$$

and integrate by parts. Since the integrated terms vanish, we obtain

$$\frac{t}{x}\int_0^\infty \frac{K_2(xu)\,K(tu)}{u^2}\,du = \frac{2}{x}\int_0^\infty \frac{K_2(xu)\,K_1(tu)}{u^3}\,du - \min(x, t),$$

and, by uniform convergence,

$$\begin{aligned}\int_0^\infty \frac{K_2(xu)\,K_1(tu)}{u^3}\,du &= \int_0^x dy \int_0^\infty \frac{K_1(yu)\,K_1(tu)}{u^2}\,du \\ &= \tfrac{1}{2}x^2 \quad (x \leqslant t), \\ &= (x - \tfrac{1}{2}t)t \quad (x \geqslant t);\end{aligned}$$

and therefore the lemma is established.

3. LEMMA 2. *If*

(i) $L(x)$ *and* $L''(x)$ *are bounded,*

(ii) $\psi(x)$ *tends to zero when* x *tends to infinity,*

(iii) $M(x) = \int_0^\infty \frac{L(xu)\,\psi(u)}{u^2}\,du$ *is differentiable,*

then

$$M'(x) = \int_0^\infty \frac{L'(xu)\,\psi(u)}{u}\,du,$$

the integral being a Cauchy integral at the upper limit.

Proof. Since

$$L\{u(x+h)\} - L(ux) = uhL'(xu) + O(u^2h^2),$$

it follows that

$$\begin{aligned}\frac{M(x+h)-M(x)}{h} &= \int_0^n \{L(xu+hu)-L(xu)\}\,\frac{\psi(u)}{hu^2}\,du \\ &\quad + \int_n^\infty \frac{L(xu+hu)\,\psi(u)}{hu^2}\,du - \int_n^\infty \frac{L(xu)\,\psi(u)}{hu^2}\,du \\ &= \int_0^n \frac{L'(xu)\,\psi(u)}{u}\,du + O\left(h\int_0^n |\psi(u)|\,du\right) + O\left(\frac{1}{nh}\max_{u\geqslant n}\{|\psi(u)|\}\right).\end{aligned}$$

From this, on choosing n so that

$$nh = \sqrt{\left(\max_{u \geqslant n} \{|\psi(u)|\}\right)}$$

say, we see that

$$M'(x) = \lim_{n \to \infty} \int_0^n u^{-1} L'(xu)\, \psi(u)\, du.$$

4. *Proof of Theorem* 1. Let us write

$$\psi(x) = \int_0^\infty \phi(y)\, K(xy)\, dy; \tag{4.1}$$

then we have

$$\begin{aligned} \int_0^\infty \frac{K_2(xu)\,\psi(u)}{u^2}\, du &= \int_0^\infty \frac{K_2(xu)}{u^2} \int_0^\infty K(uy)\,\phi(y)\, dy\, du \\ &= \int_0^\infty \phi(y) \int_0^\infty \frac{K_2(xu)\, K(yu)}{u^2}\, du\, dy \\ &= \int_0^x (x-y)\,\phi(y)\, dy, \end{aligned} \tag{4.2}$$

on appeal to Lemma 1, the inversion being justified by a theorem given by Hobson†.

We show next that

$$g(u) = \lim_{X \to \infty} \int_u^X \frac{\psi(v)}{v}\, dv \tag{4.3}$$

exists. In the first place,

$$\begin{aligned} \int_u^X \frac{\psi(v)}{v}\, dv &= \int_u^X \frac{1}{v} \int_0^\infty K(vy)\,\phi(y)\, dy\, dv \\ &= \int_0^\infty \phi(y) \int_u^X \frac{K(vy)}{v}\, dv\, dy, \end{aligned} \tag{4.4}$$

where we appeal again to the result in Hobson, just quoted; and, secondly,

$$\int_u^X \frac{K(vy)}{v}\, dv = \left[\frac{K_1(vy)}{vy}\right]_u^X + \int_u^X \frac{K_1(vy)}{v^2 y}\, dv$$

is boundedly convergent when X tends to infinity, for all $u \geqslant u_0 > 0$. We

† E. W. Hobson, *Theory of functions*, 2 (1926), 347.

may therefore proceed to the limit, X tending to infinity, on both sides of (4.4), and we have

$$g(u) = \int_u^\infty \frac{\psi(v)}{v}\,dv = \int_0^\infty \phi(y)\,\lambda(uy)\,dy, \tag{4.5}$$

where

$$\lambda(u) = \int_u^\infty \frac{K(x)}{x}\,dx = -\frac{K_1(u)}{u} + \int_u^\infty \frac{K_1(x)}{x^2}\,dx. \tag{4.6}$$

Concerning the behaviour of $g(u)$, we see that

$$\begin{cases} g(u) = O\left(\log \dfrac{1}{u}\right) & (u \to 0), \\ g(u) = O(u^{-\frac{1}{2}}) & (u \to \infty). \end{cases} \tag{4.7}$$

The first is trivial, since ψ is bounded. We shall prove the second. From the conditions of boundedness we see, on applying Schwarz's inequality, that

$$\lambda(u) = -\frac{K_1(u)}{u} + \int_u^\infty \frac{K_1(x)}{x^2}\,dx = O(u^{-\frac{1}{2}}),$$

and therefore

$$|g(u)| = \left| \int_0^\infty \phi(y)\,\lambda(uy)\,dy \right| < A \int_0^\infty |\phi(y)|\,(uy)^{-\frac{1}{2}}\,dy = O(u^{-\frac{1}{2}}).$$

From Lemma 2 applied to (4.2) and from (4.5), we now conclude that

$$\begin{aligned} \int_0^x \phi(y)\,dy &= -\int_0^\infty K_1(xu)\,g'(u)\,du \\ &= \lim_{\delta \to 0,\ \Delta \to \infty} \left(-\Big[K_1(xu)\,g(u) \Big]_\delta^\Delta + x \int_\delta^\Delta K(xu)\,g(u)\,du \right), \end{aligned}$$

and the integrated terms vanish in the limit. Thus

$$f(x) = \frac{1}{x} \int_0^x \phi(y)\,dy = \int_0^\infty K(xu)\,g(u)\,du, \tag{4.8}$$

where the integral is a Cauchy integral at both limits.

We have, combining (4.5) and (4.6),

$$g(u) = \int_0^\infty \phi(v) \left(\int_{uv}^\infty \frac{K_1(t)}{t^2}\,dt - \frac{K_1(uv)}{uv} \right) dv, \tag{4.9}$$

and we may separate the terms of this integral, since

$$\int_0^\infty \phi(v) \int_{uv}^\infty \frac{K_1(t)}{t^2}\, dt\, dv, \qquad \int_0^\infty \phi(v)\, \frac{K_1(uv)}{uv}\, dv$$

are Lebesgue integrals. Thus

$$(4.10) \qquad g_1(u) = \int_0^\infty \phi(v) \int_{uv}^\infty \frac{K_1(t)}{t^2}\, dt\, dv = \int_0^\infty \frac{K_1(t)}{t^2} \int_0^{tu^{-1}} \phi(v)\, dv\, dt$$

$$= \int_0^\infty \frac{K_1(t)}{t^2}\, \frac{t}{u}\, f\left(\frac{t}{u}\right) dt = \frac{1}{u} \int_0^\infty \frac{K_1(ut)}{t}\, f(t)\, dt,$$

and

$$(4.11) \quad g_2(u) = \int_0^\infty \phi(v)\, \frac{K_1(uv)}{uv}\, dv$$

$$= \lim_{\delta \to 0,\ \Delta \to \infty} \left(\left[v f(v)\, \frac{K_1(uv)}{uv} \right]_\delta^\Delta - \int_\delta^\Delta v f(v) \left(\frac{K(uv)}{v} - \frac{K_1(uv)}{uv^2} \right) dv \right)$$

$$= \lim_{\delta \to 0,\ \Delta \to \infty} \int_\delta^\Delta \left(-f(v)\, K(uv) + f(v)\, \frac{K_1(uv)}{uv} \right) dv$$

$$= -\lim \int_\delta^\Delta f(v)\, K(uv)\, dv + g_1(u);$$

and therefore

$$(4.12) \qquad g(u) = \int_0^\infty f(v)\, K(uv)\, dv,$$

the integral being a Cauchy integral at both limits. This completes the proof of the theorem.

5. LEMMA 3. *If $K(x)$ is a Fourier kernel satisfying the conditions of Theorem* 2, *then*

$$\Psi(t, x;\ n) = \int_0^n \frac{K_1(xu)\, K(tu)}{u}\, du$$

is uniformly bounded for all t, x, and n.

By Lemma 1,

$$M(x) = \int_0^\infty \frac{K_2(xu)\, K(tu)}{u^2}\, du = 0 \qquad (x \leqslant t)$$

$$= x - t \quad (x \geqslant t);$$

and thus $M(x)$ has the uniformly bounded incrementary ratio

$$\frac{M(x+h)-M(x)}{h} = \int_0^\infty \frac{K(tu)}{u^2 h}\left(K_2(xu+hu)-K_2(xu)\right)du$$

$$= \int_0^n \left(\frac{K_1(xu)}{u}+hK(xu+\theta hu)\right)K(tu)du$$

$$+\int_n^\infty \frac{K_2(xu+hu)\,K(ut)}{u^2 h}\,du-\int_n^\infty \frac{K_2(xu)\,K(tu)}{u^2 h}\,du.$$

Thus

$$\Psi(t, x;\ n) = \frac{\Delta M(x)}{\Delta(x)}-\int_0^n hK(xu+\theta hu)\,K(tu)\,du$$

$$-\int_n^\infty \frac{K_2(xu+hu)\,K(ut)}{u^2 h}\,du+\int_n^\infty \frac{K_2(xu)\,K(tu)}{u^2 h}\,du$$

$$= O(1)+O(nh)+O\left(\frac{1}{nh}\right) = O(1),$$

provided that we take $h = 1/n$.

6. LEMMA 4. *If*

(i) $f(x)$ *is a function of bounded variation,*

(ii) $\{g_n(x)\}$ *is a sequence of functions such that*

$$\int g_n(x)\,df(x)$$

exists for all n,

(iii) *there is a* $g^*(x)$ *such that, for all* n *and* x,

$$|g_n(x)| \leqslant g^*(x),$$

(iv) $\int g^*(x)\,|df(x)|$ *exists,*

(v) $g_n(x)$ *tends to* $g(x)$ *almost everywhere, and in particular at all points of discontinuity of* $f(x)$,

then

$$\int g_n(x)\,df(x) \to \int g(x)\,df(x).$$

This is a familiar result in the theory of the Lebesgue-Stieltjes integral.

7. We suppose now that $f(x)$ belongs to L^2 $(0, \infty)$. Then, by the general transform theory, $f(x)$ has a transform $g(x)$, also of $L^2(0, \infty)$, and Parseval's equation is true.

Lemmas 5 and 6 and Theorem A are taken, with appropriate modifications, from the paper by Kaczmarz quoted above.

LEMMA 5. *If*

(i) $v(x)$ *is a positive, increasing, unbounded function,*

(ii) $$\int_0^\infty g^2(x)\,v(x)\,dx < \infty,$$

(iii) $$k \leqslant v(n_k) < k+1;$$

then, almost everywhere,

$$f_{n_k}(x) = \int_0^{n_k} g(u)\,K(xu)\,du$$

tends to $f(x)$ *as* k *tends to infinity.*

We write

$$r_n = \int_0^\infty \big(f(x)-f_n(x)\big)^2\,dx = \int_n^\infty g^2(u)\,du$$

(using Parseval's theorem), and therefore

$$\sum_{k=1}^{\infty} r_{n_k} = \sum_{1}^{\infty} (r_{n_k}-r_{n_{k+1}})\,k = \sum_{1}^{\infty} k \int_{n_k}^{n_{k+1}} g^2(u)\,du < \int_{n_1}^{\infty} g^2(u)\,v(u)\,du < \infty.$$

It follows without difficulty that, almost everywhere, $f_{n_k}(x)$ tends to $f(x)$.

LEMMA 6. *There exists a function* $\delta(x) = \delta_n(x)$ *such that*

(i) $$|f_k(x)-f_{2^n}(x)| \leqslant \delta_n(x), \quad k = 2^n+1,\ 2^n+2,\ \ldots,\ 2^{n+1}-1;$$

(ii) $$\int_0^\infty \delta_n{}^2(x)\,dx = n^2 \int_{2^n}^{2^{n+1}} g^2(u)\,du.$$

Let us divide the range $(2^n, 2^{n+1})$ into unit intervals by repeated bisection; then

$$t_k(x) = f_k(x)-f_{2^n}(x) = \sum_{j=1}^{n} \epsilon_j p_j(x) \quad (\epsilon_j = 0,\ 1),$$

where

$$p_j(x) = \int_J g(u)\,K(xu)\,du$$

and J is *one* interval of the j-th bisection. Further,

$$p_j{}^2(x) \leqslant \sum_J \left(\int_J g(u)\,K(xu)\,du\right)^2 = A_j(x),$$

say, where the summation is over *all* intervals of the j-th bisection. Now, from Schwarz's inequality,

$$t_k^2(x) \leqslant n \sum_{j=1}^{n} p_j^2(x) \leqslant n \sum_{j=1}^{n} A_j(x) = \delta_n^2(x),$$

this equation being the definition of $\delta_n(x)$, and therefore

$$\begin{aligned}\int_0^\infty \delta_n^2(x)\,dx &= \int_0^\infty n \sum_{j=1}^{n} \sum_J \left(\int_J g(u)\,K(xu)\,du\right)^2 dx \\ &= n \sum_{j=1}^{n} \sum_J \int_0^\infty \left(\int_J g(u)\,K(xu)\,du\right)^2 dx \\ &= n \sum_{j=1}^{n} \sum_J \int_J g^2(u)\,du \\ &= n^2 \int_{2^n}^{2^{n+1}} g^2(u)\,du.\end{aligned}$$

THEOREM A. *If $g(x)\log x$ belongs to $L^2(0, \infty)$, then, almost everywhere,*

$$f_n(x) = \int_0^n g(u)\,K(xu)\,du \to f(x)$$

when n tends to infinity through integral values.

By writing $v(x) = \log_2 x$ in Lemma 5, we find that the sequence $\{f_{2^n}(x)\}$ converges to $f(x)$ almost everywhere. We have, in the notation of Lemma 6,

$$\sum_{n=1}^{\infty} \int_0^\infty \delta_n^2(x)\,dx = \sum_1^\infty n^2 \int_{2^n}^{2^{n+1}} g^2(u)\,du < C \int_2^\infty \left(g(u)\log u\right)^2 du < \infty,$$

from which it follows that $\Sigma\delta_n^2(x)$ is convergent almost everywhere, or that $\delta_n(x)$ tends to zero almost everywhere. Recalling the first property of $\delta_n(x)$, we conclude that $f_n(x)$ tends to $f(x)$ almost everywhere.

COROLLARY. *If $K(x)$ is bounded, we may take n to be a continuous variable.*

8. LEMMA 7. *If $u^{-1}K_1(u)\log u$ belongs to $L^2(0, \infty)$, then, for fixed x and almost all t,*

$$\lim_{n\to\infty} \Psi(t, x;\ n) = \psi(t, x) = \begin{cases} 1 & (t < x), \\ \frac{1}{2} & (t = x), \\ 0 & (t > x). \end{cases}$$

This is an immediate corollary of Theorem A, in which we write

$$f(t) = \psi(t, x);$$

for then $$g(u) = u^{-1} K_1(ux).$$

9. *Proof of Theorem* 2.

$$f_n(x) = \int_0^n g(u) K(xu)\, du = \int_0^n K(xu) \int_0^\infty f(t) K(ut)\, dt\, du$$

$$= \int_0^\infty f(t) \int_0^n K(ux) K(ut)\, du\, dt = \int_0^\infty f(t)\, d\left(\int_0^n K(ux) \frac{K_1(ut)}{u}\, du \right)$$

$$= -\int_0^\infty \left(\int_0^n K(xu) \frac{K_1(tu)}{u}\, du \right) df(t).$$

Now the inner integral is bounded, and, as we see by a partial integration, is equal to

$$\frac{K_1(xn) K_1(tn)}{nx} + \frac{1}{x} \int_0^n \frac{K_1(xu) K_1(tu)}{u^2}\, du - \frac{t}{x} \int_0^n \frac{K_1(xu) K(tu)}{u}\, du;$$

it therefore converges for almost all values of t to

$$\frac{1}{x} \min(x, t) - \frac{t}{x} \psi(t, x) = \psi(x, t),$$

when n tends to infinity. Therefore, by Lemma 4, since $f(x)$ is continuous,

$$f_n(x) \to -\int_0^\infty \psi(x, t)\, df(t) = f(x).$$

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AN EXTENSION OF THE FOURIER-BESSEL INTEGRAL

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The standard theorem in this subject is given by Watson* in the form:

THEOREM A. *If x is an interior point of an interval in which the function $f(t)$ has bounded variation, and if*

$$\int_0^\infty |f(t)|\,t^{\frac{1}{2}}\,dt < \infty, \tag{1}$$

then, for $\nu \geqslant -\frac{1}{2}$,

$$\int_0^\infty J_\nu(ux)\,u\,du \int_0^\infty f(t)\,J_\nu(ut)\,t\,dt = \tfrac{1}{2}\{f(x+0)+f(x-0)\}.$$

In a foot-note to the theorem, Watson suggests the possibility that the result may be valid for $\nu > -1$. It is the aim of the present paper to establish this extension under a (necessary) further restriction on $f(t)$. We prove

THEOREM B. *If x is an interior point of an interval in which $f(t)$ has bounded variation, and if*

$$\int_0^1 |f(t)|\,t^{\nu+1}\,dt < \infty \quad \text{and} \quad \int_0^\infty |f(t)|\,t^{\frac{1}{2}}\,dt < \infty, \tag{2}$$

then, for $\nu > -1$,

$$\int_0^\infty J_\nu(ux)\,u\,du \int_0^\infty f(t)\,J_\nu(ut)\,t\,dt = \tfrac{1}{2}\{f(x+0)+f(x-0)\}.$$

* Watson, *Theory of Bessel functions* (1922), 456. References to this work will be fairly frequent, and will generally be made to the appropriate paragraph and page, as Watson, § 14.4, 456.

The extra condition in (2) says nothing unless $\nu < -\frac{1}{2}$, so that, in the proof following, we may confine our attention to the case $-1 < \nu < -\frac{1}{2}$.

The method of the proof adheres fairly closely to Watson's account save for a divergence indicated in Lemma 3 below, which applies equally well in the whole range $\nu > -1$, and which seems to make for simplicity and clarity.

It is convenient to divide some of the detail into lemmas. The first is a restatement of Watson, §14.41:

LEMMA 1. *If $\nu \geqslant -\frac{1}{2}$, and $g(t)$ satisfies condition* (1), *then*

$$\int_a^\beta g(t)\, J_\nu(\lambda t)\, t\, dt = o(\lambda^{-\frac{1}{2}})$$

as $\lambda \to \infty$, where $0 \leqslant a < \beta \leqslant \infty$.

LEMMA 2. *If* $\nu < -\frac{1}{2}$, and $g(t)$ satisfies condition* (2), *then*

$$\int_a^\beta g(t)\, J_\nu(\lambda t)\, t\, dt = o(\lambda^{-\frac{1}{2}})$$

as $\lambda \to \infty$, where $0 \leqslant a < \beta \leqslant \infty$.

If $a = 0$, we argue that, in $0 < t < \delta$, λt takes both small and large values, so that

$$|J_\nu(\lambda t)| < A\lambda^\nu t^\nu + A\lambda^{-\frac{1}{2}} t^{-\frac{1}{2}} < A\lambda^{-\frac{1}{2}} t^\nu,$$

since $\nu < -\frac{1}{2}$. Thus

$$\left| \int_0^\delta g(t)\, J_\nu(\lambda t)\, t\, dt \right| < A\lambda^{-\frac{1}{2}} \int_0^\delta |g(t)|\, t^{\nu+1}\, dt < \epsilon \lambda^{-\frac{1}{2}}$$

by choice of δ, sufficiently small.

If $\beta = \infty$,

$$\left| \int_\Delta^\infty g(t)\, J_\nu(\lambda t)\, t\, dt \right| < A\lambda^{-\frac{1}{2}} \int_\Delta^\infty |g(t)|\, t^{\frac{1}{2}}\, dt < \epsilon \lambda^{-\frac{1}{2}}$$

by choice of Δ.

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* There is no lower bound to the permissible values of ν, though in applications in the present paper ν will always exceed -1.

In the interval (δ, Δ), $\lambda^{\frac{1}{2}} t^{\frac{1}{2}} J_\nu(\lambda t)$ satisfies the conditions of Hobson's convergence theorem*, so that

$$\lambda^{\frac{1}{2}} \int_\delta^\Delta g(t)\, J_\nu(\lambda t)\, t\, dt = o(1)$$

as $\lambda \to \infty$. This completes the proof of Lemma 2.

LEMMA 3. *If $x > 0$ is fixed, and $y = y(\lambda) > x$ may vary boundedly as $\lambda \to \infty$, then, for $\nu > -1$, the integral*

$$\int_0^\lambda J_\nu(ux)\, u\, du \int_x^y J_\nu(ut)\, t^{\nu+1}\, dt \tag{3}$$

is bounded as $\lambda \to \infty$. If, further, y is fixed, the integral converges, and

$$\int_0^\infty J_\nu(ux)\, u\, du \int_x^y J_\nu(ut)\, t^{\nu+1}\, dt = \tfrac{1}{2} x^\nu. \tag{4}$$

This is the special case of Theorem B obtained by taking $f(t) = t^\nu$ for $x \leqslant t \leqslant y$, and zero otherwise. The lemma provides a Gegenbeispiel for any projected extension of the theorem to values of ν less than -1, by the imposition of further conditions on $f(t)$, since, although the function here satisfies any conceivable conditions at $t = 0$, yet the integral (4) fails to converge at the lower limit of the outer integration, if $\nu < -1$, as will appear in the course of the proof.

We have†

$$\int_x^y J_\nu(ut)\, t^{\nu+1}\, dt = \frac{1}{u} [y^{\nu+1} J_{\nu+1}(uy) - x^{\nu+1} J_{\nu+1}(ux)],$$

and (3) is expressed as the difference

$$y^{\nu+1} \int_0^\lambda J_\nu(ux)\, J_{\nu+1}(uy)\, du - x^{\nu+1} \int_0^\lambda J_\nu(ux)\, J_{\nu+1}(ux)\, du \tag{5}$$

of two integrals, of which the second is independent of y. Since both integrands are uniformly $O(u^{2\nu+1})$ for small u, and are thus integrable if‡ $\nu > -1$, it is sufficient, in discussing the boundedness of (3), to consider

$$\int_c^\lambda J_\nu(ux)\, J_{\nu+1}(uy)\, du,$$

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* Hobson, *Theory of functions of a real variable*, 2 (1926), 422.

† Watson, § 5.1 (1), 132.

‡ The limitation is obviously necessary at this point.

where c is sufficiently large. If we substitute the asymptotic expansions of the Bessel functions in the form

$$J_\nu(z) = Az^{-\frac{1}{2}} \cos(z - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi) + O(z^{-\frac{3}{2}}),$$

all the resulting integrals are absolutely convergent, independently of y, except the leading term

$$Ay^{-\frac{1}{2}} \int_c^\lambda \cos(ux - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi) \cos(uy - \tfrac{1}{2}\nu\pi - \tfrac{3}{4}\pi)\, u^{-1}\, du$$

$$= Ay^{-\frac{1}{2}} \int_c^\lambda [\sin u(y-x) - \cos\{u(y+x) - \nu\pi\}]\, u^{-1}\, du$$

$$= Ay^{-\frac{1}{2}} \int_{c(y-x)}^{\lambda(y-x)} \frac{\sin u}{u}\, du - Ay^{-\frac{1}{2}} \int_{c(y+x)}^{\lambda(y+x)} \frac{\cos(u - \nu\pi)}{u}\, du.$$

Both parts of the last expression converge for fixed y, and are in any case bounded for bounded y.

To evaluate the infinite integral (4), we use Weber's discontinuous integral*:

$$\int_0^\infty J_\nu(ux)\, J_{\nu+1}(uy)\, du = \begin{cases} x^\nu/y^{\nu+1} & (y > x), \\ 1/(2x) & (y = x). \end{cases}$$

Making λ tend to ∞ in (5), we obtain, for the value of (4),

$$\frac{y^{\nu+1}\, x^\nu}{y^{\nu+1}} - \frac{x^{\nu+1}}{2x} = \tfrac{1}{2}x^\nu,$$

which is the result stated.

Lemma 3 is equivalent to § 14.44 in Watson's account, but the use of $t^{\nu+1}$ in (4), instead of $t^{\frac{1}{2}}$, simplifies the evaluation of the integral†.

We are now in a position to prove Theorem B. The integrand of

$$\int_0^\lambda J_\nu(ux)\, u\, du \int_0^\infty f(t)\, J_\nu(ut)\, t\, dt$$

is majorised by

$$A\,|f|\,\{t^{\frac{1}{2}} + t^{\nu+1}\}\{1 + u^{2\nu+1}\},$$

which is absolutely integrable in $0 < u < \lambda$, $t > 0$, so that we may invert

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* Watson, § 13.42 (8), 406.

† See Watson, § 14.45, 463. The details are rather heavy.

the order of integration. The inner integral is then known*, namely

$$\int_0^\lambda J_\nu(ux)\,J_\nu(ut)\,u\,du = \frac{\lambda}{t^2-x^2}\{t\,J_{\nu+1}(\lambda t)\,J_\nu(\lambda x) - xJ_{\nu+1}(\lambda x)\,J_\nu(\lambda t)\}.$$

Denoting by (α, β) an interval which is either $(0, x-\delta)$ or $(x+\delta, \infty)$ (or any other interval for which x is an exterior point), we have

$$\int_0^\lambda J_\nu(ux)\,u\,du \int_\alpha^\beta f(t)\,J_\nu(ut)\,t\,dt$$

$$= \lambda J_\nu(\lambda x)\int_\alpha^\beta f(t)\,J_{\nu+1}(\lambda t)\,\frac{t^2}{t^2-x^2}\,dt - \lambda x\,J_{\nu+1}(\lambda x)\int_\alpha^\beta f(t)\,J_\nu(\lambda t)\,\frac{t}{t^2-x^2}\,dt$$

$$= I_1 - I_2.$$

But, when $f(t)$ satisfies the conditions of Theorem B, $I_1 = o(1)$ as $\lambda \to \infty$, by Lemma 1, and $I_2 = o(1)$ by Lemma 2. Thus

$$\int_0^\lambda J_\nu(ux)\,u\,du \int_\alpha^\beta f(t)\,J_\nu(ut)\,u\,du = o(1) \tag{6}$$

as $\lambda \to \infty$.

There remains the interval $(x-\delta, x+\delta)$. In $(x, x+\delta)$, the function $t^{-\nu}f(t)$ has bounded variation and so is the difference of two functions, $t^{-\nu}f_1(t)$ and $t^{-\nu}f_2(t)$, which are positive and monotone. It is sufficient to deal with one of these. By the second mean-value theorem, writing y for $x+\delta$, we have

$$\int_x^y f_1(t)\,t^{-\nu}t^{\nu+1}\,dt \int_0^\lambda J_\nu(ux)\,J_\nu(ut)\,u\,du$$

$$= x^{-\nu}f_1(x+0)\int_x^y t^{\nu+1}\,dt \int_0^\lambda J_\nu(ux)\,J_\nu(ut)\,u\,du$$

$$+ \{y^{-\nu}f_1(y-0) - x^{-\nu}f_1(x+0)\}\int_\xi^y t^{\nu+1}\,dt \int_0^\lambda J_\nu(ux)\,J_\nu(ut)\,u\,du,$$

where $x < \xi < y$. By Lemma 3, the first term tends to $\frac{1}{2}f_1(x+0)$ as $\lambda \to \infty$, while the second (expressing the outer integral as the difference of two integrals to which Lemma 3 may be applied) is a bounded multiple of $y^{-\nu}f_1(y-0) - x^{-\nu}f_1(x+0)$, which may be made arbitrarily small by choice of y sufficiently near to x.

Treating the interval $(x-\delta, x)$ in the same way, we conclude that

$$\left| \int_{x-\delta}^{x+\delta} f(t)\,t\,dt \int_0^\lambda J_\nu(ux)\,J_\nu(ut)\,u\,du - \tfrac{1}{2}\{f(x+0)+f(x-0)\} \right| < \epsilon,$$

* Watson, § 5.11 (8), 134.

if λ is sufficiently large and δ sufficiently small. Using (6) to incorporate the intervals $(0, x-\delta)$ and $(x+\delta, \infty)$, we have

$$\left| \int_0^\infty f(t)\, t\, dt \int_0^\lambda J_\nu(ux)\, J_\nu(ut)\, u\, du - \tfrac{1}{2}\{f(x+0)+f(x-0)\} \right| < \epsilon + o(1)$$

The left-hand side being independent of δ, its limit as $\lambda \to \infty$ is zero; that is, re-inverting the repeated integral, we have

$$\lim_{\lambda\to\infty} \left| \int_0^\lambda J_\nu(ux)\, u\, du \int_0^\infty f(t)\, J_\nu(ut)\, t\, dt - \tfrac{1}{2}\{f(x+0)+f(x-0)\} \right| = 0,$$

which is the assertion of Theorem B.

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ON THE PRODUCT OF TWO LEGENDRE POLYNOMIALS WITH DIFFERENT ARGUMENTS

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1. Formulae have been given* which express $(\frac{1}{2}z)^{\kappa-\mu-\nu}J_\mu(az)J_\nu(bz)$ as a Neumann series of Bessel functions and also as a series of the type $\Sigma c_n z^n J_{\kappa+n}(z)$. The best known result of the first type is no doubt Bateman's well-known expansion† of $\frac{1}{2}zJ_\mu(z\cos\phi\cos\Phi)J_\nu(z\sin\phi\sin\Phi)$ as a Neumann series, the coefficients involving products of ordinary hypergeometric series.

Recently Dhar‡ gave a formula expressing the product of two parabolic cylinder functions $D_m(y\sin\alpha)D_n(y\cos\alpha)$ as a sum of the form $\Sigma c_r D_{n+m-2r}(y)$, where c_r involves a hypergeometric series.

In the case of Legendre polynomials there is the classical formula§ of Neumann and Adams which expresses the product $P_m(x)P_n(x)$ as a sum of Legendre polynomials, but, so far as I know, no attempt has been made to express the product of two Legendre polynomials with different arguments as a sum involving Legendre polynomials. It may consequently be of interest to note some formulae of this type. In each formula the Legendre polynomials occurring in the product are of the same order.

2. In the case of the first formula it is just as simple to work with Jacobi polynomials, which will be defined by

$$P_n^{(\alpha,\beta)}(x) = (-1)^n \frac{(\alpha+1)_n}{n!} F[-n,\ n+\alpha+\beta+1;\ \alpha+1;\ \tfrac{1}{2}(1-x)].$$

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* See Bailey (1), where further references are given.

† Watson (5), § 11.6.

‡ Dhar (3), § 3.

§ Hobson (4), 85.

Using the formula*

$$(2.1)\quad F_4[-n,\ \beta+n;\ \gamma,\ \beta-\gamma+1;\ zZ,\ (1-z)(1-Z)]$$
$$=\frac{(-1)^n(\gamma)_n}{(\beta-\gamma+1)_n}F[-n,\ \beta+n;\ \gamma;\ z]F[-n,\ \beta+n;\ \gamma;\ Z],$$

we have

$$P_n^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(y)=\left\{\frac{(\alpha+1)_n}{n!}\right\}^2 F[-n,\ n+\alpha+\beta+1;\ \alpha+1;\ \tfrac{1}{2}(1-x)]$$
$$\times F[-n,\ n+\alpha+\beta+1;\ \alpha+1;\ \tfrac{1}{2}(1-y)]$$
$$=\frac{(-1)^n(\alpha+1)_n(\beta+1)_n}{(n!)^2}\sum_{r=0}^{r+s\leqslant n}\sum_{s=0}\frac{(-n)_{r+s}(n+\alpha+\beta+1)_{r+s}}{r!\,s!\,(\alpha+1)_r(\beta+1)_s}$$
$$\times\{\tfrac{1}{4}(1-x)(1-y)\}^r\{\tfrac{1}{4}(1+x)(1+y)\}^s.$$

If we write $s=m-r$, this becomes

$$P_n^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(y)=\frac{(-1)^n(\alpha+1)_n(\beta+1)_n}{(n!)^2}\sum_{m=0}^{n}\frac{(-n)_m(n+\alpha+\beta+1)_m}{m!\,(\beta+1)_m}$$
$$\times\{\tfrac{1}{4}(1+x)(1+y)\}^m F\left[-m,\ -\beta-m;\ \alpha+1;\ \frac{(1-x)(1-y)}{(1+x)(1+y)}\right]$$
$$=\frac{(-1)^n(\alpha+1)_n(\beta+1)_n}{(n!)^2}\sum_{m=0}^{n}\frac{(-n)_m(n+\alpha+\beta+1)_m}{2^m m!\,(\beta+1)_m}(x+y)^m$$
$$\times F\left[-m,\ 1+\alpha+\beta+m;\ \alpha+1;\ -\frac{(1-x)(1-y)}{2(x+y)}\right],$$

on using the transformation

$$F[a,\ b;\ c;\ z]=(1-z)^{-a}F[a,\ c-b;\ c;\ -z/(1-z)].$$

It follows that

$$(2.2)\quad P_n^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(y)=\frac{(-1)^n(\alpha+1)_n(\beta+1)_n}{n!}$$
$$\times\sum_{m=0}^{n}\frac{(n+\alpha+\beta+1)_m}{2^m(n-m)!\,(\alpha+1)_m(\beta+1)_m}(x+y)^m P_m^{(\alpha,\beta)}\left(\frac{1+xy}{x+y}\right).$$

Since $P_n^{(0,0)}(x)=(-1)^n P_n(x)$, the corresponding result for Legendre polynomials is

$$(2.3)\quad P_n(x)P_n(y)=\sum_{m=0}^{n}\frac{(-1)^{n+m}(n+m)!}{2^m(m!)^2(n-m)!}(x+y)^m P_m\left(\frac{1+xy}{x+y}\right).$$

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* Watson (6), § 2.

3. The second expansion is obtained by using the formulae*

$$(3.1)\qquad P_n(x) = \frac{(2n)!}{2^n (n!)^2}\, x^n\, F[-\tfrac{1}{2}n,\ \tfrac{1}{2}(1-n);\ \tfrac{1}{2}-n;\ 1/x^2],$$

$$(3.2)\qquad F[\alpha,\ \beta;\ \gamma;\ z]\, F[\alpha,\ \beta;\ \gamma;\ Z]$$

$$= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(\alpha)_{r+s} (\beta)_{r+s} (\gamma-\alpha)_r (\gamma-\beta)_r}{r!\, s!\, (\gamma)_r (\gamma)_{2r+s}}\, (zZ)^r (z+Z-zZ)^s.$$

We thus find that

$$P_n(x)\, P_n(y) = \frac{\{(2n)!\}^2}{2^{2n}(n!)^4} \sum_{r=0}^{r+s\leqslant \frac{1}{2}n} \sum_{s=0} \frac{(-\tfrac{1}{2}n)_{r+s} (\tfrac{1}{2}-\tfrac{1}{2}n)_{r+s} (\tfrac{1}{2}-\tfrac{1}{2}n)_r (-\tfrac{1}{2}n)_r}{r!\, s!\, (\tfrac{1}{2}-n)_r (\tfrac{1}{2}-n)_{2r+s}}\, x^n y^n$$

$$\times \left(\frac{1}{x^2 y^2}\right)^r \left(\frac{1}{x^2} + \frac{1}{y^2} - \frac{1}{x^2 y^2}\right)^s$$

$$= \frac{\{(2n)!\}^2}{2^{2n}(n!)^4} \sum_{r=0}^{\leqslant \frac{1}{2}n} \frac{\{(-n)_{2r}\}^2 x^n y^n}{2^{4r} r! (\frac{1}{2}-n)_r (\frac{1}{2}-n)_{2r}} \left(\frac{1}{xy}\right)^{2r}$$

$$\times F\left[-\tfrac{1}{2}n+r,\ \tfrac{1}{2}-\tfrac{1}{2}n+r;\ \tfrac{1}{2}-n+2r;\ \frac{x^2+y^2-1}{x^2 y^2}\right].$$

and so

$$(3.3)\qquad P_n(x)\, P_n(y) = \frac{(2n)!}{2^n (n!)^2} \sum_{r=0}^{\leqslant \frac{1}{2}n} \frac{(-n)_{2r}}{2^{2r} r!\, (\frac{1}{2}-n)_r}$$

$$\times (x^2+y^2-1)^{\frac{1}{2}n-r}\, P_{n-2r}\left(\frac{xy}{\sqrt{(x^2+y^2-1)}}\right).$$

The same method could evidently be used to obtain the corresponding result for the product of two Gegenbauer polynomials.

4. A third expansion of a different type can be obtained in a similar way by using the formula

$$P_n(x) = F[\tfrac{1}{2}(1+n),\ -\tfrac{1}{2}n;\ 1;\ 1-x^2]$$

together with (3.2). We find that

$$(4.1)\qquad P_n(x)\, P_n(y) = \sum_{r=0}^{\leqslant \frac{1}{2}n} \frac{(n+1)_{2r} (-n)_{2r}}{2^{2r} (r!)^2} \left\{\frac{(x^2-1)(y^2-1)}{x^2 y^2 - 1}\right\}^r P_n^{-2r}(xy).$$

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* For (3.2) see Watson (6), § 4.

More generally, it can be shown that

$$(4.2)\quad P_n^{-m}(x)\,P_n^{-m}(y) = \sum_{r=0}^{\frac{1}{2}(n-m)} \frac{(n+m+1)_{2r}\,(-n+m)_{2r}}{2^{m+2r}\,r!\,(m+r)!} \times \left\{\frac{(x^2-1)(y^2-1)}{x^2y^2-1}\right\}^{\frac{1}{2}m+r} P_n^{-m-2r}(xy),$$

where m and n are positive integers and $n > m$. This result was suggested by expanding I_μ in the formula*

$$(4.3)\quad P_{\nu-\frac{1}{2}}^{-\mu}(\cosh\phi)\,P_{\nu-\frac{1}{2}}^{-\mu}(\cosh\psi) = \frac{1}{\sqrt{(\frac{1}{2}\pi)}\,\Gamma(\mu+\nu+\frac{1}{2})\,\Gamma(\mu-\nu+\frac{1}{2})} \times \int_0^\infty t^{-\frac{1}{2}}\,I_\mu(t\sinh\phi\sinh\psi)\,K_\nu(t)\,e^{-t\cosh\phi\cosh\psi}\,dt$$

in powers of t, and integrating term by term. The method is purely formal in the case of (4.1) unless the integral is replaced by a contour integral to avoid the infinity in the integrand when $t = 0$. In the case of Legendre polynomials the appropriate form of (4.3) is

$$(4.4)\quad P_n(x)\,P_n(y) = \frac{1}{2\pi i}\,\frac{(-1)^n}{\sqrt{(\frac{1}{2}\pi)}} \times \int_\infty^{(0+)} t^{-\frac{1}{2}}\,I_0[t\sqrt{\{(x^2-1)(y^2-1)\}}]\,K_{n+\frac{1}{2}}(t)\,e^{-txy}\,dt.$$

Since the integrand is single-valued in the t-plane, the contour can be replaced by a contour encircling the origin once in the positive sense. This gives the curious result that $P_n(x)\,P_n(y)$ is the term independent of t in the Laurent expansion of

$$\frac{(-1)^n}{\sqrt{(\frac{1}{2}\pi)}}\,t^{\frac{1}{2}}\,K_{n+\frac{1}{2}}(t)\,I_0[t\sqrt{\{(x^2-1)(y^2-1)\}}]\,e^{-txy}.$$

5. Further interesting results can be obtained from (3.2). Thus, if we replace z, Z by z/β, Z/β and let $\beta \to \infty$, we obtain

$$(5.1)\quad {}_1F_1[\alpha;\ \gamma;\ z]\,{}_1F_1[\alpha;\ \gamma;\ Z] = \sum_{r=0}^\infty \sum_{s=0}^\infty \frac{(-1)^r(\alpha)_{r+s}(\gamma-\alpha)_r}{r!\,s!\,(\gamma)_r\,(\gamma)_{2r+s}}\,(zZ)^r\,(z+Z)^s = \sum_{r=0}^\infty \frac{(-1)^r(\alpha)_r(\gamma-\alpha)_r}{r!\,(\gamma)_r\,(\gamma)_{2r}}\,(zZ)^r\,{}_1F_1[\alpha+r;\ \gamma+2r;\ z+Z].$$

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* Bailey (2), (2.4).

This formula will be used only when α is a negative integer, and in that case no justification of the limiting process is necessary, but the justification in the general case presents no difficulty.

Again, from (5.1), if we replace z, Z by z/α, Z/α and let $\alpha \to \infty$, we obtain a formula equivalent to

$$(5.2) \qquad J_\nu(z\cos\theta)\,J_\nu(z\sin\theta) = \sum_{r=0}^{\infty} \frac{(\frac{1}{4}z\sin 2\theta)^{\nu+2r}}{r!\,\Gamma(\nu+r+1)}\,J_{\nu+2r}(z).$$

This expansion does not appear to have been stated explicitly before, but it is the limiting case, when $a^2+b^2=1$, of the formula*

$$(5.3) \qquad J_\nu(az)\,J_\nu(bz) = \sum_{n=0}^{\infty} \frac{(abz\sinh A)^{\nu+n}}{n!}\,J_{\nu+n}(z)\,P_{\nu+n}^{-\nu}(\coth A),$$

where $2ab\cosh A = 1-a^2-b^2$.

The relation (5.1) has an interesting connection with Laguerre polynomials† defined by

$$L_n^{(\alpha)}(x) = \frac{e^x x^{-\alpha}}{n!}\,\frac{d^n(e^{-x}x^{n+\alpha})}{dx^n} = \sum_{r=0}^{n}\binom{n+\alpha}{n-r}\frac{(-x)^r}{r!}$$

$$= \frac{(1+\alpha)_n}{n!}\,{}_1F_1[-n;\ 1+\alpha;\ x].$$

In this case (5.1) gives the relation

$$(5.4) \qquad L_n^{(\alpha)}(x)\,L_n^{(\alpha)}(y) = \frac{\Gamma(1+\alpha+n)}{n!}\sum_{r=0}^{n}\frac{(xy)^r\,L_{n-r}^{\alpha+2r}(x+y)}{r!\,\Gamma(1+\alpha+r)}.$$

The formula (5.1) can also be used to give a simple proof of the formula‡

$$(5.5) \qquad \sum_{n=0}^{\infty} t^n\,\frac{n!}{\Gamma(n+\alpha+1)}\,e^{-\frac{1}{2}(x+y)}(xy)^{\frac{1}{2}\alpha}\,L_n^{(\alpha)}(x)\,L_n^{(\alpha)}(y)$$
$$= \frac{t^{-\frac{1}{2}\alpha}}{1-t}\exp\left\{-\tfrac{1}{2}(x+y)\,\frac{1+t}{1-t}\right\}\,I_\alpha\left(\frac{2\sqrt{(xyt)}}{1-t}\right),$$

where $|t|<1$, due to Hille, Hardy, and Kogbetliantz. For, using (5.1), we have

$$\sum_{n=0}^{\infty}\frac{t^n\,n!}{(\alpha+1)_n}\,L_n^{(\alpha)}(x)\,L_n^{(\alpha)}(y)$$
$$= \sum_{n=0}^{\infty}\sum_{r=0}^{r+s\leqslant n}\sum_{s=0}\frac{(-n)_{r+s}(1+\alpha+n)_r(-1)^r}{r!\,s!\,(1+\alpha)_r\,(1+\alpha)_{2r+s}}\,\frac{(1+\alpha)_n}{n!}\,(xy)^r\,(x+y)^s\,t^n.$$

* Bailey (1), (5.1).

† See Watson (7).

‡ See Watson (7) where further references are given.

Putting $n = r+s+m$, we get

$$\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^s(1+\alpha+2r+s)_m}{m!\,r!\,s!\,(1+\alpha)_r}(xyt)^r\{t(x+y)\}^s t^m$$

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^s(xyt)^r\{t(x+y)\}^s(1-t)^{-1-\alpha-2r-s}}{r!\,s!\,(1+\alpha)_r}$$

$$=(1-t)^{-1-\alpha}\sum_{s=0}^{\infty}\frac{1}{s!}\left\{-\frac{t(x+y)}{1-t}\right\}^s\sum_{r=0}^{\infty}\frac{1}{r!(1+\alpha)_r}\left\{\frac{xyt}{(1-t)^2}\right\}^r$$

$$=\frac{\Gamma(1+\alpha)(xyt)^{-\frac{1}{2}\alpha}}{1-t}\exp\left\{-\frac{t(x+y)}{1-t}\right\}I_\alpha\left(\frac{2\sqrt{(xyt)}}{1-t}\right),$$

and this proves the formula. The analysis is simpler than that given by Watson, though the argument is not quite so elementary. On the other hand, (5.1) can be proved by comparing coefficients of $z^m Z^n$ and using Saalschütz's theorem, and then the argument becomes as elementary as Watson's.

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THEOREMS ABOUT THE MAXIMUM MODULUS OF POLYNOMIALS

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G. Aumann† has proved the following theorem:

Let $f(x)$ and $g(x)$ be polynomials of degrees m, n, respectively, and let $\mathfrak{M}$ be any closed bounded continuum in the x-plane. Then

$$(1) \qquad \Big(\max_{x \text{ in } \mathfrak{M}} |f(x)g(x)|\Big) : \Big(\max_{x \text{ in } \mathfrak{M}} |f| \max_{x \text{ in } \mathfrak{M}} |g|\Big) \geqslant c(m, n) > 0,$$

where $c(m, n)$ does not depend on $\mathfrak{M}$ or on the coefficients of the polynomials, but only on their degrees m, n.

We can express the theorem in a more convenient form. From (1) follows the existence of x^* in $\mathfrak{M}$ with

$$|f(x^*)g(x^*)| \geqslant c \max_{\mathfrak{M}} |f| \max_{\mathfrak{M}} |g|,$$

and therefore

$$(2) \qquad \begin{cases} |f(x^*)| \geqslant c \max\limits_{x \text{ in } \mathfrak{M}} |f(x)|, \\ |g(x^*)| \geqslant c \max\limits_{x \text{ in } \mathfrak{M}} |g(x)|. \end{cases}$$

(2) asserts that there is always a point in $\mathfrak{M}$ where both f and g reach at least a certain fixed proportion of their respective maxima in $\mathfrak{M}$. This proportion depends only on the degrees m and n.

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† *Sitzungsberichte der Preussischen Akademie der Wissenschaften, Phys.-Math. Klasse* (1933), 924–931. See also H. Kneser, *ibid.* (1934), 426–431, where the best possible constant $c(m, n)$ of (1) is determined.

On the other hand, (1) with c^2 instead of c obviously follows from (2).

This paper will deal with generalisations of (2). Theorem 1 generalises (2) in the following direction. We are given a finite number of polynomials $f_\kappa(x)$ whose product is of degree n, and a number ϑ such that $0 \leqslant \vartheta < 1$. Then, corresponding to any closed bounded continuum $\mathfrak{M}$, we prove that there exists a circle C with centre in $\mathfrak{M}$ and radius (diameter of $\mathfrak{M}) \times \vartheta/(2n)$ with the property that

$$|f_\kappa(x')| \geqslant c_n(\vartheta) \max_{x \text{ in } \mathfrak{M}} |f_\kappa(x)| \tag{3}$$

for all κ and every point x' in C, but not necessarily in $\mathfrak{M}$. Here $c_n(\vartheta)$ depends only on n and ϑ. For $\vartheta = 1$ the theorem is false. Denoting by $c_n^*(\vartheta)$ the best possible constant of this theorem, we have

$$\text{const. } (1-\vartheta)^2 (\tfrac{1}{6})^n \leqslant c_n^*(\vartheta) \leqslant \text{const. } (1-\vartheta)(\tfrac{1}{2})^n. \tag{4}$$

The proof of I, though it uses similar principles, is different from Aumann's proof, being much shorter and giving a numerical value for c. It also gives a finite process for finding the circle C.

Theorem II is an easy generalisation to the case where $\mathfrak{M}$ is not a continuum but contains a subset which is a continuum. In § 2 we give a geometrical proof of I for the special case $\vartheta = 0$. This proof, though giving for general polynomials f_κ a value $c_n(0)$ in (3) which is much smaller than the constant given in Theorem I, will enable us to determine the best possible constant $c_n^*(0)$ in the special case where every $f_\kappa(x)$ is linear. Our result for linear polynomials is

$$c_n^*(0) = \frac{1-\alpha_n}{1+\alpha_n},$$

where α_n is the positive root of the equation

$$x^{[\frac{1}{2}(n+1)]} + x^{[\frac{1}{2}n]} = 1.$$

We have

$$\frac{1-\alpha_n}{1+\alpha_n} = \frac{\log 2}{n} \{1+o(1)\}.$$

Comparing with (4), we see that the worst case of Theorem I, that is the case where only a very small c has the property (3), does not occur in the two extreme cases where either the degree n is divided up among n linear polynomials, or where there is only one polynomial of degree n.

In § 3, Theorems I and II are generalised to the case of polynomials in more than one variable.

1. THEOREM I. *Given a number* ϑ $(0 \leqslant \vartheta < 1)$ *and a positive integer* n, *there is a positive number* $c_n(\vartheta)$ *with the following property. Corresponding to any polynomials* $f_1(x)$, $f_2(x)$, ..., $f_k(x)$, *whose product is of degree* n, *and to any closed bounded continuum* $\mathfrak{M}$ *in the* x-*plane with diameter* $d(\mathfrak{M})$, *there exists an* x^* *in* $\mathfrak{M}$ *with*

$$|f_\kappa(x')| \geqslant c_n(\vartheta) \max_{x \text{ in } \mathfrak{M}} |f_\kappa(x)|$$

for

$$1 \leqslant \kappa \leqslant k, \quad |x'-x^*| \leqslant \frac{\vartheta}{2n} d(\mathfrak{M}).$$

A possible value is

$$c_n(\vartheta) = \tfrac{1}{2}(1-\vartheta)^2 n^{-\frac{3}{2}} \left(\frac{1}{2e}\right)^n. \tag{1}$$

Denoting by $c_n^*(\mathfrak{M})$ *the best constant* $c_n(0)$ *belonging to a fixed* $\mathfrak{M}$, *we have, for segments* S *of straight lines and for circles* C,

$$A_1 (\tfrac{1}{3})^n < c_n^*(S) < A_2 (\tfrac{1}{2})^n < A_3 \frac{1}{n} \left\{\tfrac{1}{2}\sqrt{\left(\frac{\pi}{e}\right)}\right\}^n < c_n^*(C) < A_4 \left(\frac{1}{\sqrt{e}}\right)^n,$$

where the A_ν *are absolute constants.*

For the circle, the better inequalities

$$A_5 \frac{1}{n} \prod_{1 \leqslant \nu \leqslant \frac{1}{4}n} \tan^2 \frac{\pi\nu}{4} \leqslant c_n^*(C) \leqslant A_6 \prod_{1 \leqslant \nu \leqslant \frac{1}{4}n} \tan^2 \frac{\pi\nu}{4}$$

hold.

For $\vartheta = 1$, Theorem I is false, as is shown by the example

$$\mathfrak{M} = \langle \tfrac{1}{2},\ n+\tfrac{1}{2} \rangle,$$

$f_\kappa(x)$ arbitrary with

$$f_1(x) f_2(x) \ldots f_n(x) = (x-1)(x-2)\ldots(x-n).$$

Here, any circle of radius $d(\mathfrak{M})/(2n)$ and centre in $\mathfrak{M}$ contains a zero of one of the f_κ.

There are two alternative ways of expressing Theorem I. These are, in the case $\vartheta = 0$, the following.

THEOREM I'. *Under the hypotheses and with the notation of* I,

$$\min\big(|f_1(x)|,\ |f_2(x)|,\ \ldots,\ |f_k(x)|\big) \leqslant A$$

for every x of $\mathfrak{M}$ implies the existence of κ_0 with

$$|f_{\kappa_0}(x)| \leqslant c_n(0)\,A$$

for every x of $\mathfrak{M}$.

THEOREM I''. *Under the hypotheses and with the notation of* I, *we have*

$$\max_{x \text{ in } \mathfrak{M}} \min_{\kappa} |f_\kappa(x)| \geqslant c_n(0) \min_{\kappa} \max_{x \text{ in } \mathfrak{M}} |f_\kappa(x)|.$$

The equivalence of I' and I'' with I in the case $\vartheta = 0$ is easily shown. We begin by proving that

$$c_n^*(S) \leqslant 3(\tfrac{1}{2})^n. \tag{2}$$

We take

$$k = 2, \quad f_1(x) = \prod_{\nu=1}^{m} (x-\nu), \quad f_2(x) = \prod_{m+1}^{2m} (x-\nu), \quad \mathfrak{M} = \langle \tfrac{1}{2}, 2m+\tfrac{1}{2} \rangle.$$

Here

$$M_\kappa = \max_{\mathfrak{M}} |f_\kappa(x)| = \prod_{m+1}^{2m} (\nu-\tfrac{1}{2}),$$

while, on the other hand, for any x in $\mathfrak{M}$, at least one of the $f_\kappa(x)$ is not greater than

$$\prod_{1}^{m} (\nu-\tfrac{1}{2}).$$

Therefore, obviously,

$$c_{2m}^*(S) \leqslant \prod_{1}^{m} (\nu-\tfrac{1}{2}) \prod_{m+1}^{2m} (\nu-\tfrac{1}{2})^{-1} = Q_m.$$

Using Stirling's formula, we have

$$Q_m = \frac{[(2m)!]^3}{m!\,m!\,(4m)!} \leqslant 3 \cdot 2^{-2m}.$$

A similar argument proves (2) for odd values of n.

Examples of the same type prove the assertions in I concerning upper bounds of $c_n^*(C)$ and also the right-hand part of inequality (4) of the introduction.

To prove the essential part of I, the existence of a positive $c_n(\vartheta)$, we may assume, without loss of generality, that

$$d(\mathfrak{M}) = 1, \quad k > 1, \quad M_\kappa = 1, \quad n_\kappa > 0 \quad (1 \leqslant \kappa \leqslant k).$$

Let the number $\epsilon > 0$ have the following property. Given any x^* of $\mathfrak{M}$, there is always an $x' = x'(x^*)$ and a $\kappa = \kappa(x^*)$ with

$$|x'-x^*| \leqslant \frac{\vartheta}{2n}, \quad |f_\kappa(x')| < \epsilon.$$

We have to show that ϵ cannot be too small; in particular,

$$\epsilon > \tfrac{1}{2}(1-\vartheta)^2 n^{-\frac{3}{2}}(2e)^{-n}.$$

There exist a, b in $\mathfrak{M}$ with

$$|a-b| = 1.$$

Since $\mathfrak{M}$ is connected there exist points x_ν in $\mathfrak{M}$ ($x_0 = a$, $x_n = b$) with

$$|x_\nu - a| = \frac{\nu}{n} \quad (0 \leqslant \nu \leqslant n),$$

for the point a of $\mathfrak{M}$ lies inside and the point b of $\mathfrak{M}$ not inside the circle

$$|x-a| = \frac{\nu}{n} \quad (0 < \nu < n).$$

We shall show that at least one of the x_ν may be used as x^* in the theorem.

Our assumption about ϵ, applied to the point $x^* = x_\nu$, implies the existence of x_ν' with

$$|x_\nu' - x_\nu| \leqslant \frac{\vartheta}{2n}, \quad |f_{\kappa_\nu}(x_\nu')| < \epsilon \quad (0 \leqslant \nu \leqslant n),$$

where κ_ν is one of the numbers 1, 2, ..., k.

For fixed κ, let n_κ' be the number of ν's for which $\kappa_\nu = \kappa$. Then

$$\sum_{\kappa=1}^{k} (n_\kappa' - n_\kappa) = (n+1)-n > 0.$$

Therefore there is a κ^* with $n_{\kappa^*}' > n_{\kappa^*}$, that is,

$$|f_{\kappa^*}(x_\nu')| < \epsilon \tag{3}$$

for certain $n_{\kappa^*}+1$ different values of ν.

For $\alpha \neq \beta$, we have

$$|x_\alpha' - x_\beta'| \geqslant |x_\alpha - x_\beta| - 2\frac{\vartheta}{2n} \geqslant \frac{1}{n} - \frac{\vartheta}{n}. \tag{4}$$

n_{κ^*} is the degree of the polynomial f_{κ^*}. By means of Lagrange's interpolation formula, $f_{\kappa^*}(x)$ is uniquely determined by its values at these $n_{\kappa^*}+1$ points x_ν'. Using the special structure of this formula, the inequalities (3), (4) and the definition of $d(\mathfrak{M})$ and x_ν', we obtain for any x of $\mathfrak{M}$, by rough estimation,

$$|f_{\kappa^*}(x)| < (n_{\kappa^*}+1)\left(1+\frac{\vartheta}{2n}\right)^{n_{\kappa^*}}\left(\frac{n}{1-\vartheta}\right)^{n_{\kappa^*}}\epsilon \leqslant (n+1)\,e\,n^n(1-\vartheta)^{-n}\,\epsilon.$$

Now, choosing for x a point where $|f_{\kappa^*}(x)|$ assumes its maximum value 1, we find

$$1 < (n+1)\,e\,n^n(1-\vartheta)^{-n}\,\epsilon,$$

$$\epsilon > e^{-1}(1-\vartheta)^n(n+1)^{-1}n^{-n},$$

which establishes the existence of a positive $c_n(\vartheta)$.

We get the better result (1) by estimating a little more carefully. We observe that the points x_ν lie on concentric circles whose radii differ by at least $1/n$. Then we arrive at the stronger inequality

$$1 < \epsilon \sum_{\nu=0}^{m}\left(1+\frac{\vartheta}{2n}\right)^m\left[\frac{\eta}{n}\left(\frac{\eta}{n}+\frac{1}{n}\right)\dots\left(\frac{\eta}{n}+\frac{\nu-1}{n}\right)\right]^{-1}$$

$$\times\left[\frac{\eta}{n}\left(\frac{\eta}{n}+\frac{1}{n}\right)\dots\left(\frac{\eta}{n}+\frac{m-\nu-1}{n}\right)\right]^{-1},$$

where $\eta = 1-\vartheta$, $m = n_{\kappa^*}$, and the terms with $\nu = 0$ and $\nu = m$ have their obvious meaning. It follows that

$$(5)\quad 1 < \epsilon\,e\,n^m \sum_{\nu=0}^{m}[\eta(\eta+1)\dots(\eta+\nu-1)]^{-1}[\eta(\eta+1)\dots(\eta+m-\nu-1)]^{-1}$$

$$= \epsilon\,e\,n^m\,S.$$

For $m \geqslant 2$, we have

$$S \leqslant \frac{2}{\eta}\,\frac{1}{(m-1)!}+\frac{1}{\eta^2}\sum_{\nu=1}^{m-1}\frac{1}{(\nu-1)!\,(m-\nu-1)!}$$

$$= \frac{2}{\eta(m-1)!}+\frac{2^{m-2}}{\eta^2(m-2)!} \leqslant \frac{2^m}{\eta^2(m-2)!},$$

$$S \leqslant \frac{2^m m^2}{\eta^2 m!}.$$

The last inequality also holds for $m = 1$. Using Stirling's formula, we finally deduce from (5)

$$\frac{1}{\epsilon} < e\, n^m \frac{2^m m^2}{\eta^2 m!} \leqslant e \frac{n^2}{\eta^2} \frac{(2n)^m}{m!} \leqslant e \frac{n^2}{\eta^2} \frac{(2n)^n}{n!}$$

$$\leqslant e \frac{n^2}{\eta^2} (2n)^n n^{-n-\frac{1}{2}} e^n (2\pi)^{-\frac{1}{2}} \leqslant 2\eta^{-2} n^{\frac{3}{2}} (2e)^n.$$

The special lower bounds for c_n^* in the cases of a segment or a circle are deduced in a similar way by choosing appropriate points

$$x_0, \quad x_1, \quad \ldots, \quad x_n.$$

If $\mathfrak{M}$ is not connected, Theorem I is false, even if we replace $c_n(\vartheta)$ by a number depending on everything except on $\mathfrak{M}$. This is shown by the case where $\mathfrak{M}$ consists of the two points $+1$, -1,

$$k = 2, \quad f_1 = x+1, \quad f_2 = x-1.$$

Here, in every point of $\mathfrak{M}$, one of the f_κ vanishes, while $M_1 = M_2 = 2$.

But the proof of I makes it clear that, if $\mathfrak{M}$ contains $n+1$ different points, Theorem I still holds if $c_n(\vartheta)$ is replaced by $c_n(\vartheta, \mathfrak{M})$ and $d(\mathfrak{M})$ by some constant $d_n'(\mathfrak{M})$. We simply choose as x_ν these $n+1$ different points of $\mathfrak{M}$, and proceed as above.

We note a special case, a direct generalisation of I, where $c_n(\vartheta, \mathfrak{M})$ depends on $\mathfrak{M}$ in a very simple manner.

THEOREM II. *Let $\mathfrak{M}$ be a closed bounded set in the x-plane with $d(\mathfrak{M}) > 0$ which contains a closed continuum $\mathfrak{C}$. Then (with notation as in* I) *there is an x^* in $\mathfrak{C}$ with*

$$|f_\kappa(x')| \geqslant c_n(\vartheta) \left(\frac{d(\mathfrak{C})}{d(\mathfrak{M})}\right)^n \max_{x \text{ in } \mathfrak{M}} |f_\kappa(x)|$$

for

$$1 \leqslant \kappa \leqslant k, \quad |x'-x^*| \leqslant \frac{\vartheta}{2n} d(\mathfrak{C}).$$

$c_n(\vartheta)$ may be chosen as in (1).

To prove II, we use almost the same argument as above, the only difference being that now a, b, x_ν are chosen with respect to $\mathfrak{C}$, and not, as above, with respect to $\mathfrak{M}$.

The inequalities for $c_n^*(\mathfrak{M})$ given in Theorem I and the method of proving them suggest that for fixed n the segment S and the circle

C are the two extreme cases, *i.e.* that

$$c_n^*(S) \leqslant c_n^*(\mathfrak{M}) \leqslant c_n^*(C)$$

for any connected set $\mathfrak{M}$. I have as yet been unable to prove this conjecture.

2. There is an alternative way of proving I for the special case $\vartheta = 0$, which will be explained in this paragraph. This second proof, though giving a constant c which is smaller than the constant given in Theorem I, shows that I is, essentially, a very simple geometrical lemma about circles.

LEMMA 1. *Given m circles*

$$Z_\mu: \quad |x-a_\mu| \leqslant r_\mu \quad (1 \leqslant \mu \leqslant m),$$

which, taken together, form a connected set Z, there is always a μ_0 (with $1 \leqslant \mu_0 \leqslant m$) such that the whole set Z lies within the circle

$$|x-a_{\mu_0}| \leqslant (2m-1)\, r_{\mu_0}.$$

Proof of the lemma. We choose μ_0 so that

$$r_{\mu_0} = \max(r_1, r_2, \ldots, r_m).$$

There is a point x_0 in Z with

$$|x-a_{\mu_0}| \leqslant |x_0-a_{\mu_0}|$$

for every x of Z. Using the connectivity of Z, we can find l and indices $\mu_1, \mu_2, \ldots, \mu_l$ $(0 \leqslant l < m)$ such that x_0 lies in Z_{μ_l} and for $0 \leqslant \lambda < l$ the circles Z_{μ_λ} and $Z_{\mu_{\lambda+1}}$ have a common part. Then, according to the definition of μ_0 and x_0, we have, for any x in Z,

$$\begin{aligned} |x-a_{\mu_0}| &\leqslant |x_0-a_{\mu_0}| \leqslant |x_0-a_{\mu_l}| + |a_{\mu_l}-a_{\mu_{l-1}}| + \ldots + |a_{\mu_1}-a_{\mu_0}| \\ &\leqslant r_{\mu_l} + (r_{\mu_l}+r_{\mu_{l-1}}) + \ldots + (r_{\mu_1}+r_{\mu_0}) \leqslant (2l+1)\, r_{\mu_0} \leqslant (2m-1)\, r_{\mu_0}. \end{aligned}$$

We now deduce I (with $\vartheta = 0$) from the lemma as follows.
Without loss of generality we may assume

$$f_\kappa(x) = \prod_{\nu=1}^{n_\kappa} (x-a_{\kappa\nu}), \quad M_{\kappa\nu} = \max_{x \text{ in } \mathfrak{M}} |x-a_{\kappa\nu}| > 0.$$

Let $\epsilon > 0$ have the following property. Given any x of $\mathfrak{M}$ there are always $\kappa = \kappa(x)$ and $\nu = \nu(x)$ with

$$|x - a_{\kappa\nu}| \leqslant \epsilon M_{\kappa\nu}. \tag{1}$$

Then the set Z' of the n circles (1) contains $\mathfrak{M}$. Z' is the sum of a certain number of connected components, each of which consists of some of the circles (1). Since $\mathfrak{M}$ itself is connected, it lies entirely within one of these connected sets of circles, say Z. The lemma now establishes the existence of κ_0, ν_0 such that Z, and therefore *a fortiori* $\mathfrak{M}$, is contained in the circle

$$|x - a_{\kappa_0\nu_0}| \leqslant (2m-1)\,\epsilon M_{\kappa_0\nu_0},$$

where $m \leqslant n$ is the number of circles building up Z. On the other hand, $|x - a_{\kappa_0\nu_0}|$ attains the value $M_{\kappa_0\nu_0}$ somewhere in $\mathfrak{M}$. Therefore

$$M_{\kappa_0\nu_0} \leqslant (2m-1)\,\epsilon M_{\kappa_0\nu_0},$$

$$\epsilon \geqslant \frac{1}{2m-1} \geqslant \frac{1}{2n-1}.$$

Consequently for $\epsilon = 1/(2n)$ our assumption is false, *i.e.* there is an x^* in $\mathfrak{M}$ with

$$|x^* - a_{\kappa\nu}| > \frac{1}{2n} M_{\kappa\nu} \quad \text{for all } \kappa,\ \nu\ (1 \leqslant \kappa \leqslant k;\ 1 \leqslant \nu \leqslant n_\kappa).$$

Then

$$|f_\kappa(x^*)| \geqslant (2n)^{-n_\kappa} \prod_{\nu=1}^{n_\kappa} M_{\kappa\nu} \geqslant (2n)^{-n} M_\kappa \quad \text{for} \quad 1 \leqslant \kappa \leqslant k.$$

This is Theorem I for the case $\vartheta = 0$ with $c_n(0) = (2n)^{-n}$.

For the rest of this paragraph we shall confine ourselves to the case of linear polynomials, that is

$$n_1 = n_2 = \ldots = n_k = 1, \quad n = k.$$

For fixed n we shall determine the greatest value c_n of $c_n(0)$ in Theorem I for this special case of linear polynomials.

Let C_m be the smallest constant which still has the property assigned to $2m-1$ in the lemma. It is easily shown that

$$c_n = C_n^{-1}. \tag{2}$$

For, first, the proof given above shows that

$$c_n \geqslant C_n^{-1}. \tag{3}$$

On the other hand, let Z be a connected set of n circles

$$Z_\nu: \quad |x-a_\nu| \leqslant r_\nu \quad (1 \leqslant \nu \leqslant n). \tag{4}$$

Applying I to

$$\mathfrak{M} = Z, \quad f_\nu(x) = x-a_\nu, \quad \vartheta = 0,$$

we get the existence of x^* in $\mathfrak{M}$ with

$$|x^*-a_\nu| \geqslant c_n M_\nu \quad (1 \leqslant \nu \leqslant n).$$

Now

$$|x^*-a_{\nu_0}| \leqslant r_{\nu_0}$$

for a certain ν_0. Therefore

$$|x-a_{\nu_0}| \leqslant M_{\nu_0} \leqslant \frac{1}{c_n}|x^*-a_{\nu_0}| \leqslant \frac{1}{c_n} r_{\nu_0} \quad \text{for any } x \text{ of } Z. \tag{5}$$

(5) shows that $C_n \leqslant 1/c_n$, which together with (3) proves (2).

From now onwards the letter Z always denotes a connected set of n circles Z_ν, given by (4), where $r_\nu \geqslant 0$ and not all $r_\nu = 0$. Similarly for $Z^{(0)}$, Z', Z^*, ...

For every Z we define

$$\phi_\nu(Z) = \frac{1}{r_\nu} \max_{x \text{ in } Z} |x-a_\nu|, \quad \phi(Z) = \min_{1 \leqslant \nu \leqslant n} \phi_\nu(Z),$$

$$\phi_\nu(Z) = +\infty \quad \text{if} \quad r_\nu = 0.$$

Then Lemma 1 simply reads

$$\phi(Z) \leqslant 2n-1 \quad \text{for any } Z,$$

and we have

$$C_n = \text{upper bound of } \phi(Z) \text{ for varying } Z.$$

We shall show that, apart from trivial transformations, there exists exactly one set $Z = Z^*$ with

$$\phi(Z^*) = C_n. \tag{6}$$

To show that (6) has at least one solution we choose a sequence

$$Z^{(1)}, \ Z^{(2)}, \ \ldots$$

with

$$\phi(Z^{(\lambda)}) \to C_n.$$

Obviously it is possible to choose $Z^{(\lambda)}$ so that

$$a_1^{(\lambda)} = 0, \quad r_\nu^{(\lambda)} \leqslant r_1^{(\lambda)} = 1 \quad \text{for} \quad 1 \leqslant \nu \leqslant n; \ \lambda = 1, 2, \ldots.$$

Then, in view of the connectivity of $Z^{(\lambda)}$,

$$|a_\nu^{(\lambda)}| \leqslant 2n.$$

Therefore, replacing the sequence $Z^{(\lambda)}$ by an appropriate subsequence, we may assume

$$\lim_{\lambda \to \infty} a_\nu^{(\lambda)} = a_\nu^*, \quad \lim_{\lambda \to \infty} r_\nu^{(\lambda)} = r_\nu^*.$$

Then the set Z^* evidently fulfils (6).

Now let Z be any set with

$$\phi(Z) = C_n. \tag{7}$$

We shall first prove that it follows from (7) that the circles of Z have collinear centres and touch one another externally.

We shall first establish a lemma which will be used later.

LEMMA 2. *Let the $a_\nu^{(0)}$ be real and let the circles $Z_\nu^{(0)}$ touch one another externally. Further, let*

$$\phi(Z^{(0)}) = C_n.$$

Then $$\phi_\nu(Z^{(0)}) = C_n \quad \textit{for} \quad 1 \leqslant \nu \leqslant n.$$

In particular, it follows that

$$r_\nu^{(0)} > 0 \quad \textit{for} \quad 1 \leqslant \nu \leqslant n.$$

Proof of the lemma. We may assume that

$$a_\nu^{(0)} = 2(r_1^{(0)} + r_2^{(0)} + \ldots + r_\nu^{(0)}) - r_1^{(0)} - r_\nu^{(0)} \quad (1 \leqslant \nu \leqslant n).$$

If the lemma was not true, we should have

$$\phi_{\nu_0}(Z^{(0)}) > C_n$$

for a certain ν_0.

We choose $\epsilon > 0$ and take

$$\begin{aligned}
r_\nu^{(1)} &= r_\nu^{(0)} + \epsilon \quad \text{for} \quad \nu = \nu_0, \\
&= r_\nu^{(0)} \quad \text{for} \quad \nu \neq \nu_0, \\
a_\nu^{(1)} &= a_\nu^{(0)} - \epsilon \quad \text{for} \quad 1 \leqslant \nu < \nu_0, \\
&= a_\nu^{(0)} \quad \text{for} \quad \nu = \nu_0, \\
&= a_\nu^{(0)} + \epsilon \quad \text{for} \quad \nu > \nu_0.
\end{aligned}$$

$Z^{(1)}$ is again connected and, if ϵ is sufficiently small, it satisfies

$$\phi(Z^{(1)}) = C_n,$$

$$\phi_1(Z^{(1)}) > C_n, \quad \phi_n(Z^{(1)}) > C_n.$$

We fix ϵ. Choosing $\eta > 0$, we take

$$\begin{aligned} r_\nu^{(2)} &= r_\nu^{(1)} + \eta \quad \text{for} \quad \nu = 1 \text{ and } \nu = n, \\ &= r_\nu^{(1)} \qquad \text{for} \quad 1 < \nu < n, \\ a_\nu^{(2)} &= a_\nu^{(1)} - \eta \quad \text{for} \quad \nu = 1, \\ &= a_\nu^{(1)} \qquad \text{for} \quad 1 < \nu < n, \\ &= a_\nu^{(1)} + \eta \quad \text{for} \quad \nu = n. \end{aligned}$$

Then $Z^{(2)}$ is connected. For sufficiently small η it is evident that

$$\phi(Z^{(2)}) > C_n,$$

which contradicts the definition of C_n. This establishes the lemma.

Now let Z be any solution of (7). We say that the circles

$$Z_{a_1}, \ Z_{a_2}, \ \ldots, \ Z_{a_m}$$

form a *chain* $\mathfrak{K}$, if the indices a_μ are different from each other and

$$|a_{a_\mu} - a_{a_{\mu+1}}| \leqslant r_{a_\mu} + r_{a_{\mu+1}} \quad \text{for} \quad 1 \leqslant \mu < m.$$

We define the *length* of $\mathfrak{K}$ by

$$L(\mathfrak{K}) = 2(r_{a_1} + \ldots + r_{a_m}).$$

We have

$$(8) \qquad |a_{a_1} - a_{a_m}| \leqslant \sum_{\mu=1}^{m-1} |a_{a_\mu} - a_{a_{\mu+1}}| \leqslant \sum_{\mu=1}^{m-1} (r_{a_\mu} + r_{a_{\mu+1}})$$

$$= L(\mathfrak{K}) - r_{a_1} - r_{a_m}.$$

We number the circles Z_ν in such a way that

$$\mathfrak{K}_0: \ Z_1, \ Z_2, \ \ldots, \ Z_{n_0}$$

is a chain of maximal length among all chains in Z†. We define $Z^{(0)}$ by

$$r_\nu^{(0)} = r_\nu,$$

$$a_\nu^{(0)} = 2(r_1+\ldots+r_\nu)-r_1-r_\nu.$$

Then we shall show that

$$\phi_\nu(Z^{(0)}) \geqslant \phi_\nu(Z) \quad \text{for} \quad 1 \leqslant \nu \leqslant n. \tag{9}$$

Let ν be some fixed index with $1 \leqslant \nu \leqslant n$. In consequence of the definition of $\phi_\nu(Z)$, there exists ρ with $1 \leqslant \rho \leqslant n$ and

$$\frac{1}{r_\nu}(|a_\nu - a_\rho| + r_\rho) = \phi_\nu(Z).$$

Since Z is connected, there exists a chain

$$Z_{\alpha_1},\ Z_{\alpha_2},\ \ldots,\ Z_{\alpha_l}$$

with $\alpha_1 = \nu$, $\alpha_l = \rho$.

Case 1. $\qquad 1 \leqslant \nu \leqslant n_0.$

Then there exists a maximum index $k \leqslant l$ with

$$1 \leqslant \alpha_k \leqslant n_0.$$

Then, according as $\nu \leqslant \alpha_k$ or $\alpha_k < \nu$, we have either the chain

$$(a) \qquad Z_1,\ \ldots,\ Z_\nu,\ Z_{\nu+1},\ \ldots,\ Z_{\alpha_k},\ Z_{\alpha_{k+1}},\ \ldots,\ Z_{\alpha_l}$$

or the chain

$$(b) \qquad Z_{n_0},\ \ldots,\ Z_\nu,\ Z_{\nu-1},\ \ldots,\ Z_{\alpha_k},\ Z_{\alpha_{k+1}},\ \ldots,\ Z_{\alpha_l}.$$

Using the definition of $\mathfrak{R}_0$, we have in case (a)

$$L(Z_1,\ \ldots,\ Z_\nu,\ Z_{\nu+1},\ \ldots,\ Z_{\alpha_k},\ Z_{\alpha_{k+1}},\ \ldots,\ Z_{\alpha_l}) \leqslant L(Z_1,\ \ldots,\ Z_{n_0}),$$

i.e.

$$\begin{aligned} L(Z_\nu,\ Z_{\nu+1},\ \ldots,\ Z_{\alpha_k},\ Z_{\alpha_{k+1}},\ \ldots,\ Z_{\alpha_l}) &\leqslant L(Z_\nu,\ Z_{\nu+1},\ \ldots,\ Z_{n_0}) \\ &= L(Z_\nu^{(0)},\ \ldots,\ Z_{n_0}^{(0)}) \\ &\leqslant L(Z_\nu^{(0)},\ \ldots,\ Z_n^{(0)}). \end{aligned}$$

† For this idea of a chain of maximal length and the proof of (9), I am indebted to Mr. J. W. Sutherland of Magdalene College, Cambridge.

Therefore, making use of (8), we have

$$\phi_\nu(Z) = \frac{1}{r_\nu}(|a_\nu - a_\rho| + r_\rho) \leqslant \frac{1}{r_\nu}\{L(Z_\nu, \ldots, Z_{a_k}, \ldots, Z_{a_l}) - r_\nu\}$$

$$\leqslant \frac{1}{r_\nu^{(0)}}\{L(Z_\nu^{(0)}, Z_{\nu+1}^{(0)}, \ldots, Z_n^{(0)}) - r_\nu^{(0)}\}$$

$$= \frac{1}{r_\nu^{(0)}}(|a_\nu^{(0)} - a_n^{(0)}| + r_n^{(0)}) \leqslant \phi_\nu(Z^{(0)}).$$

A similar argument applies to the case (*b*).

Case 2. $n_0 < \nu \leqslant n.$

Then, using again (8) and the definition of $\mathfrak{R}_0$, we have

$$\phi_\nu(Z) = \frac{1}{r_\nu}(|a_\iota - a_\rho| + r_\rho) \leqslant \frac{1}{r_\nu}\{L(Z_{a_1}, \ldots, Z_{a_l}) - r_\nu\}$$

$$\leqslant \frac{1}{r_\nu}\{L(Z_1, \ldots, Z_{n_0}) - r_\nu\} = \frac{1}{r_\nu^{(0)}}\{L(Z_1^{(0)}, \ldots, Z_{n_0}^{(0)}) - r_\nu^{(0)}\}$$

$$\leqslant \frac{1}{r_\nu^{(0)}}\{L(Z_1^{(0)}, \ldots, Z_\nu^{(0)}) - r_\nu^{(0)}\} \tag{10}$$

$$\leqslant \phi_\nu(Z^{(0)}).$$

This proves (9).

Now it follows from the definition of C_n that

$$\phi(Z^{(0)}) = C_n.$$

Then Lemma 2, applied to $Z^{(0)}$, gives

$$\phi_\nu(Z^{(0)}) = C_n \quad \text{for all } \nu. \tag{11}$$

In particular, it follows that $r_\nu^{(0)} = r_\nu > 0$ for every ν.

If $n_0 < n$, then in (10) we can replace "$\leqslant$" by "$<$" for $\nu = n$, and therefore

$$\phi_n(Z^{(0)}) > \phi_n(Z) \geqslant C_n,$$

which contradicts (11). Therefore $n_0 = n$.

If the sets Z and $Z^{(0)}$ were not congruent, then, obviously,

$$\phi_1(Z^{(0)}) > \phi_1(Z) \geqslant C_n,$$

which again contradicts (11). Hence Z and $Z^{(0)}$ are congruent, and Z has the desired form.

Now it is easy to show that all sets Z satisfying (11) are similar to one another. Clearly we have

$$C_n = \phi_\nu(Z) = \frac{1}{r_\nu} \max \{-r_\nu + 2(r_1 + \ldots + r_\nu),\ -r_\nu + 2(r_\nu + \ldots + r_n)\}$$

$$= -1 + \frac{2}{r_\nu} \max \left(\sum_1^\nu r_\lambda,\ \sum_\nu^n r_\lambda \right),$$

$$\max \left(\sum_1^\nu r_\lambda,\ \sum_\nu^n r_\lambda \right) = \frac{r_\nu}{2} (C_n + 1) \quad (1 \leqslant \nu \leqslant n). \tag{12}$$

First we show that, for every ν,

$$r_\nu = r_{n+1-\nu}. \tag{13}$$

Taking $\nu = 1$ and $\nu = n$ in (12), we get

$$r_1 = r_n = \frac{2}{C_n + 1} \sum_1^n r_\lambda. \tag{14}$$

If (13) is true for $1 \leqslant \nu \leqslant \nu_0$, where

$$1 \leqslant \nu_0 < n - \nu_0,$$

then

$$\sum_1^{\nu_0+1} r_\lambda = r_{\nu_0+1} + \sum_1^{\nu_0} = r_{\nu_0+1} + \sum_{n+1-\nu_0}^{n} \leqslant \sum_{\nu_0+1}^{n} r_\lambda,$$

$$\sum_1^{n-\nu_0} r_\lambda = \sum_1^{\nu_0} + \sum_{\nu_0+1}^{n-\nu_0} = \sum_{n+1-\nu_0}^{n} + \sum_{\nu_0+1}^{n-\nu_0} = \sum_{\nu_0+1}^{n} \geqslant \sum_{n-\nu_0}^{n} r_\lambda,$$

and, therefore, by (12),

$$r_{\nu_0+1} = r_{n-\nu_0} = \frac{2}{C_n + 1} \sum_{\nu_0+1}^{n} r_\lambda.$$

It is thus proved by induction that

$$r_\nu = r_{n+1-\nu} = \frac{2}{C_n + 1} \sum_\nu^n r_\lambda \quad \text{for} \quad 1 \leqslant \nu \leqslant \left[\frac{n+1}{2} \right]. \tag{15}$$

Subtracting (15) for two consecutive values of ν, we get

$$r_\nu - r_{\nu+1} = \frac{2}{C_n + 1} r_\nu, \quad r_{\nu+1} = a_n r_\nu \quad \left(1 \leqslant \nu \leqslant \left[\frac{n-1}{2} \right] \right),$$

with

$$a_n = \frac{C_n - 1}{C_n + 1},$$

and, finally,

$$r_\nu = r_{n+1-\nu} = r_1\, a_n^{\nu-1} \quad \left(1 \leqslant \nu \leqslant \left[\frac{n+1}{2}\right]\right). \tag{16}$$

Thus the ratio of any two of the r_ν is uniquely determined.

To determine C_n we substitute (16) in (14) and get the following equations.

(a) When n is even,

$$\begin{aligned} r_1 &= (1-a_n)\, r_1 (1+a_n+\ldots+a_n^{\frac{1}{2}n-1}+a_n^{\frac{1}{2}n-1}+a_n^{\frac{1}{2}n-2}+\ldots+a_n+1) \\ &= 2r_1(1-a_n^{\frac{1}{2}n}), \end{aligned}$$

$$2\, a_n^{\frac{1}{2}n} = 1.$$

(b) When n is odd and greater than 1,

$$\begin{aligned} r_1 &= (1-a_n)\, r_1 (1+a_n+\ldots+a_n^{\frac{1}{2}(n-1)}+a_n^{\frac{1}{2}(n-3)}+\ldots+a_n+1) \\ &= r_1(1-a_n^{\frac{1}{2}(n+1)}+1-a_n^{\frac{1}{2}(n-1)}), \end{aligned}$$

$$a_n^{\frac{1}{2}(n+1)}+a_n^{\frac{1}{2}(n-1)} = 1.$$

Asymptotically, we have

$$C_n = \frac{1+a_n}{1-a_n} = \frac{1}{\log 2}\, n + O(1).$$

This shows that the result $C_n \leqslant 2n-1$ of Lemma 1 is not too far from the ultimate truth.

We sum up in the following theorem:

THEOREM III. *Given a closed bounded continuum $\mathfrak{M}$ in the x-plane and linear polynomials $f_1(x)$, ..., $f_n(x)$, there is always a point x^* in $\mathfrak{M}$ with*

$$|f_\nu(x^*)| \geqslant \frac{1-a_n}{1+a_n} \max_{x \text{ in } \mathfrak{M}} |f_\nu(x)| \quad (\nu = 1, 2, \ldots, n), \tag{17}$$

where a_n $(n > 1)$ is the positive root of

$$t^{[\frac{1}{2}(n+1)]}+t^{[\frac{1}{2}n]} = 1,$$

and $a_1 = 0$.

The constant in (17) *is best possible. The limiting cases, i.e. the cases where, for every point x^* of $\mathfrak{M}$,*

$$|f_\nu(x^*)| \leqslant \frac{1-a_n}{1+a_n} \max_{x \text{ in } \mathfrak{M}} |f_\nu(x)|$$

for some $\nu = \nu(x^*)$, *are characterised, for* $n > 1$, *by the following properties*:

(i) $\mathfrak{M}$ *is contained in the sum* Z *of* n *circles* Z_ν *with radii*

$$r_\nu = r_0\, a_n^{-|\nu - \frac{1}{2}(n+1)|} \quad (\nu = 1, 2, \ldots, n;\ r_0 > 0,\ \textit{arbitrary})$$

and centres a_ν *which, at the same time, are the zeros of the polynomials* $f_\nu(x)$.

(ii) *The circles* $Z_1, Z_2, \ldots, Z_n$ *have as diameters consecutive segments of a straight line.*

(iii) $\mathfrak{M}$ *and* Z *have the same diameter.*

The assertions about the limiting cases follow immediately from our considerations above.

3. We shall generalise I to the case of polynomials in more than one variable. This is not possible without further restriction on the admitted sets $\mathfrak{M}$, now sets of pairs (x, y) of complex numbers, as is shown by the example

$$k = 2;\quad f_1(x, y) = x;\quad f_2(x, y) = y;$$

$$\mathfrak{M} = \text{set of all } (x, y) \text{ with } 0 \leqslant x, y \leqslant 1;\quad xy = 0.$$

Here we have $M_1 = M_2 = 1$, while, on the other hand, at every point of $\mathfrak{M}$ at least one of the f_κ vanishes.

But, as we shall see, it is sufficient to suppose that $\mathfrak{M}$ is the "product" of two continua, one in the x-plane and one in the y-plane. For the sake of simplicity, we state only the generalisation of the case $\vartheta = 0$. There is no difficulty in generalising the whole of Theorem I by allowing a certain range $\langle 0, \vartheta_n \rangle$ for ϑ, where the positive number ϑ_n depends only on the sum n of the degrees of the polynomials.

THEOREM IV. *Corresponding to every pair of positive integers* n, r *there exists* $c_r(n) > 0$ *with the following properties. Given polynomials* $f_\kappa(x^{(1)}, \ldots, x^{(r)})$ *of total degree* n_κ $(1 \leqslant \kappa \leqslant k)$ *and* r *closed bounded continua* $\mathfrak{M}^{(\rho)}$ *in the complex plane with diameter* $d(\mathfrak{M}^{(\rho)})$, *there exist always points* $x^{(\rho)*}$ *in* $\mathfrak{M}^{(\rho)}$ *with*

$$|f_\kappa(x^{(1)*}, \ldots, x^{(r)*})| \geqslant c_r(n) \max_{x^{(\rho)} \text{ in } \mathfrak{M}^{(\rho)}} |f_\kappa(x^{(1)}, \ldots, x^{(r)})| \quad \text{for} \quad 1 \leqslant \kappa \leqslant k$$

$$(n = n_1 + n_2 + \ldots + n_k).$$

As analogue to Theorem II we have the more general theorem.

THEOREM V. *Let $\mathfrak{M}$ be a closed bounded aggregate of systems $(x^{(1)}, \ldots x^{(r)})$ (every $x^{(\rho)}$ complex). Let $\mathfrak{M}^{(\rho)}$ be the set of all $x^{(\rho)}$ which occur as the ρ-th component of a system of $\mathfrak{M}$. Let $d(\mathfrak{M}^{(\rho)}) > 0$ for every ρ. Let $\mathfrak{C}^{(\rho)}$ be a closed bounded continuum in the complex plane with the property that, if $x^{(\rho)}$ lies in $\mathfrak{C}^{(\rho)}$ for every ρ, then*

$$(x^{(1)}, \ldots, x^{(r)}) \text{ in } \mathfrak{M}.$$

Let $f_\kappa(x^{(1)}, \ldots, x^{(r)})$ be a polynomial of total degree n_κ $(1 \leqslant \kappa \leqslant k)$. Then there is an $x^{(\rho)*}$ in $\mathfrak{C}^{(\rho)}$ with

$$|f_\kappa(x^{(1)*}, \ldots, x^{(r)*})| \geqslant c_r(n)\left(\prod_{\rho=1}^{r} \frac{d(\mathfrak{C}^{(\rho)})}{d(\mathfrak{M}^{(\rho)})}\right)^n \max_{(x^{(1)}, \ldots, x^{(r)}) \text{ in } \mathfrak{M}} |f_\kappa(x^{(1)}, \ldots, x^{(r)})|$$

$$\text{for } 1 \leqslant \kappa \leqslant k \quad (n = n_1 + n_2 + \ldots + n_k).$$

The proof of V depends on an extension of the combinatorial considerations of p. 225.

We introduce the following notation. If $\mathfrak{A}$ is a distribution of the elements $\alpha, \beta, \gamma, \ldots$ into classes, the congruence

$$\alpha \equiv \beta \pmod{\mathfrak{A}}$$

means that in this distribution the elements α and β are in the same class.

Now the combinatorial lemma reads:

LEMMA 3. *Given integers $k, n, r > 0$, there is an integer $A(k, n, r) > 0$ with the following property. If $\mathfrak{A}$ is any distribution of all systems $(\mu^{(1)}, \ldots, \mu^{(r)})$ with integers $\mu^{(\rho)} \geqslant 0$ into k classes, there are always integers $c_\nu^{(\rho)}$ with*

$$0 \leqslant c_0^{(\rho)} < c_1^{(\rho)} < \ldots < c_n^{(\rho)} \leqslant A \quad (1 \leqslant \rho \leqslant r), \tag{1}$$

$$(c_{\nu_1}^{(1)}, \ldots, c_{\nu_r}^{(r)}) \equiv (c_0^{(1)}, \ldots, c_0^{(r)}) \pmod{\mathfrak{A}} \tag{2}$$

for any ν_ρ with $0 \leqslant \nu_\rho \leqslant n$.

A possible value of A is

$$A(k, n, r) = (kn)^{r!\,(n+1)^{r-1}}. \tag{3}$$

The lemma, without the numerical statement (3), follows at once from a deeper theorem† which asserts that it is even possible to find the $c_\nu^{(\rho)}$

† R. Rado, *Sitzungsberichte der Preussischen Akademie der Wissenschaften, Phys.-Math. Klasse* (1933), 589–596.

such that for every ρ the numbers

$$c_0^{(\rho)},\ c_1^{(\rho)},\ \ldots,\ c_n^{(\rho)}$$

form an arithmetical progression.

We shall prove the lemma independently.

Obviously, we can take, in accordance with (3),

$$A(k,\ n,\ 1) = kn.$$

Let us assume that $A(k,\ n,\ r-1) = A_{r-1}$ is already defined for some $k,\ n,\ r$ with $r > 1$. If we can show that it is possible to put

$$A_r = A(k,\ n,\ r) = A_{r-1}^{r(n+1)},$$

the lemma will be proved.

Let $\mathfrak{A}$ be a distribution of all systems $(\mu^{(1)},\ \ldots,\ \mu^{(r)})$ $(\mu^{(\rho)} \geqslant 0)$ into k classes. For every integer $\lambda \geqslant 0$ we define the distribution $\mathfrak{A}_\lambda$ by saying that

$$(\mu_1^{(2)},\ \ldots,\ \mu_1^{(r)}) \equiv (\mu_2^{(2)},\ \ldots,\ \mu_2^{(r)}) \pmod{\mathfrak{A}_\lambda}$$

is equivalent to

$$(\lambda,\ \mu_1^{(2)},\ \ldots,\ \mu_1^{(r)}) \equiv (\lambda,\ \mu_2^{(2)},\ \ldots,\ \mu_2^{(r)}) \pmod{\mathfrak{A}}.$$

$\mathfrak{A}_\lambda$ has, like $\mathfrak{A}$, k classes. The definition of A_{r-1} gives the existence of integers $c_r^{(\rho)}(\lambda)$ such that

$$0 \leqslant c_0^{(\rho)}(\lambda) < c_1^{(\rho)}(\lambda) < \ldots < c_n^{(\rho)}(\lambda) \leqslant A_{r-1} \quad (2 \leqslant \rho \leqslant r),$$

$$\left(c_{\nu_2}^{(2)}(\lambda),\ \ldots,\ c_{\nu_r}^{(r)}(\lambda)\right) \equiv \left(c_0^{(2)}(\lambda),\ \ldots,\ c_0^{(r)}(\lambda)\right) \pmod{\mathfrak{A}_\lambda}, \tag{4}$$

for any ν_ρ with $0 \leqslant \nu_\rho \leqslant n$.

Now we define the distribution $\mathfrak{A}^*$ by the rule:

$$\lambda \equiv \lambda' \pmod{\mathfrak{A}^*}$$

if, and only if,

$$c_\nu^{(\rho)}(\lambda) = c_\nu^{(\rho)}(\lambda') \quad \text{for} \quad 2 \leqslant \rho \leqslant r,\ 0 \leqslant \nu \leqslant n,$$

$$\left(\lambda,\ c_0^{(2)}(\lambda),\ \ldots,\ c_0^{(r)}(\lambda)\right) \equiv \left(\lambda',\ c_0^{(2)}(\lambda'),\ \ldots,\ c_0^{(r)}(\lambda')\right) \pmod{\mathfrak{A}}.$$

Obviously, $\mathfrak{A}^*$ has not more than

$$A_{r-1}^{(r-1)(n+1)} k = K$$

classes. Therefore, since

$$A(K,\ n,\ 1) = Kn,$$

there are integers $c_\nu^{(1)}$ such that

$$0 \leqslant c_0^{(1)} < c_1^{(1)} < \ldots < c_n^{(1)} \leqslant K n,$$

$$c_{\nu_1}^{(1)} \equiv c_0^{(1)} \pmod{\mathfrak{A}^*} \quad \text{for} \quad 0 \leqslant \nu_1 \leqslant n. \tag{5}$$

It follows from (5) and the definition of $\mathfrak{A}^*$ that

$$c_\nu^{(\rho)}(c_{\nu_1}^{(1)}) = c_\nu^{(\rho)}(c_0^{(1)}) = c_\nu^{(\rho)}, \quad \text{say,}$$

$$(2 \leqslant \rho \leqslant r,\ 0 \leqslant \nu \leqslant n,\ 0 \leqslant \nu_1 \leqslant n),$$

$$(c_{\nu_1}^{(1)},\ c_0^{(2)},\ \ldots,\ c_0^{(r)}) \equiv (c_0^{(1)},\ c_0^{(2)},\ \ldots,\ c_0^{(r)}) \pmod{\mathfrak{A}} \tag{6}$$

$$(0 \leqslant \nu_1 \leqslant n).$$

In (4) we put $\lambda = c_{\nu_1}^{(1)}$. Then, using the definition of $\mathfrak{A}_\lambda$, we get

$$(c_{\nu_1}^{(1)},\ c_{\nu_2}^{(2)},\ \ldots,\ c_{\nu_r}^{(r)}) \equiv (c_{\nu_1}^{(1)},\ c_0^{(2)},\ \ldots,\ c_0^{(r)}) \pmod{\mathfrak{A}} \tag{7}$$

$$(0 \leqslant \nu_\rho \leqslant n).$$

The congruence (2) follows from (6) and (7). Further,

$$0 \leqslant c_0^{(\rho)} < c_1^{(\rho)} < \ldots < c_n^{(\rho)} \leqslant K n = (A_{r-1})^{(r-1)(n+1)} kn \leqslant A_{r-1}^{r(n+1)} = A_r$$

$$(1 \leqslant \rho \leqslant r),$$

which establishes the lemma.

To prove Theorem V we need Lagrange's interpolation formula for the case of r variables. We state the formula in the special form in which it will be used. Let $f(x^{(1)}, \ldots, x^{(r)})$ be a polynomial of total degree not greater than n. For any ρ $(1 \leqslant \rho \leqslant r)$, let the numbers

$$z_0^{(\rho)},\ z_1^{(\rho)},\ \ldots,\ z_n^{(\rho)}$$

be different from one another. Then

$$f(x^{(1)}, \ldots, x^{(r)}) = \sum_{\substack{0 \leqslant \nu_1 \leqslant n \\ \cdots\cdots\cdots \\ 0 \leqslant \nu_r \leqslant n}} f(z_{\nu_1}^{(1)}, \ldots, z_{\nu_r}^{(r)}) \prod_{\substack{1 \leqslant \rho \leqslant r \\ 0 \leqslant \nu \leqslant n \\ \nu \neq \nu_\rho}} \frac{x^{(\rho)} - z_\nu^{(\rho)}}{z_{\nu_\rho}^{(\rho)} - z_\nu^{(\rho)}}.$$

After these preliminaries we can give the proof of V on exactly the same lines as in the case of Theorem I.

We may suppose that no f_κ is constant. Then, obviously,

$$k \leqslant n.$$

Further, there is no loss of generality in assuming that

$$d(\mathfrak{C}^{(\rho)}) = 1, \quad M_\kappa = \max_{\mathfrak{M}} |f_\kappa(x^{(1)}, \ldots, x^{(r)})| = 1.$$

Let us now suppose that the number $\epsilon > 0$ has the following property. Given any $x^{(\rho)}$ such that

$$x^{(\rho)} \text{ in } \mathfrak{C}^{(\rho)} \quad (1 \leqslant \rho \leqslant r),$$

there is always a $\kappa = \kappa(x)$ satisfying

$$|f_\kappa(x^{(1)}, \ldots, x^{(r)})| < \epsilon.$$

We have to show that in these circumstances

$$\epsilon > c_r(n) > 0.$$

There are $a^{(\rho)}$, $b^{(\rho)}$ in $\mathfrak{C}^{(\rho)}$ with

$$|a^{(\rho)} - b^{(\rho)}| = 1 \quad (1 \leqslant \rho \leqslant r).$$

Since $\mathfrak{C}^{(\rho)}$ is connected, there exist points $x_\mu^{(\rho)}$ in $\mathfrak{C}^{(\rho)}$ $(x_0^{(\rho)} = a^{(\rho)}, x_m^{(\rho)} = b^{(\rho)})$ with

$$|x_\mu^{(\rho)} - a^{(\rho)}| = \frac{\mu}{m} \quad (1 \leqslant \rho \leqslant r, \ 0 \leqslant \mu \leqslant m),$$

where m will be defined later.

Our assumption concerning ϵ gives

$$|f_{\kappa_\mu}(x_{\mu_1}^{(1)}, \ldots, x_{\mu_r}^{(r)})| < \epsilon \quad (0 \leqslant \mu_\rho \leqslant m),$$

where $\kappa_\mu = \kappa_{\mu_1 \mu_2 \ldots \mu_r}$ is one of the numbers 1, 2, ..., k. We define the distribution $\mathfrak{A}$ by:

$$(\mu_1, \ldots, \mu_r) \equiv (\mu_1', \ldots, \mu_r') \pmod{\mathfrak{A}},$$

if, and only if, $$\kappa_\mu = \kappa_{\mu'}.$$

$\mathfrak{A}$ has k classes.

Now, in particular, let

$$m = A(k, n, r),$$

where A is given by (3). Then the lemma asserts the existence of integers $c_\nu^{(\rho)}$ satisfying (1) and (2). Consequently, by the definition of $\mathfrak{A}$,

$$\kappa_{c_{\nu_1}^{(1)}, \ldots, c_{\nu_r}^{(r)}} = \kappa_{c_0^{(1)}, \ldots, c_0^{(r)}} = \kappa^*, \quad \text{say} \quad (0 \leqslant \nu_\rho \leqslant n).$$

We put $$x_{c_\nu^{(\rho)}}^{(\rho)} = z_\nu^{(\rho)}.$$

Then the definition of κ_ν gives

$$|f_{\kappa^*}(z_{\nu_1}^{(1)}, \ldots, z_{\nu_r}^{(r)})| < \epsilon \quad (0 \leqslant \nu_\rho \leqslant n). \tag{8}$$

We apply the interpolation formula to f_{κ^*}. Using (8) and the definitions of $d(\mathfrak{M}^{(\rho)})$, $x_\mu^{(\rho)}$, we get, for every $(x^{(1)}, \ldots, x^{(r)})$ of $\mathfrak{M}$,

$$|f_{\kappa^*}(x^{(1)}, \ldots, x^{(r)})| < (n+1)^r \prod_{\rho=1}^{r} \left[\frac{d(\mathfrak{M}^{(\rho)})}{1/m}\right]^n \epsilon.$$

Since the left-hand side attains its maximum 1 somewhere in $\mathfrak{M}$, we get

$$1 < (n+1)^r m^{nr} \left(\prod_\rho d(\mathfrak{M}^{(\rho)})\right)^n \epsilon,$$

which gives the desired result.

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THE GENERALIZATION OF A THEOREM ON BOREL SUMMABILITY

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1. *Introduction.*

The series Σa_n is said to be summable (V, α) to the sum s, if*

$$\lim_{\mu\to\infty} \frac{\mu^{-\alpha}}{\sqrt{(2\pi)}} \sum_{n=-\mu}^{\infty} s_{n+\mu} \exp(-\tfrac{1}{2}n^2\mu^{-2\alpha}) = s.$$

I shall be concerned in this paper with the particular case $\alpha = \frac{1}{2}$, and I require also an auxiliary method of summability $(C\text{-}V, \frac{1}{2})$ which is defined by means of the limit

$$\lim_{x\to\infty} \frac{1}{\sqrt{(2\pi x)}} \sum_{0}^{\infty} s_n \exp\{-\tfrac{1}{2}(x-n)^2 x^{-1}\}.$$

It will be observed that, when x tends to infinity through integral values, summability $(C\text{-}V, \frac{1}{2})$ reduces to summability $(V, \frac{1}{2})$.

Hardy and Littlewood† have proved the following result:

If $s_n = o(n^{\frac{1}{2}})$, then summability by Borel's method (B) implies summability $(V, \frac{1}{2})$, and conversely.

The proof consists in showing that, when $s_n = o(n^{\frac{1}{2}})$,

$$\lim_{\mu\to\infty} \left\{ e^{-\mu} \sum_{0}^{\infty} \frac{\mu^n}{n!} s_n - \frac{1}{\sqrt{(2\pi\mu)}} \sum_{-\mu}^{\infty} s_{n+\mu} \exp(-\tfrac{1}{2}n^2\mu^{-1}) \right\} = 0,$$

and the fact that Σa_n is summable (B) or summable $(V, \frac{1}{2})$ is not essential.

* This is a particular case of a more general method, see G. Valiron, *Rend. di Palermo*, 42 (1917), 267–284.

† G. H. Hardy and J. E. Littlewood, *Rend. di Palermo*, 41 (1916), 36–53.

It will be shown in this paper that the theorem remains true when the condition $s_n = o(n^{\frac{1}{2}})$ is replaced by $s_n = O(n^\lambda)$, λ being any positive number. In order to counter the effect of this widening of the magnitude condition it is necessary to postulate the summability of Σa_n by one of the methods concerned. A much closer approximation to the function

$$e^{-\mu}\frac{\mu^n}{n!}$$

is also required, and this is the point of the functions u and R which are introduced in Lemma 5.

From this theorem some connections between Cesàro's method and the methods (B) and $(V, \frac{1}{2})$ are deduced. The region of summability $(V, \frac{1}{2})$ of the power series Σz^n is also considered, and this throws some light on the necessity of the condition $s_n = O(n^\lambda)$.

In a previous paper*, I have proved similar theorems for summability by the methods of de la Vallée Poussin and Weierstrass, and I shall omit proofs which may readily be constructed by referring to this source.

2. *Preliminary lemmas.*

LEMMA 1. *If ρ, λ are positive or zero, $0 < \delta < \frac{1}{2}$, and if*

$$M = [x - x^{\frac{1}{2}+\delta}], \quad N = [x + x^{\frac{1}{2}+\delta}],$$

then, as x tends to infinity,

(i) $$\sum_N^\infty (n-x)^\rho n^\lambda \exp\{-\tfrac{1}{2}(n-x)^2 x^{-1}\} = o(1),$$

(ii) $$\sum_0^M (x-n)^\rho n^\lambda \exp\{-\tfrac{1}{2}(x-n)^2 x^{-1}\} = o(1),$$

(iii) $$\sum_0^\infty |x-n|^\rho n^\lambda \exp\{-\tfrac{1}{2}(x-n)^2 x^{-1}\} = O\{x^{\lambda+\frac{1}{2}(\rho+1)}\}.$$

By considering the derivative, it is easy to see that, if $\sigma > 0$,

$$(x-n)^\sigma \exp\{-\tfrac{1}{2}(x-n)^2 x^{-1}\},$$

regarded as a function of n, is increasing or decreasing according as

$$(x-n)^2 \gtrless \sigma x.$$

* J. M. Hyslop, *Proc. London Math. Soc.* (2), 40 (1935), 449–467.

We therefore have

$$\sum_N^\infty (n-x)^\rho n^\lambda \exp\{-\tfrac{1}{2}(n-x)^2 x^{-1}\}$$

$$= O\Big[x^\lambda \sum_N^{2x} (n-x)^\rho \exp\{-\tfrac{1}{2}(n-x)^2 x^{-1}\}\Big]$$

$$+ O\Big[\sum_{2x}^\infty (n-x)^{\rho+\lambda} \exp\{-\tfrac{1}{2}(n-x)^2 x^{-1}\}\Big]$$

$$= o(1) + O\left\{\int_x^\infty y^{\rho+\lambda} \exp(-\tfrac{1}{2}y^2 x^{-1})\, dy\right\} = o(1),$$

which proves (i).

The proof of (ii) is similar.

To prove (iii) we observe that, by (i),

$$\sum_0^\infty |x-n|^\rho n^\lambda \exp\{-\tfrac{1}{2}(n-x)^2 x^{-1}\}$$

$$= \sum_0^{2x} |x-n|^\rho n^\lambda \exp\{-\tfrac{1}{2}(x-n)^2 x^{-1}\} + o(1)$$

$$= O\left\{x^\lambda \int_0^x y^\rho \exp(-\tfrac{1}{2}y^2 x^{-1})\, dy\right\} + O(x^{\lambda+\frac{1}{2}\rho}) + o(1)$$

$$= O\{x^{\lambda+\frac{1}{2}(\rho+1)}\}.$$

LEMMA 2. *If ϕ and ψ are increasing functions satisfying the relation $\phi = O(\psi)$, and if $f^{(r)}(x)$ is continuous,*

$$f = o(\phi), \quad x^r f^{(r)} = O(\psi),$$

then, for $0 < s < r$,

$$x^s f^{(s)}(x) = o\{\phi^{(r-s)/r} \psi^{s/r}\}.$$

This result was proved by Hardy and Littlewood*. It is fundamental so far as this paper is concerned.

LEMMA 3. *If $s_n = O(n^\lambda)$, and if the series Σa_n is summable $(C\text{-}V, \frac{1}{2})$ to the sum zero, then*

$$\sum_0^\infty (x-n)^\rho s_n \exp\{-\tfrac{1}{2}(x-n)^2 x^{-1}\} = o\{x^{\frac{1}{2}(\rho+1)+\eta}\},$$

where ρ is a positive integer and λ, η are any positive numbers.

* G. H. Hardy and J. E. Littlewood, *Proc. London Math. Soc.* (2), 11 (1913), 411–478.

I shall merely sketch the proof of this lemma. Choose η and let

$$p = [2\lambda+\rho], \quad r = \max\left(p, \left[\frac{\lambda\rho}{\eta}\right]+1\right).$$

If $f(x)$ denotes the function

$$\sum_0^\infty s_n \exp\{-\tfrac{1}{2}(x-n)^2 x^{-1}\},$$

then it is easy to verify that $f^{(r)}(x)$ is of the form

$$\sum_0^\infty s_n \exp\{-\tfrac{1}{2}(x-n)^2 x^{-1}\}\left\{\sum_{\nu=0}^{r} a_{\nu,r}(x-n)^\nu x^{-r+[\frac{1}{2}(r-\nu)]} + \sum_{\nu=r+1}^{2r} a_{\nu,r}(x-n)^\nu x^{-\nu}\right\},$$

where $a_{\nu,r}$ is a polynomial in x^{-1} and $a_{r,r}$ has a non-zero constant term. Using the relation $s_n = O(n^\lambda)$ and applying Lemma 1 to each term in this sum, we obtain

$$f^{(r)}(x) = O\{x^{\lambda-\frac{1}{2}(r-1)}\}.$$

By hypothesis,

$$f(x) = o(x^{\frac{1}{2}}),$$

so that, by Lemma 2,

$$f^{(s)}(x) = o\{x^{-\frac{1}{2}(s-1)+(\lambda s/r)}\} \quad (0 < s < r).$$

If $\phi_s(x)$ denotes the function

$$\sum_0^\infty (x-n)^s s_n \exp\{-\tfrac{1}{2}(x-n)^2 x^{-1}\},$$

we have the following equations:

$$\sum_{\nu=1}^{s} a_{\nu,s} x^{-s+[\frac{1}{2}(s-\nu)]}\phi_\nu + \sum_{\nu=s+1}^{2s} a_{\nu,s} x^{-\nu}\phi_\nu = o\{x^{-\frac{1}{2}(s-1)+(\lambda s/r)}\},$$

where $s = 1, 2, \ldots, p$. In virtue of our choice of p and r, we may discard certain terms from these equations and rewrite them as

$$\sum_{\nu=1}^{s} a_{\nu,s} x^{-s+[\frac{1}{2}(s-\nu)]}\phi_\nu + \sum_{\nu=s+1}^{\beta} a_{\nu,s} x^{-\nu}\phi_\nu = o\{x^{-\frac{1}{2}(\gamma-1)+\eta}\},$$

where $\beta = \max(p, 2s)$ and γ is equal to s or ρ according as $s \leqslant \rho$ or $s > \rho$. We have now p equations in the p unknowns $\phi_1, \phi_2, \ldots, \phi_p$, and an examination of the determinant of the equations shows at once that

$$\phi_\rho = o\{x^{\frac{1}{2}(\rho+1)+\eta}\}.$$

LEMMA 4. *With the same conditions as in Lemma* 3,

$$\sum_{M}^{N}(x-n)^{\rho}s_n\exp\{-\tfrac{1}{2}(x-n)^2x^{-1}\}=o\{x^{\frac{1}{2}(\rho+1)+\eta}\}.$$

This follows immediately from Lemmas 1 and 3.

LEMMA 5. *If* $0<H<1$, $(1-H)x\leqslant n\leqslant(1+H)x$, *and* ν *is any positive integer, then, as* x *tends to infinity,*

$$e^{-x}\frac{x^n}{n!}=\frac{1}{\sqrt{(2\pi x)}}\exp\{-\tfrac{1}{2}(x-n)^2x^{-1}+u(n,x)+R(n,x)\},$$

where

$$R(n,x)=O\{|x-n|^{2\nu+1}x^{-2\nu}\},$$

and $u(n,x)$ *is of the form*

$$\sum_{\alpha=1}^{2\nu+1}\sum_{\beta=\gamma}^{2\nu}{}'\ b_{\alpha,\beta}(x-n)^{\alpha}x^{-\beta},$$

where $b_{\alpha,\beta}$ *is a constant,* $\gamma=\max(1,\alpha-1)$, *and the dash denotes the omission of the term for which* $\alpha=2$, $\beta=1$.

This may be proved quite easily by Stirling's theorem.

LEMMA 6. *If* M, N, *and* H *are defined as in previous lemmas, then*

(i) $u+R<\frac{1}{4}(x-n)^2x^{-1}$ *when* $N\leqslant n\leqslant(1+H)x$,

(ii) $u\to0$, $R\to0$ *as* $x\to\infty$ *when* $M\leqslant n\leqslant N$,

(iii) $|u+R|<1$ *when* $M\leqslant n\leqslant N$ *and for sufficiently large values of* x.

These may be deduced readily by examining the nature of the functions u and R.

LEMMA 7. *With the same conditions as in Lemma* 1, *we have, as* x *tends to infinity,*

(i) $$e^{-x}\sum_{0}^{M}(x-n)^{\rho}n^{\lambda}\frac{x^n}{n!}=o(1),$$

(ii) $$e^{-x}\sum_{N}^{\infty}(n-x)^{\rho}n^{\lambda}\frac{x^n}{n!}=o(1),$$

(iii) $$e^{-x}\sum_{0}^{\infty}|x-n|^{\rho}n^{\lambda}\frac{x^n}{n!}=O(x^{\lambda+\frac{1}{2}\rho}).$$

(i) We have, by Lemmas 5 and 6,

$$e^{-x}\sum_0^M (x-n)^\rho n^\lambda \frac{x^n}{n!} = O\left(x^{\rho+\lambda+1}e^{-x}\frac{x^M}{M!}\right)$$

$$= O[x^{\rho+\lambda+\frac{1}{2}}\exp\{-\tfrac{1}{2}(x-M)^2x^{-1}+u(M,x)+R(M,x)\}]$$

$$= o(1).$$

(ii) Write

$$e^{-x}\sum_N^\infty (n-x)^\rho n^\lambda \frac{x^n}{n!} = e^{-x}\sum_N^{(1+H)x}(n-x)^\rho n^\lambda \frac{x^n}{n!} + e^{-x}\sum_{(1+H)x}^\infty (n-x)^\rho n^\lambda \frac{x^n}{n!} = s_1+s_2.$$

By Lemmas 5 and 6,

$$s_1 = O\left[\sum_N^{(1+H)x} n^{\rho+\lambda}\exp\{-\tfrac{1}{2}(x-n)^2x^{-1}+u(n,x)+R(n,x)\}\right]$$

$$= O\left[\sum_N^\infty n^{\rho+\lambda}\exp\{-\tfrac{1}{4}(x-n)^2x^{-1}\}\right] = o(1),$$

and, if $p = [(1+H)x]$,

$$s_2 \leqslant p^{\rho+\lambda}e^{-x}\left\{\frac{x^p}{p!}+\left(\frac{p+1}{p}\right)^{\rho+\lambda}\frac{x^{p+1}}{(p+1)!}+\ldots\right\}$$

$$= O[x^{\rho+\lambda-\frac{1}{2}}\exp\{-\tfrac{1}{4}(x-p)^2x^{-1}\}] = o(1).$$

The proof that $s_2 = o(1)$ is given in more detail in a paper by Hardy and Littlewood.*

(iii) By Lemmas 1, 5, and 6,

$$e^{-x}\sum_0^\infty |x-n|^\rho n^\lambda \frac{x^n}{n!} = e^{-x}\sum_M^N |x-n|^\rho n^\lambda \frac{x^n}{n!} + o(1)$$

$$= O\left\{x^\lambda e^{-x}\sum_M^N |x-n|^\rho \frac{x^n}{n!}\right\} + o(1)$$

$$= O\left[x^{\lambda-\frac{1}{2}}\sum_M^N |x-n|^\rho \exp\{-\tfrac{1}{2}(x-n)^2x^{-1}+u+R\}\right] + o(1)$$

$$= O\left[x^{\lambda-\frac{1}{2}}\sum_M^N |x-n|^\rho \exp\{-\tfrac{1}{2}(x-n)^2x^{-1}\}\right] + o(1)$$

$$= O(x^{\lambda+\frac{1}{2}\rho}).$$

* G. H. Hardy and J. E. Littlewood, *Proc. London Math. Soc.* (2), 11 (1913), 1–16.

3. *The equivalence theorems.*

THEOREM I. *If the series Σa_n is summable $(C\text{-}V, \frac{1}{2})$, and if $s_n = O(n^\lambda)$, then it is summable (B) to the same sum.*

We take the sum of the series in the $(C\text{-}V, \frac{1}{2})$ sense to be zero and consider

$$V(x) = \left| \sum_0^\infty \left[e^{-x} \frac{x^n}{n!} - \frac{1}{\sqrt{(2\pi x)}} \exp\{-\tfrac{1}{2}(x-n)^2 x^{-1}\} \right] s_n \right|.$$

We may write $$V(x) \leqslant V_1 + V_2 + V_3,$$

where, by Lemmas 1 and 7,

$$V_1 = \left| \sum_0^M \left[e^{-x} \frac{x^n}{n!} - \frac{1}{\sqrt{(2\pi x)}} \exp\{-\tfrac{1}{2}(x-n)^2 x^{-1}\} \right] s_n \right|$$

$$= O\left[\sum_0^M n^\lambda e^{-x} \frac{x^n}{n!} \right] + O\left[x^{-\frac{1}{2}} \sum_0^M n^\lambda \exp\{-\tfrac{1}{2}(x-n)^2 x^{-1}\} \right] = o(1),$$

$$V_2 = \left| \sum_N^\infty \left[e^{-x} \frac{x^n}{n!} - \frac{1}{\sqrt{(2\pi x)}} \exp\{-\tfrac{1}{2}(x-n)^2 x^{-1}\} \right] s_n \right|$$

$$= O\left[\sum_N^\infty e^{-x} n^\lambda \frac{x^n}{n!} \right] + O\left[x^{-\frac{1}{2}} \sum_N^\infty n^\lambda \exp\{-\tfrac{1}{2}(x-n)^2 x^{-1}\} \right] = o(1),$$

and $$V_3 = \left| \sum_M^N \left[e^{-x} \frac{x^n}{n!} - \frac{1}{\sqrt{(2\pi x)}} \exp\{-\tfrac{1}{2}(x-n)^2 x^{-1}\} \right] s_n \right|.$$

Using Lemma 5, we have

$$V_3 = \left| \frac{1}{\sqrt{(2\pi x)}} \sum_M^N s_n \exp\{-\tfrac{1}{2}(x-n)^2 x^{-1}\} [e^{u+R} - 1] \right|$$

$$\leqslant V_{3,1} + V_{3,2} + V_{3,3},$$

where

$$V_{3,1} = \left| \frac{1}{\sqrt{(2\pi x)}} \sum_M^N \left\{ u + \frac{u^2}{2!} + \ldots + \frac{u^{2\mu}}{(2\mu)!} \right\} s_n \exp\{-\tfrac{1}{2}(x-n)^2 x^{-1}\} \right|,$$

$$V_{3,2} = \left| \frac{1}{\sqrt{(2\pi x)}} \sum_M^N \left\{ \frac{(u+R)-u}{1!} + \ldots + \frac{(u+R)^{2\mu} - u^{2\mu}}{(2\mu)!} \right\} s_n \exp\{-\tfrac{1}{2}(x-n)^2 x^{-1}\} \right|,$$

$$V_{3,3} = \left| \frac{1}{\sqrt{(2\pi x)}} \sum_M^N \left\{ \frac{(u+R)^{2\mu+1}}{(2\mu+1)!} + \ldots \right\} s_n \exp\{-\tfrac{1}{2}(x-n)^2 x^{-1}\} \right|,$$

and μ is a positive integer at our disposal.

By Lemma 4 each term in $V_{3,1}$ tends to zero as x tends to infinity, so that $V_{3,1} = o(1)$.

$V_{3,2}$ is majorized by a sum of terms of the form

$$\sigma = x^{-\frac{1}{2}} \sum_{M}^{N} n^{\lambda} R^{s} |u|^{t} \exp\{-\tfrac{1}{2}(x-n)^2 x^{-1}\},$$

where $s \geqslant 1$, $s+t \leqslant 2\mu$. By Lemma 1,

$$\sigma = O\{x^{-\frac{1}{2}+\lambda-s(\nu-\frac{1}{2})-\frac{1}{2}t+\frac{1}{2}}\} = O\{x^{\lambda-\nu+\frac{1}{2}}\} = o(1),$$

if ν is chosen larger than $\lambda+\frac{1}{2}$.

By Lemma 6,

$$V_{3,3} = O\left[x^{-\frac{1}{2}} \sum_{M}^{N} n^{\lambda} |u+R|^{2\mu+1} \exp\{-\tfrac{1}{2}(x-n)^2 x^{-1}\}\right]$$

$$= O\left[x^{-\frac{1}{2}} \sum_{M}^{N} n^{\lambda} x^{-4\mu-2} |x-n|^{6\mu+3} \exp\{-\tfrac{1}{2}(x-n)^2 x^{-1}\}\right]$$

$$+ O\left[x^{-\frac{1}{2}} \sum_{M}^{N} n^{\lambda} x^{-2\mu-1} |x-n|^{2\mu+1} \exp\{-\tfrac{1}{2}(x-n)^2 x^{-1}\}\right]$$

$$= O\{x^{-\frac{1}{2}-\mu+\lambda}\} = o(1),$$

if μ is chosen larger than $\lambda-\frac{1}{2}$.

It follows that $V(x) = o(1)$, and the theorem is proved.

It remains to prove that summability $(V, \frac{1}{2})$ implies summability $(C\text{-}V, \frac{1}{2})$ when $s_n = O(n^{\lambda})$. To do this we require a lemma similar to Lemma 3.

LEMMA 8. *If the series Σa_n is summable $(V, \frac{1}{2})$ to the sum zero, and if $s_n = O(n^{\lambda})$, then*

$$\sum_{0}^{\infty} (\mu-n)^{\rho} s_n \exp\{-\tfrac{1}{2}(\mu-n)^2 \mu^{-1}\} = o(\mu^{\rho+\frac{1}{2}}),$$

where ρ is a positive integer.

Let

$$f(\mu) = \sum_{0}^{\infty} s_n \exp\{-\tfrac{1}{2}(\mu-n)^2 \mu^{-1}\},$$

$$\psi_s(\mu) = \mu^{-s} \sum_{0}^{\infty} (\mu-n)^{s} s_n \exp\{-\tfrac{1}{2}(\mu-n)^2 \mu^{-1}\} \quad (s = 1, 2, \ldots).$$

It may be verified that $\Delta^s f(\mu)$ is of the form

$$\mu^{-1} \sum_{\alpha=1}^{s-1} c_{s,\alpha}\psi_\alpha + \sum_{\alpha=s}^{p} c_{s,\alpha}\psi_\alpha + o(1),$$

where $p > 2\lambda+1$, $s = 1, 2, 3, \ldots, p$, $c_{s,\alpha}$ is a polynomial in μ^{-1} and $c_{s,s}$ is not zero for large values of μ.

By the hypothesis,

$$\Delta^s f(\mu) = o(\mu^{\frac{1}{2}})$$

for every value of s, so that we obtain p equations of the form

$$\mu^{-1} \sum_{\alpha=1}^{s-1} c_{s,\alpha}\psi_\alpha + \sum_{\alpha=s}^{p} c_{s,\alpha}\psi_\alpha = o(\mu^{\frac{1}{2}}) \quad (s = 1, 2, \ldots, p);$$

and, on examining the determinant of these equations, we see that

$$\psi_\rho(\mu) = o(\mu^{\frac{1}{2}}),$$

which leads to the required result.

THEOREM II. *If $s_n = O(n^\lambda)$, and the series Σa_n is summable $(V, \frac{1}{2})$, then it is summable $(C\text{-}V, \frac{1}{2})$ to the same sum.*

Take the sum of the series in the $(V, \frac{1}{2})$ sense to be zero, and let $x = \mu+\theta$, $0 < \theta < 1$. It is not difficult to see that

$$\left| \sum_0^\infty s_n [\exp\{-\tfrac{1}{2}(x-n)^2 x^{-1} - \exp\{-\tfrac{1}{2}(\mu-n)^2\mu^{-1}\}] \right|$$

$$= |A_1\psi_1 + A_2\psi_2 + \ldots + A_p\psi_p| + o(1),$$

where p is defined as in Lemma 8, and $A_1, A_2, \ldots, A_p$ are polynomials in μ^{-1}. The result now follows by the hypothesis and Lemma 8.

THEOREM III. *If $s_n = O(n^\lambda)$, and if Σa_n is summable $(V, \frac{1}{2})$, then it is summable (B) to the same sum.*

This follows at once from Theorems I and II.

LEMMA 9. *If $s_n = O(n^\lambda)$, and if Σa_n is summable (B) to the sum zero, then*

$$e^{-x} \sum_0^\infty s_n (x-n)^\rho \frac{x^n}{n!} = o(x^{\frac{1}{2}\rho+\eta}),$$

where ρ is a positive integer and η is any positive number.

The proof of this lemma is similar to the proof of Lemma 3.

THEOREM IV. *If* $s_n = O(n^\lambda)$, *and if* Σa_n *is summable* (B), *then it is summable* $(C\text{-}V, \frac{1}{2})$ *to the same sum.*

The proof may be constructed by referring to my paper* already mentioned, or by following the lines of the proof of Theorem I and using Lemma 9 instead of Lemma 4.

THEOREM V. *If* Σa_n *is summable* $(C\text{-}V, \frac{1}{2})$, *then it is summable* $(V, \frac{1}{2})$ *to the same sum.*

This is, of course, obvious.

THEOREM VI. *If* $s_n = O(n^\lambda)$, *and if* Σa_n *is summable* (B), *then it is summable* $(V, \frac{1}{2})$ *to the same sum.*

This follows at once from Theorems IV and V.

4. *Connections with Cesàro's method.*

THEOREM VII. *If* $s_n = O(n^\lambda)$, *and if* Σa_n *is summable* $(V, \frac{1}{2})$, *then it is summable to the same sum by Cesàro means of sufficiently high order.*

Hardy and Littlewood† have proved a similar theorem for Borel summability, and from this and Theorem III the above follows immediately.

It will be observed that other theorems on Borel summability of a similar character apply also in the case of summability $(V, \frac{1}{2})$.

We also have the following generalization of a theorem due to Hardy‡.

THEOREM VIII. *If* p *is a positive integer, and if the* p*-th Cesàro mean of* Σa_n *is such that*

$$c_n^{(p)} = s + o(n^{-\frac{1}{2}p}),$$

then Σa_n *is summable* (B) *to the sum* s.

It has been proved§ that, subject to the above hypothesis, the series

* J. M. Hyslop, *loc. cit.*, 454–457.

† G. H. Hardy and J. E. Littlewood, *Proc. London Math. Soc.* (2), 11 (1913), 1–16.

‡ G. H. Hardy, *Quart. J. of Math.*, 35 (1904), 22–66.

§ This is a particular case of a more general theorem; see J. M. Hyslop, *Proc. Edinburgh Math. Soc.* (2), 4 (1936), 218–223.

Σa_n is summable $(V, \frac{1}{2})$ to the sum s. Theorem VIII is an immediate consequence of this result and Theorem III.

5. *The necessity of the condition* $s_n = O(n^\lambda)$.

LEMMA 10. *If m is any positive integer and μ_0 is positive, then the series*

$$\sum_m^\infty \exp(-2\mu n^2\pi^2)\cos(2\mu n\pi \log z)$$

converges uniformly for $\mu \geqslant \mu_0$, and for all finite values of z.

Write $z = re^{i\theta}$, $-\pi < \theta \leqslant \pi$. Then it is easy to see that

$$\sum_m^\infty |\exp(-2\mu n^2\pi^2)\cos(2\mu n\pi \log z) < A\sum_m^\infty \exp\{-2\mu\pi^2(n^2-n)\}$$

$$< A\sum_m^\infty \exp\{-2\mu_0\pi^2(n^2-n)\},$$

where A is a constant independent of z and μ. This series converges, so that the result follows.

LEMMA 11. *The function*

$$F(z, \mu) = 2\sum_2^\infty \exp(-2\mu n^2\pi^2)\cos(2\mu n\pi \log z)$$

tends uniformly to zero as μ tends to infinity for all finite values of z.

Each term in this series tends uniformly to zero as μ tends to infinity, so that the result follows from Lemma 10.

In what follows, C denotes the curve

$$r = \exp\{\sqrt{(1+\theta^2)}-1\}, \quad -\pi < \theta \leqslant \pi.$$

THEOREM IX. *The series $\sum_0^\infty z^n$ is summable $(V, \frac{1}{2})$ to the sum $1/(1-z)$ for every point z lying within the curve C.*

It is clear that every point of C, except the point $z = 1$, lies outside the unit circle $|z| = 1$.

When $|z| < 1$, the series is summable $(V, \frac{1}{2})$ to the sum $1/(1-z)$, by the consistency theorem* for summability $(V, \frac{1}{2})$. When $z = 1$, the series is

* J. M. Hyslop, *loc. cit.*, 220.

properly divergent. We may therefore confine our attention to values of z for which $|z| \geqslant 1$, $z \neq 1$.

We have

$$\frac{1-z}{\sqrt{(2\pi\mu)}} \sum_{-\mu}^{\infty} s_{n+\mu} \exp(-\tfrac{1}{2}n^2\mu^{-1})$$

$$= \frac{1}{\sqrt{(2\pi\mu)}} \sum_{-\mu}^{\infty} (1-z^{n+\mu}) \exp(-\tfrac{1}{2}n^2\mu^{-1})$$

$$= \frac{1}{\sqrt{(2\pi\mu)}} \sum_{-\mu}^{\infty} \exp(-\tfrac{1}{2}n^2\mu^{-1}) - \frac{z^\mu}{\sqrt{(2\pi\mu)}} \sum_{-\infty}^{\infty} \exp(n\log z - \tfrac{1}{2}n^2\mu^{-1})$$

$$+ \frac{1}{\sqrt{(2\pi\mu)}} \sum_{-\infty}^{-\mu} z^{n+\mu} \exp(-\tfrac{1}{2}n^2\mu^{-1})$$

$$= s_1 - s_2 + s_3.$$

It is easy to prove that $s_1 \to 1$ as $\mu \to \infty$. Also

$$|s_3| \leqslant \frac{1}{\sqrt{(2\pi\mu)}} \sum_{\mu}^{\infty} |z|^{-(n-\mu)} \exp(-\tfrac{1}{2}n^2\mu^{-1})$$

$$\leqslant \frac{1}{\sqrt{(2\pi\mu)}} \sum_{\mu}^{\infty} \exp(-\tfrac{1}{2}n^2\mu^{-1}).$$

Hence s_3 tends uniformly to zero as μ tends to infinity. It remains to prove that s_2 behaves similarly.

By the transformation formula for the theta-function we have

$$s_2 = z^\mu \sum_{-\infty}^{\infty} \exp\left\{-2\mu\left(\frac{\log z}{2i} - n\pi\right)^2\right\}$$

$$= \phi(z, \mu)\{1 + G(z, \mu) + F(z, \mu)\}, \tag{1}$$

where $$\phi(z, \mu) = \exp[\mu\{\log z + \tfrac{1}{2}(\log z)^2\}],$$

$$G(z, \mu) = 2\exp(-2\mu\pi^2)\cos(2\mu\pi\log z),$$

and $F(z, \mu)$ is the function defined in Lemma 11.

If z lies between the unit circle and C, that is, if

$$0 \leqslant \log r < \sqrt{(1+\theta^2)} - 1, \quad 0 < |\theta| \leqslant \pi,$$

we have $$|\phi| = \exp[\tfrac{1}{2}\mu\{(\log r)^2 + 2\log r - \theta^2\}],$$

which tends to zero as μ tends to infinity.

Also, it is clear that G is bounded for such values of z.

It follows that s_2 tends to zero as μ tends to infinity, and the theorem is therefore proved.

THEOREM X. *The series* $\sum_0^\infty z^n$ *is not summable* $(V, \frac{1}{2})$ *for any point* z *lying on or outside the curve* C.

Two cases have to be considered.

Case (i). $0 < |\theta| < \pi, \quad \log r \geqslant \sqrt{(1+\theta^2)} - 1.$

From Theorem IX we have

$$G = \exp\{2\mu(-\pi^2 - \pi\theta + i\pi \log r)\} + \exp\{2\mu(-\pi^2 + \pi\theta - i\pi \log r)\},$$

which tends to zero as μ tends to infinity. Hence, from (1),

$$\lim_{\mu \to \infty} \frac{s_2}{\phi} = 1.$$

Clearly, ϕ does not tend to a limit, and therefore s_2 does not tend to a limit. It follows that the series is not summable $(V, \frac{1}{2})$ in this case.

Case (ii). $\theta = \pi, \quad \log r \geqslant \sqrt{(1+\pi^2)} - 1.$

In this case,

$$\begin{aligned} G &= \exp\{2\mu(-2\pi^2 + i\pi \log r)\} + \exp(-2\mu i\pi \log r) \\ &= H(z, \mu) + \exp(-2\mu i\pi \log r), \end{aligned}$$

where H tends uniformly to zero as μ tends to infinity. Hence, from (1),

$$\frac{s_2}{\phi} = 1 + H + F + \exp(-2\mu i\pi \log r).$$

As μ tends to infinity through the values $m/\log r$, m being a positive integer, the right-hand side of this equation tends to 2. Clearly, ϕ does not tend to a limit as μ tends to infinity through these values, and so neither does s_2.

The theorem is therefore proved.

THEOREM XI. *There exist series with* $s_n = O(e^{An})$, $A > 0$, *which are summable* (B) *but not summable* $(V, \frac{1}{2})$.

It is well-known that the series $\sum_0^\infty z^n$ is summable (B) throughout the region $R(z) < 1$.

Let $$f(\theta) = \log\cos\theta + \sqrt{(1+\theta^2)} - 1 \quad (0 \leqslant \theta < \tfrac{1}{2}\pi).$$

Then $$f'(\theta) = -\tan\theta + \frac{\theta}{\sqrt{(1+\theta^2)}} < 0.$$

It follows that

$$\exp\{\sqrt{(1+\theta^2)} - 1\} < \sec\theta \quad (0 < \theta < \tfrac{1}{2}\pi),$$

so that the curve C lies wholly to the left of $R(z) = 1$. The theorem therefore follows.

This theorem shows that a further generalization of Theorem III is not improbable, but that very little, if any, generalization is possible in the case of Theorem VI.

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NOTES ON THE THEORY OF SERIES (XX): ON LAMBERT SERIES

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1. A series Σa_n may be said to be "Lambert summable", or "summable (L)", to sum l, if

$$g(y) = \sum_1^\infty \frac{nye^{-ny}}{1-e^{-ny}} a_n \to l \tag{1.1}$$

when $y \to 0$ through positive values. This method of summation is one of the most interesting of the less familiar methods because of its curious connections with the analytic theory of numbers†.

There are known theorems which state simple relations between Lambert summability and summability by more usual methods.

THEOREM 1‡. *If Σa_n is summable (C, k), for any k, then it is summable (L) to the same sum.*

THEOREM 2§. *If Σa_n is summable (L), then it is summable (A) to the same sum.*

In fact $(C) \prec (L) \prec (A)$; the Lambert method includes the Cesàro methods and is included in the Abel-Poisson method. The proof of Theorem 1 is straightforward‖; but that of Theorem 2 demands the "prime number theorem" or an equivalent. Both theorems are "Abelian" in character; they state relations of complete inclusion, without supplementary conditions.

† See Hardy and Littlewood (4); Wiener (10, 39; 11, 112).

‡ Hardy (2), Theorem 2.

§ Hardy and Littlewood (4).

‖ It is a corollary of general theorems of Bohr, Bromwich, and Hardy, concerning series $\Sigma a_n \chi_n(y)$.

2. It is natural to ask for "Tauberian" theorems corresponding to Theorem 2: in what circumstances does summability (A) imply summability (L)? The supplementary or "Tauberian" conditions may be expected to bear, not directly on the coefficients a_n, but upon the function

$$f(y) = \sum_1^\infty a_n e^{-ny}. \tag{2.1}$$

One such theorem has been proved by Ananda Rau†.

THEOREM 3. *If*

$$f(y) \to l \tag{2.2}$$

and

$$f'(y) = O\big(\phi(y)\big), \tag{2.3}$$

where $\phi(y)$ is a positive decreasing function such that

$$\int_0 \phi(y)\, dy < \infty, \tag{2.4}$$

then

$$g(y) \to l. \tag{2.5}$$

That is to say, summability (A) implies summability (L) under the supplementary condition (2.3). We shall give a new proof of this theorem in §3, and see later (§8) that it is a "best possible" theorem of its kind. There are, however, other theorems (neither including Theorem 3 nor included by it) in which the supplementary condition has a simpler form.

3. THEOREM 4. *If $f(y) \to l$ and*

$$f''(y) > -\frac{C}{y^2}, \tag{3.1}$$

then $g(y) \to l$‡.

We may suppose $l = 0$. Then, since

$$g(y) = \sum_{m=1}^\infty \frac{mye^{-my}}{1-e^{-my}} a_m = y \sum_{m=1}^\infty \sum_{n=1}^\infty m a_m e^{-mny}$$

$$= y \sum_{n=1}^\infty \sum_{m=1}^\infty m a_m e^{-mny} = -y \sum_{n=1}^\infty f'(ny),$$

† Ananda Rau (1), Theorem 2.2.

‡ Here, and elsewhere, C is a positive number independent of y and (when n occurs) of n.

(2.5) is equivalent to

$$h(y) = -\frac{g(y)}{y} = \sum_1^\infty f'(ny) = o\left(\frac{1}{y}\right). \tag{3.2}$$

Now
$$f(y) = -\int_y^\infty f'(t)\,dt = -\sum_{n=1}^\infty \int_{ny}^{(n+1)y} f'(t)\,dt.$$

Hence

$$f(y)-g(y) = f(y)+yh(y) = \sum_{n=1}^\infty u_n(y) = S(y), \tag{3.3}$$

where

$$u_n(y) = \int_{ny}^{(n+1)y} \Big(f'(ny)-f'(t)\Big)\,dt. \tag{3.4}$$

Similarly,

$$f(y)-yf'(y)-g(y) = f(y)-yf'(y)+yh(y) = \sum_{n=1}^\infty v_n(y) = T(y), \tag{3.5}$$

where

$$v_n(y) = \int_{ny}^{(n+1)y} \Big(f'\big((n+1)y\big)-f'(t)\Big)\,dt. \tag{3.6}$$

We use both (3.3) and (3.5) in proving Theorem 4.

The conditions of Theorem 4 imply that†

$$f'(y) = o\left(\frac{1}{y}\right), \tag{3.7}$$

so that each term of $S(y)$ or $T(y)$ tends to zero. If the conditions of Theorem 3 are satisfied, then $S(y)$ is majorized by

$$y\Sigma\phi(ny) < y\int_0^\infty \phi(ty)\,dt < C,$$

so that $S(y)\to 0$ and therefore $g(y)\to 0$. We thus obtain an immediate proof of Ananda Rau's theorem.

Returning to Theorem 4, we write

$$S(y) = \sum_1^\infty u_n(y) = \sum_1^N u_n(y) + \sum_{N+1}^\infty u_n(y) = S_1(y)+S_2(y).$$

In $S_2(y)$ we have

$$f'(ny)-f'(t) = -(t-ny)f''(\tau) \quad (ny<\tau<t),$$

and so
$$u_n(y) \leqslant y\,.\,y\,.\,\frac{C}{n^2y^2} = \frac{C}{n^2}.$$

† Hardy and Littlewood (3), or Landau (5, 58).

Hence $$S_2(y) \leqslant C \sum_N^\infty \frac{1}{n^2} < \epsilon$$

for $N = N(\epsilon)$. Also $S_1(y) \to 0$ when N is fixed. It follows that

$$\overline{\lim}\, S(y) \leqslant 0,$$

and therefore, from (3.3), that

$$\overline{\lim}\, yh(y) \leqslant 0. \tag{3.8}$$

Similarly we prove that

$$\underline{\lim}\, T(y) \geqslant 0,$$

and therefore, from (3.5) and (3.7), that

$$\underline{\lim}\, yh(y) \geqslant 0. \tag{3.9}$$

Finally, (3.8) and (3.9) show that $yh(y) \to 0$.

4. There is another theorem of some interest which neither includes nor is included by Theorem 4.

We write

$$f^\alpha(y) = f_{-\alpha}(y) = \Sigma n^\alpha a_n e^{-ny}, \tag{4.1}$$

for all real α. When α is a positive integer, $f^\alpha(y)$ is, apart from sign, the α-th derivative of $f(y)$. Also

$$f_\beta(y) = \frac{1}{\Gamma(\beta)} \int_y^\infty (t-y)^{\beta-1} f(t)\, dt \tag{4.2}$$

when $\beta > 0$, and

$$f^{\alpha-\gamma}(y) = \frac{1}{\Gamma(\gamma)} \int_y^\infty (t-y)^{\gamma-1} f^\alpha(t)\, dt \tag{4.3}$$

when $\gamma > 0$. Our formulae embody the natural definitions of the derivatives of arbitrary order.

THEOREM 5. *If* $f(y) \to l$ *and*

$$f^{1+\delta}(y) = O(y^{-1-\delta}) \tag{4.4}$$

for some positive δ, *then* $g(y) \to l$.

It will be observed that (4.4) is not a "one-sided" condition, as (3.1) is; we return to this point in a moment.

We use the identity (3.3). The hypotheses again involve (3.7)†, and $u_n(y) \to 0$ for every n.

Next, by (4.3),

$$-f'(t) = f^1(t) = \frac{1}{\Gamma(\delta)} \int_t^\infty (u-t)^{\delta-1} f^{1+\delta}(u)\, du,$$

and so

$$f'(ny) - f'(t) = -\frac{1}{\Gamma(\delta)} \int_{ny}^t (u-ny)^{\delta-1} f^{1+\delta}(u)\, du$$

$$+ \frac{1}{\Gamma(\delta)} \int_t^\infty \left\{ (u-t)^{\delta-1} - (u-ny)^{\delta-1} \right\} f^{1+\delta}(u)\, du = J_1 + J_2,$$

say. Here

$$J_1 = O\left((ny)^{-1-\delta} \int_{ny}^{(n+1)y} (u-ny)^{\delta-1} du \right) = O\left((ny)^{-1-\delta} y^\delta \right) = O(n^{-1-\delta} y^{-1}),$$

uniformly for $ny < t < (n+1)y$ and all n; and

$$J_2 = O\left((ny)^{-1-\delta} \int_t^\infty \left\{ (u-t)^{\delta-1} - (u-ny)^{\delta-1} \right\} du \right)$$

$$= O\left((ny)^{-1-\delta} (t-ny)^\delta \right) = O(n^{-1-\delta} y^{-1}).$$

Hence $$|u_n(y)| < Cn^{-1-\delta},$$

and $S(y)$ is dominated by $C \Sigma n^{-1-\delta}$. It follows that $S(y) \to 0$.

5. It might be thought that there should be a more general theorem with

$$f^{1+\delta}(y) > -Cy^{-1-\delta} \tag{5.1}$$

in place of (3.1) or (4.4); such a theorem would include both Theorem 4 and Theorem 5. It seems that this is not so; that (5.1) is not a sufficient condition unless $\delta \geqslant 1$; and that each of Theorems 4 and 5 is in a sense "best possible". We do not prove this formally, but the remarks which follow will probably persuade the reader.

There are well-known relations between the orders of magnitude of the derivatives of $f(y)$. Let us suppose for the moment that $f(y)$ is a general

† See M. Riesz (7). Riesz works with a different definition of the fractional derivatives, but the difference is of no importance here.

function of the real variable y†. Then

$$f(y)\to l \tag{5.2}$$

and

$$f^{\alpha}(y) = o(y^{-\alpha}) \tag{5.3}$$

imply

$$f^{\beta}(y) = o(y^{-\beta}) \tag{5.4}$$

for $0<\beta<\alpha$‡. If α and β are integers, (5.3) may be replaced by

$$f^{\alpha}(y) > -Cy^{-\alpha}\text{§}, \tag{5.5}$$

but this is not true without reservation for general α and β; (5.2) and (5.5) imply (5.4) when $0<\beta\leqslant\alpha-1$, but not necessarily when

$$\alpha-1<\beta<\alpha.$$

Suppose, for example, that $\alpha=2$ and $C=0$, so that (5.5) is

$$f''(y)>0. \tag{5.6}$$

Then (5.4) is true for $0<\beta<1$, but $f^{\beta}(y)$ may have infinities, when $\beta>1$, in every neighbourhood of the origin; in this case $f''(y)$ will also have infinities, but this is consistent with (5.6). Thus (5.2) and (5.6) do not imply

$$f^{\beta}(y) = O\big(\phi(y)\big)$$

for any monotonic ϕ, however rapidly ϕ may tend to infinity, when $\beta>1$.

Here we are concerned with power series (so that infinities are impossible), but the limitations on theorems of this kind are effectively the same. Thus

$$f(y)=\Sigma a_n e^{-ny}\to l, \quad f''(y)=\Sigma n^2 a_n e^{-ny}>0$$

---

† In this case our definitions of the derivatives must be framed differently. We must define $f_{\beta}(y)$, for $\beta>0$, by (4.2), and

$$f^{k-\beta}(y)=f_{\beta-k}(y) \quad (k=1,\,2,\,\ldots)$$

by

$$f_{\beta-k}(y)=(-1)^k\left(\frac{d}{dy}\right)^k f_{\beta}(y).$$

The functions are no longer power-series, and the question of " consistency " requires attention, but the general situation is much the same.

‡ See Hardy and Littlewood (3), M. Riesz (7). Theorems of this type are by now familiar.

§ See Hardy and Littlewood (3).

do not imply

$$f^{\beta}(y) = \Sigma n^{\beta} a_n e^{-ny} = o(y^{-\beta})$$

when $\beta > 1$. Hence Theorem 4 is not (as we might have expected it to be) a corollary of Theorem 5. And (5.1), for some positive δ, does not, even when $C = 0$, imply (4.4) for all smaller positive δ, so that we cannot replace (4.4) by (5.1), in Theorem 5, in the way which is naturally suggested†.

Negative results.

6. We shall not attempt to prove formally that Theorems 3, 4, 5 have all a best possible character (though this is the natural lesson of our analysis). But it is desirable to show by definite examples that one particular condition, viz.

$$f(y) = o\left(\frac{1}{y}\right), \tag{6.1}$$

is not a sufficient substitute for (2.3), (3.1), or (4.4); and this will lead us to the study of some special series which have an independent interest.

The proposition to be negatived is

$$f(y) \to l \,.\, f'(y) = o\left(\frac{1}{y}\right) \,.\!\!\rightarrowtail g(y) \to l \ddagger. \tag{A}$$

With this we may associate two other propositions, viz.

$$f(y) = O(1) \,.\, f'(y) = O\left(\frac{1}{y}\right) \,.\!\!\rightarrowtail g(y) = O(1); \tag{A_1}$$

and

$$|f(y)| < C \,.\, |f'(y)| < \frac{D}{y} \,.\!\!\rightarrowtail |g(y)| < E, \tag{A_2}$$

where $E = E(C, D)$. We may call (A_1) the "O form" and (A_2) the "inequality form" of (A). The three propositions stand or fall together; if we can prove (A_2) false, for example, we can deduce the falsity of (A)

† It may be shown that (5.2) and (5.5) imply

$$\underline{\lim}\, y^{\beta} f^{\beta}(y) \geqslant 0$$

when $0 < \alpha - 1 < \beta < \alpha$.

The significance of the difference 1 between α and β becomes clearer if we suppose that $f^{s}(y) > 0$ and observe that

$$f^{\beta}(y) = \frac{1}{\Gamma(\alpha - \beta)} \int_y^{\infty} (t-y)^{\alpha-\beta-1} f^{\alpha}(t)\, dt$$

increases (when y decreases) if $\alpha - \beta \geqslant 1$, but not necessarily if $\alpha - \beta < 1$.

‡ $\rightarrowtail$ is Hilbert's symbol for implication.

by a familiar, though possibly tiresome, routine†. In what follows we shall consider (A_1) or (A_2), and leave the transition to (A) to the reader's imagination.

We have two ways of proving (A_1) or (A_2) false. The first depends, at bottom, on the facts that

$$h(y) = \Sigma f'(ny) = y^{-s}\zeta(s)$$

when

$$f(y) = \frac{y^{1-s}}{1-s}$$

and that $\zeta(s)$ is not bounded on the line $s = 1+it$. This is the most natural line of proof, but there are complications because (i) y^{-s} is not a power series in e^{-y}, and (ii) Σn^{-s} is not convergent when $s = 1+it$.

We take

$$f(y) = -\frac{(1-e^{-y})^{-ia}}{ia}, \quad f'(y) = e^{-y}(1-e^{-y})^{-1-ia}, \tag{6.2}$$

where

$$a = [Y^{-\frac{1}{2}}]$$

and Y is small, so that $a > 1$. This $f(y)$ obviously satisfies the hypotheses of (A_2), with $C = D = 1$. We consider its behaviour for $y = Y$.

We have

$$\begin{aligned} -g(Y) = Yh(Y) &= Y\Sigma f'(nY) = Y\Sigma e^{-nY}(1-e^{-nY})^{-1-ia} \\ &= Y\sum_{n\leqslant a} + Y\sum_{a<n\leqslant 1/Y} + Y\sum_{n>1/Y} = S_1+S_2+S_3, \end{aligned} \tag{6.3}$$

say.

In S_1, $nY = \eta$ is small, and

$$e^{-\eta} = 1+O(\eta), \quad 1-e^{-\eta} = \eta+O(\eta^2), \quad (1-e^{-\eta})^{-1-ia} = \eta^{-1-ia}+O(a).$$

Hence

$$\begin{aligned} S_1 &= Y\sum_{n\leqslant a}\Big((nY)^{-1-ia}+O(a)\Big) = Y^{-ia}\sum_{n\leqslant a} n^{-1-ia}+O(Ya^2) \\ &= Y^{-ia}\Big(\zeta(1+ia)+O(1)\Big)+O(1)\ddagger = Y^{-ia}\zeta(1+ia)+O(1). \end{aligned} \tag{6.4}$$

---

† Compare, for example, the three false propositions:

(a) if $f(x)\to l$ when $x\to 0$, then

$$J_n = \frac{2}{\pi}\int_0^\delta f(x)\,\frac{\sin nx}{x}\,dx \to l;$$

(a_1) if $f(x)$ is bounded, then J_n is bounded;

(a_2) if $|f(x)|<C$, then $|J_n|<E(C)$.

‡ It is familiar that $\zeta(1+it) = \sum_{n\leqslant t} n^{-1-it}+O(1)$. See, for example, Titchmarsh (8), 13.

Next

$$(6.5)\quad S_2 = \Sigma\, Ye^{-nY}(1-e^{-Y})^{-1-ia} = \Sigma\, u_n(Y) + \Sigma\, v_n(Y) = S_2' + S_2'',$$

where

$$u_n(Y) = \int_{nY}^{(n+1)Y} e^{-t}(1-e^{-t})^{-1-ia}\, dt,$$

$$v_n(Y) = \int_{nY}^{(n+1)Y} \left(e^{-nY}(1-e^{-nY})^{-1-ia} - e^{-t}(1-e^{-t})^{-1-ia}\right) dt,$$

and the summations are over $a < n \leqslant 1/Y$. Here

$$(6.6)\quad S_2' = \int_\tau^T e^{-t}(1-e^{-t})^{-1-ia}\, dt = \left[-\frac{1}{ia}\,(1-e^{-t})^{-ia}\right]_\tau^T = O\left(\frac{1}{a}\right) = O(1)$$

(the particular values of τ and T being irrelevant). The integrand in $v_n(Y)$ is

$$-\int_{nY}^{t} \frac{d}{du}\left(e^{-u}(1-e^{-u})^{-1-ia}\right) du = O\left(Y \cdot \frac{1}{nY}\right) + O\left(Y \cdot \frac{a}{n^2 Y^2}\right) = O\left(\frac{a}{n^2 Y}\right),$$

uniformly in n, and so

$$(6.7)\qquad S_2'' = O\left(aY \sum_{n>a} \frac{1}{n^2 Y}\right) = O\left(a \sum_{n>a} \frac{1}{n^2}\right) = O(1).$$

Finally, it is obvious that

$$(6.8)\qquad S_3 = O(1);$$

and (6.3)–(6.8) give

$$(6.9)\qquad -g(Y) = Y^{-ia}\zeta(1+ia) + O(1).$$

Since $\zeta(1+ia)$ is not bounded for large a, or small Y, this contradicts (A_2).

7. There is an alternative method which leads to more interesting analysis. We take as our goal the refutation of (A_1). Suppose that

$$(7.1)\quad f'(y) = \operatorname{cosech} y \cos(a \operatorname{cosech} y + \beta) = \frac{2e^{-y}}{1-e^{-2y}} \cos\left(\frac{2ae^{-y}}{1-e^{-2y}} + \beta\right).$$

Then $f(y)$ tends to a limit† and $yf'(y)$ is bounded. We wish to show that $g(y)$ is not bounded, and it is plainly enough to prove that

$$(7.2)\qquad S = \sum_{n<1/y} f'(ny) \neq O\left(\frac{1}{y}\right).$$

† Since $\int_0 f'(y)\, dy = \lim_{\epsilon \to 0} \int_\epsilon f'(y)\, dy$ exists.

If we replace the first factor cosech ny in $f'(ny)$ by $1/ny$, the error introduced into a term of S is $O(ny)$, and that in S is

$$O\Big(y \sum_{n<1/y} n\Big) = O\Big(\frac{1}{y}\Big).$$

Also

$$\cos\Big(\frac{\alpha}{\sinh ny}+\beta\Big)-\cos\Big(\frac{\alpha}{ny}+\beta\Big) = O(1)\sin\Big(O(ny)\Big) = O(ny),$$

and so

$$\sum_{n<1/y}\frac{1}{ny}\left(\cos\Big(\frac{\alpha}{\sinh ny}+\beta\Big)-\cos\Big(\frac{\alpha}{ny}+\beta\Big)\right) = \sum_{n<1/y} O(1) = O\Big(\frac{1}{y}\Big).$$

Hence (7.2) is equivalent to

$$T = \sum_{n<1/y}\frac{1}{n}\cos\Big(\frac{\alpha}{ny}+\beta\Big) \neq O(1), \tag{7.3}$$

and it is sufficient for our purpose to prove that *the functions*

$$P(x) = \sum_{n<x}\frac{1}{n}\cos\frac{x}{n}, \qquad Q(x) = \sum_{n<x}\frac{1}{n}\sin\frac{x}{n} \tag{7.4}$$

are not bounded for large x†.

8. THEOREM 6. $P(x) = \Omega(\log\log x)$‡.

If c is fixed and

$$c\sqrt{x} \leqslant \mu < \nu \leqslant x,$$

then

$$\sum_{\mu}^{\nu}\frac{1}{n}\cos\frac{x}{n} - \int_{\mu}^{\nu+1}\frac{1}{t}\cos\frac{x}{t}\,dt$$

$$= \sum_{\mu}^{\nu}\int_{n}^{n+1}\left(\frac{1}{n}\cos\frac{x}{n}-\frac{1}{t}\cos\frac{x}{t}\right)dt$$

$$= \sum_{\mu}^{\nu}\left(O\Big(\frac{1}{n^2}\Big)+O\Big(\frac{x}{n^3}\Big)\right) = O(1)+O\Big(x\sum_{n>c\sqrt{x}}\frac{1}{n^3}\Big) = O(1).$$

† Either function would suffice; but there are interesting differences between them, and we discuss both. Since

$$\sum_{n\geqslant x}\frac{1}{n}\sin\frac{x}{n} = O\Big(x\sum_{n\geqslant x}\frac{1}{n^2}\Big) = O(1),$$

the second series may equally well be extended to infinity.

‡ $|P| > C\log\log x$ for appropriate large x.

Hence

$$P(x) = \sum_{n<\mu} \frac{1}{n} \cos \frac{x}{n} + O(1) = P^*(x) + O(1), \tag{8.1}$$

the O being uniform in μ for $c\sqrt{x} \leqslant \mu \leqslant x$.

We suppose k to be a large positive integer, and take

$$x = 2\pi k! = 2\pi K, \quad x_j = jx \quad (j = 1, 2, \ldots, K), \quad \mu = K.$$

Then μ lies between $\sqrt{(x_j/2\pi)}$ and x_j for all j, and (8.1) is true when $\mu = K$ and $x = x_j$, uniformly in j.

Now

$$P^*(x_j) = \sum_{n=1}^{k} \frac{1}{n} + \sum_{n=k+1}^{K} \frac{1}{n} \cos \frac{x_j}{n} > \log k + \sum_{n=k+1}^{K} \frac{1}{n} \cos \frac{x_j}{n},$$

and so

$$\frac{1}{K} \sum_{j=1}^{K} \left(1 - \frac{j}{K}\right) P^*(x_j) > \frac{K-1}{2K} \log k + \frac{1}{K} \sum_{n=k+1}^{K} \frac{1}{n} \sum_{j=1}^{K} \left(1 - \frac{j}{K}\right) \cos \frac{jx_1}{n}.$$

But

$$\tfrac{1}{2} + \sum_{j=1}^{K} \left(1 - \frac{j}{K}\right) \cos j\theta$$

is the first Cesàro mean of the series $\frac{1}{2} + \cos\theta + \cos 2\theta + \ldots$, and is therefore non-negative for all θ; and so

$$\frac{1}{K} \sum_{j=1}^{K} \left(1 - \frac{j}{K}\right) P^*(x_j) > \frac{K-1}{2K} \log k - \frac{1}{2K} \sum_{1}^{K} \frac{1}{n} > (\tfrac{1}{2} - \epsilon) \log k$$

for every ϵ and sufficiently large k. It follows that

$$P^*(x_j) > (1 - \epsilon) \log k$$

for every ϵ, sufficiently large k, and some j. The same is true of $P(x_j)$, by (8.1). Finally,

$$x_j < 2\pi K^2 < 2\pi (k!)^2, \quad \log x_j < Ck \log k, \quad \log\log x_j < C \log k,$$

and the theorem follows.

It is plain that the same argument would prove that

$$\sum_{n<x} \frac{1}{n \log n} \cos \frac{x}{n} = \Omega(\log\log\log x),$$

and that it could be developed so as to prove Ananda Rau's theorem the best possible of its kind†.

† As indeed, no doubt, could that of § 6.

9. THEOREM 7. $Q(x) = \Omega\left((\log\log x)^{\frac{1}{4}}\right)$.

We denote by q a number of the form Πp^a, where the p are the primes $4m+1$, and take

$$K = \prod_{q \leqslant 4k+1} q,$$

$$x = \tfrac{1}{2}\pi K, \quad x_j = (4j+1)x \quad (j = 1, 2, \ldots, K).$$

As before, it is sufficient to consider

$$Q^*(x) = \sum_{n=1}^{K} \frac{1}{n} \sin\frac{x}{n}$$

for the values x_j.

If $n \mid K$, then K/n is an integer $4m+1$, and $\sin(x_j/n) = 1$. Hence

$$Q^*(x_j) = \sum_{n \mid K} \frac{1}{n} + R(x_j) = \lambda(k) + R(x_j),$$

say, where

$$R(x) = \Sigma' \frac{1}{n} \sin\frac{x}{n},$$

Σ' denoting a sum over n which do not exceed and do not divide K; and†

$$\frac{1}{K} \sum_{j=1}^{K} Q^*(x_j) = \lambda(k) + \frac{1}{K} \Sigma' \frac{1}{n} \sum_{j=1}^{K} \sin\frac{(4j+1)x}{n}. \tag{9.1}$$

In the inner sum on the right of (9.1), $2x/n$ differs from the nearest multiple of π by at least C/n, so that

$$\sum_{j=1}^{K} \sin\frac{(4j+1)x}{n} = O\left(\left|\operatorname{cosec}\frac{2x}{n}\right|\right) = O(n),$$

and the repeated sum is

$$O\left(\frac{1}{K} \sum_{n=1}^{K} \frac{1}{n} . n\right) = O(1).$$

Hence

$$\frac{1}{K} \sum_{j=1}^{K} Q^*(x_j) > (1-\epsilon)\lambda(k)$$

and

$$Q^*(x_j) > (1-\epsilon)\lambda(k),$$

for some j, in the sense of § 8.

† There is no advantage this time in taking a more elaborate mean.

Finally, it is plain that

$$\lambda(k) \geqslant \sum_{q \leqslant 4k+1} \frac{1}{q}.$$

If $N(n)$ is the number of q which do not exceed n, then

$$\sum_{q \leqslant 4k+1} \frac{1}{q} = \sum_{n \leqslant 4k+1} \frac{N(n)-N(n-1)}{n} \geqslant \sum_{n \leqslant 4k+1} \frac{N(n)}{n(n+1)}.$$

But† $$N(n) > \frac{Cn}{(\log n)^{\frac{1}{2}}},$$

and so $$Q^*(x_j) = \Omega\left((\log k)^{\frac{1}{2}}\right) = \Omega\left((\log \log x_j)^{\frac{1}{2}}\right).$$

We cannot prove that $Q(x)$ is ever as large as $\log \log x$, though this is true both of $P(x)$ and of

$$Q_1(x) = \Sigma \frac{(-1)^n}{2n+1} \sin \frac{x}{2n+1}.$$

The "O-problems" for all these sums are much like the corresponding problems for $\zeta(1+it)$.

The series are also connected by identities with series of a quite different type. Thus

$$f(z) = \sum_1^\infty \frac{1}{n}(1-e^{-z/n}) = 2\log z + 2\gamma - 2\sum_1^\infty \left\{K_0\left(\sqrt{(2n\pi iz)}\right) + K_0\left(\sqrt{(-2n\pi iz)}\right)\right\},$$

if $\Re(z) > 0$; here γ is Euler's constant and K_0 is "Basset's function"‡. Such identities lead to interesting translations of our theorems. Suppose, for example, that

$$f(s) = \sum_1^\infty n^{-\frac{1}{4}} e^{-s\sqrt{n}}$$

if $s = \sigma + it$, $\sigma > 0$, and $f(it)$ is the limit of $f(s)$ when $\sigma \to 0$. Then

$$f(it) = \Omega(|t|^{\frac{1}{4}} \log \log |t|).$$

That $f(it) = O(|t|^{\frac{1}{4}+\epsilon})$, for every positive ϵ, has been proved by Dr. Eric Phillips, but the proof has not been published.

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† Landau (6).

‡ Watson (9, 77).

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NOTES ON TWO-DIMENSIONAL ELECTRIC FIELD PROBLEMS

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Under this title four notes were given in a recent paper*. A further note is now added.

Note 5.

1. *A square within a square, diamondwise.*

Moulton† has considered the problem of the two-dimensional condenser in which the boundaries consist of two squares, one within the other, in two cases: (i) when the two squares are similarly situated, with their centres coinciding, (ii) when the inner square is situated diamondwise with respect to the outer. His solutions are given in terms of hypergeometric series whose practical value depends upon the rapidity of convergence, which itself depends upon the ratio of the sides of the squares.

The first of these problems was discussed in Note 3; the second is discussed here. As in the first problem, the solution depends upon the expression of a degenerate hyperelliptic integral as a sum of elliptic integrals with complementary moduli, but here there is the special interest that the moduli are complex.

The calculation of the capacity of the condenser, for any given ratio of the sides of the squares, depends upon q-series in which $|q| < e^{-\pi}$, so that convergence is rapid.

2. Let the plane of a cross-section of the condenser be the z-plane. By symmetry, we need only consider one-eighth of the cross-section,

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* *Proc. London Math. Soc.* (2), 39 (1935), 205.

† H. F. Moulton, *Proc. London Math. Soc.* (2), 3 (1905), 104.

$OABC$, and this is represented on a half-t-plane by the transformation

$$dz = \frac{j^{-1}\,dt}{t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}(1-k^2t)^{\frac{1}{2}}}, \tag{1}$$

where $j^{-1} = e^{-\frac{1}{2}\pi i}$: this constant is introduced to make the orientation of the figure in the z-plane agree with that shown in Fig. 1. Putting

$$t = 1+\frac{k'^2u^2}{k^2}, \tag{2}$$

we find

$$dz = \frac{k'^{\frac{1}{2}}}{k^{\frac{1}{2}}}\,\frac{2u^{\frac{1}{2}}\,du}{(1-u^2)^{\frac{1}{2}}(k^2+k'^2u^2)^{\frac{1}{2}}}. \tag{3}$$

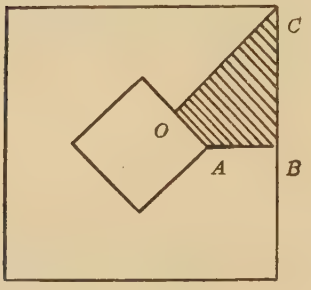

Fig. 1. z-plane.

This hyperelliptic differential reduces to the difference of two elliptic differentials with complementary moduli, by means of the substitution*

$$\zeta^2 = \frac{2(k+ik')\,u}{(1+u)(k+ik'\,u)}, \tag{4}$$

i.e.

$$\sqrt{u} = \frac{(\lambda+\lambda')\,\zeta}{\sqrt{(1-\lambda^2\zeta^2)}+\sqrt{(1-\lambda'^2\zeta^2)}}, \tag{5}$$

where

$$\lambda = \frac{\sqrt{k}+j\sqrt{k'}}{\sqrt{2}\,\sqrt{(k+ik')}}, \qquad \lambda' = \frac{\sqrt{k}-j\sqrt{k'}}{\sqrt{2}\,\sqrt{(k+ik')}}, \tag{6}$$

$$\frac{ik'}{k} = \left(\frac{\lambda-\lambda'}{\lambda+\lambda'}\right)^2, \tag{7}$$

and $j = e^{\frac{1}{2}\pi i}$, as above; for we then find that

$$dz = \frac{j^{-1}\sqrt{2}}{k^{\frac{1}{2}}(k+ik')^{\frac{1}{2}}}\left\{\frac{d\zeta}{\sqrt{\{(1-\zeta^2)(1-\lambda^2\zeta^2)\}}} - \frac{d\zeta}{\sqrt{\{(1-\zeta^2)(1-\lambda'^2\zeta^2)\}}}\right\}. \tag{8}$$

* *Cf.* Cayley, *Elliptic functions* (1895), 360.

3. We note that

$$\lambda^2 = \tfrac{1}{2} + \frac{j\sqrt{(kk')}}{k+ik'}, \qquad \lambda'^2 = \tfrac{1}{2} - \frac{j\sqrt{(kk')}}{k+ik'}, \tag{9}$$

and hence that $\lambda^2+\lambda'^2 = 1$, so that λ^2, λ'^2 are complementary moduli.

Further, if we put $k = \cos\theta$, $k' = \sin\theta$, we find that

$$4\lambda^2\lambda'^2 = \left(\frac{k-ik'}{k+ik'}\right)^2 = e^{-4i\theta}, \tag{10}$$

so that λ^2, λ'^2 are the roots of the quadratic equation

$$\lambda^4 - \lambda^2 + \tfrac{1}{4}e^{-4i\theta} = 0. \tag{11}$$

We can also express λ^2, λ'^2 in the forms

$$\lambda^2 = \tfrac{1}{2}\{k+k'+\sqrt{(2kk')}\}\, e^{i(\epsilon-2\theta)}, \tag{12}$$

$$\lambda'^2 = \tfrac{1}{2}\{k+k'-\sqrt{(2kk')}\}\, e^{-i(\epsilon+2\theta)}, \tag{13}$$

where $\epsilon = \tan^{-1}\{\sqrt{(2kk')}/(k-k')\}$.

Moreover, from (10) it follows at once that $|4\lambda^2(1-\lambda^2)| = 1$, and hence that the Argand point of λ^2 lies on the lemniscate $|4Z(1-Z)| = 1$ when it is represented on an auxiliary Z-plane. In fact, as Fig. 2 indicates, λ^2 lies on the loop along which $|Z| > \frac{1}{2}$, and λ'^2 is its image in the node $Z = \frac{1}{2}$. When

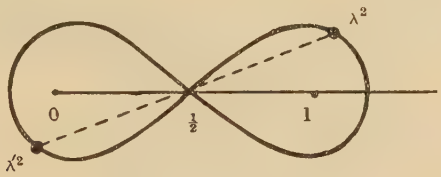

Fig. 2.

θ varies from 0 to $\frac{1}{2}\pi$, the point λ^2 describes its loop in the clockwise sense, beginning and ending at the node; at the half-way position $\theta = \frac{1}{4}\pi$, $k = k'$, and the quadrilateral $OABC$ in the z-plane is symmetrical about the diagonal AC.

4. The areas that correspond to the quadrilateral $OABC$ in the t and u planes can easily be supplied by the reader. Thus, in the t-plane the points O, A, B, C are 0, 1, $1/k^2$, ∞, and the quadrilateral is mapped on the upper half-plane; while in the u-plane the points O, A, B, C are ik/k', 0, 1, ∞, and the quadrilateral is mapped on the first quadrant.

To consider the corresponding area and its boundary in the ζ-plane, put

$$\zeta' = e^{-i\theta}\zeta^2 = \frac{2u}{(1+u)(k+ik'u)}. \tag{14}$$

Then, if we also put $\zeta' = x+iy$ for a moment, we find that the boundary in the ζ'-plane is the curve

$$2xy = (x^2+y^2)(ky-k'x), \tag{15}$$

which reduces to the strophoid when $k = k' = 1/\sqrt{2}$. It is indicated in Fig. 3 for a value of θ between 0 and $\frac{1}{4}\pi$. The point B is the point

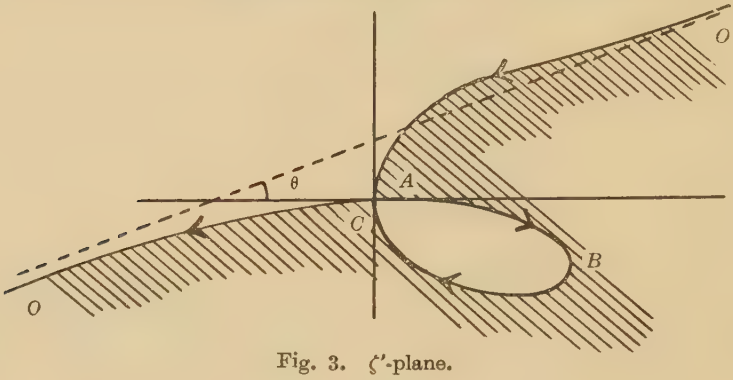

Fig. 3. ζ'-plane.

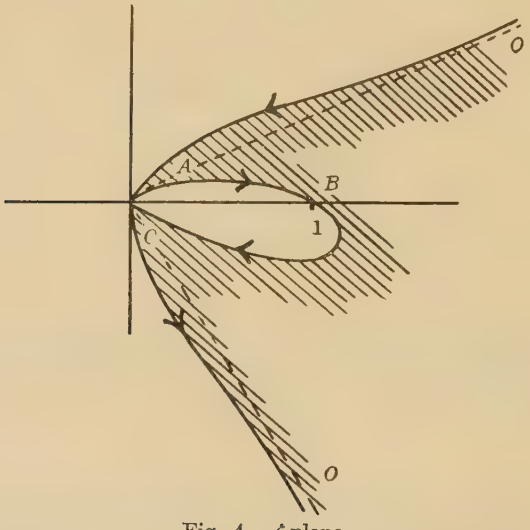

Fig. 4. ζ-plane.

$\zeta' = e^{-i\theta}$. The region that corresponds to the quadrilateral $OABC$ is that on the left-hand of the curve when it is described in the sense indicated by the arrow-heads.

The loop surrounds the point $\zeta' = e^{-i\theta}/\lambda^2$; this can be verified by means of (12), by showing that $|\zeta'| > |e^{-i\theta}/\lambda^2|$ at the point on the loop where $\operatorname{am}\zeta' = \operatorname{am}(e^{-i\theta}/\lambda^2)$. On the other hand, the point $\zeta' = e^{-i\theta}/\lambda'^2$ is outside the loop since, by (13), $\operatorname{am}(e^{-i\theta}/\lambda'^2) = \epsilon + \theta$, which lies between 0 and $\frac{3}{4}\pi$ if θ is between 0 and $\frac{1}{4}\pi$, whereas the loop is entirely within the fourth quadrant.

The corresponding curve in the ζ-plane is shown in Fig. 4; in this plane B is the point $\zeta = 1$, and the loop surrounds the branch point $1/\lambda$, but not the point $1/\lambda'$.

5. *The dimensions of the quadrilateral OABC.*

In the z-plane, put $a = AB$, $b = BC$. Then

$$a = \int_0^1 \frac{dz}{d\zeta}\, d\zeta, \quad ib = \int_1^0 \frac{dz}{d\zeta}\, d\zeta, \tag{16}$$

where $dz/d\zeta$ is given by (8). The path of integration is along the loop in the ζ-plane from A to B for the first integral, and along the loop from B to C for the second integral.

It is sufficient to take $0 < \theta < \frac{1}{4}\pi$, because, by reflection in the diagonal AC, the quadrilateral $OABC$ passes through the same sequences of shapes when θ varies from $\frac{1}{4}\pi$ to $\frac{1}{2}\pi$ as when it varies from 0 to $\frac{1}{4}\pi$ (see § 6).

Now, when $0 < \theta < \frac{1}{4}\pi$, the branch point $1/\lambda$ lies below the real axis in the ζ-plane, and for the purpose of evaluating the definite integrals (16) the loop can be reduced to the simplified path shown in Fig. 5. Hence, putting $j = e^{\frac{1}{4}\pi i}$ and $k = \cos\theta$ in (8), we find that*

$$a = \sqrt{(2\sec\theta)}\, e^{-i(\frac{1}{2}\theta + \frac{1}{4}\pi)}(L - L'), \tag{17}$$

$$ib = \sqrt{(2\sec\theta)}\, e^{-i(\frac{1}{2}\theta + \frac{1}{4}\pi)}(2iL' - L - L'), \tag{18}$$

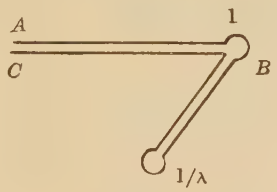

Fig. 5.

where L, L' are complete elliptic integrals of the first kind with moduli λ, λ'.

* *Cf.* Whittaker and Watson, *Modern analysis* (1915), §§ 22·3, 22·32.

6. *The capacity.*

If we put $$t = \mathrm{sn}^2(\chi, k), \tag{19}$$

the quadrilateral $OABC$ is represented in the χ-plane on a rectangle of sides K, K' through equations (1) and (19), and the capacity C of the condenser per unit length is given by

$$C = 8 \times \frac{K}{4\pi K'} = \frac{2K}{\pi K'}, \tag{20}$$

where the modulus k is given in terms of λ by equation (7), viz.

$$\frac{ik'}{k} = \left(\frac{\lambda - \lambda'}{\lambda + \lambda'}\right)^2, \tag{21}$$

and λ/λ' is determined by the dimensions of the quadrilateral as follows.

Putting $\rho = a/b$, we obtain, from (17) and (18),

$$\rho = \frac{a}{b} = \frac{L - L'}{2L' + iL + iL'}, \tag{22}$$

and hence

$$\frac{L}{L'} = \frac{1 + (2+i)\rho}{1 - i\rho} = \frac{1 + 2\rho - \rho^2 + 2\rho(1+\rho)i}{1 + \rho^2}. \tag{23}$$

It is easily seen that, when θ increases from 0 to $\frac{1}{4}\pi$, ρ increases from 0 to $\tan 22\frac{1}{2}^\circ = \sqrt{2} - 1$, and it follows that the real part of L/L' increases from 1 to $\sqrt{2}$ and hence, if we put $q' = e^{-\pi L/L'}$, that $|q'|$ varies from $e^{-\pi}$ to $e^{-\pi\sqrt{2}}$ and that the corresponding q'-series converge rapidly. In fact, neglecting $2q'^4$ ($< \cdot 000007$) compared with 1, we have*

$$\left(\frac{\lambda'}{\lambda}\right)^{\frac{1}{2}} = \frac{2q'^{\frac{1}{4}}(1 + q'^2)}{1 - 2q'}. \tag{24}$$

From (24) and (21) and known tables, the value of K/K' can thus be found and substituted in (20).

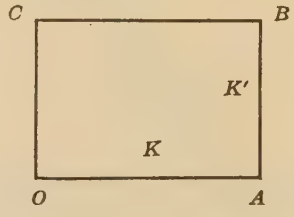

Fig. 6. χ-plane.

* *E.g.*, Whittaker and Watson, *Modern analysis* (1915), 472.

When $\frac{1}{4}\pi < \theta < \frac{1}{2}\pi$, instead of the ratio $\rho = AB/BC$, we work with the ratio OA/OC, which then lies between 0 and $\sqrt{2}-1$. The same calculation then leads to the value of K'/K whose reciprocal can be substituted in (20).

The capacity can thus be calculated with but little arithmetic for any ratio of the sides of the two squares.

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THE ABSTRACT GROUPS

$$R^m = S^m = (R^j S^j)^{p_j} = 1, \quad S^m = T^2 = (S^j T)^{2p_j} = 1,$$

$$\text{AND} \quad S^m = T^2 = (S^{-j} T S^j T)^{p_j} = 1.$$

By H. S. M. COXETER.

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In this paper we investigate the groups

$$\left.\begin{aligned} R^m = S^m = (R^j S^j)^{p_j} = 1 \\ S^m = T^2 = (S^j T)^{2p_j} = 1* \\ S^m = T^2 = (S^{-j} T S^j T)^{p_j} = 1 \end{aligned}\right\} \quad (j = 1, 2, \ldots, [\tfrac{1}{2}m]; \ p_j > 1).$$

We prove that they are finite if $p_j = 2$ for every $j < \frac{1}{2}m$, and infinite otherwise. For this purpose, we show how these groups can be derived from groups generated by reflections. In an earlier paper we developed a geometrical method which enabled us to enumerate all finite groups generated by reflections. The above theorem then follows by comparison with those results.

---

* Groups of the form $S^m = T^2 = (S^j T)^{q_j} = 1,$

where the q's may be odd, present a far more difficult problem.

The geometrical interpretation is especially applied to the case when $p_1 = 3$ while every other $p_j = 2$. In this case the reflections can be regarded as transpositions of m symbols, and from this it easily follows that the group

$$S^m = T^2 = (S^{-1}TST)^3 = (S^{-j}TS^jT)^2 = 1 \quad (j = 2,\ 3,\ \ldots,\ [\tfrac{1}{2}m];\ \ m > 2)$$

involves the symmetric group of degree m as a factor group, namely as the quotient group of an infinite Abelian subgroup. This subgroup appears in the geometrical representation as a group of translations. By magnifying these translations, we obtain a larger quotient group, which in one case is related to the symmetry of the Cartesian frame (*e.g.* when $m = 3$, to the cubic system of crystallography). A more general application of the same principle is developed in § 9.

I should like to take this opportunity to express my gratitude to Dr. O. Taussky for her helpful and constructive criticism.

1. *Preliminary remarks**.

We interpret the abstract definitions

$$\left.\begin{aligned} &(1.1) && R^m = S^m = (R^jS^j)^{p_j} = 1 \\ &(1.2) && S^m = T^2 = (S^jT)^{2p_j} = 1 \\ &(1.3) && S^m = T^2 = (S^{-j}TS^jT)^{p_j} = 1 \end{aligned}\right\} \quad (j = 1,\ 2,\ \ldots,\ [\tfrac{1}{2}m];\ \ p_j > 1)$$

as implying that the period of R^jS^j or $(S^jT)^2$ or $S^{-j}TS^jT$ (respectively) is precisely p_j (and not merely a factor of p_j). The condition $p_j > 1$ is inserted because, as we shall see in § 3, the p's can all be arbitrarily assigned so long as they are all greater than 1. To see what happens in the contrary case, we may use the following argument.

We first extend the definition of p_j (as a period), beyond the given range of j. The new values are determined by the formulae

$$p_m = 1, \quad p_{\nu m \pm j} = p_j.$$

Now, $p_j = 1$ implies that $p_i = p_{\lambda j \pm i}$ for all i and λ, and $p_j = p_k = 1$ implies that $p_{\lambda j \pm \mu k} = 1$ for all λ and μ. Let l denote the first (positive) value of j for which $p_j = 1$. Then l is a factor of m (since $p_m = 1$), and all other such values of j are just the multiples of l. (As an extreme case, we may have $l = 1$, and all $p_j = 1$.)

* The reader who is in a hurry may turn at once to § 2.

Thus the groups (1.1), (1.2), (1.3), without the restriction $p_j > 1$, are

$$\left.\begin{aligned} R^m = S^m = R^l S^l = (R^i S^i)^{p_i} = 1 \\ S^m = T^2 = (S^l T)^2 = (S^i T)^{2p_i} = 1 \\ S^m = T^2 = S^{-l} T S^l T = (S^{-i} T S^i T)^{p_i} = 1 \end{aligned}\right\} \quad (i = 1, 2, \ldots, [\tfrac{1}{2}l];\ l \mid m;\ p_i > 1),$$

or, with a change of notation,

$$\left.\begin{aligned} R^{nm} = S^{nm} = R^m S^m = (R^j S^j)^{p_j} = 1 \quad (1.4) \\ S^{nm} = T^2 = (S^m T)^2 = (S^j T)^{2p_j} = 1 \quad (1.5) \\ S^{nm} = T^2 = S^{-m} T S^m T = (S^{-j} T S^j T)^{p_j} = 1 \quad (1.6) \end{aligned}\right\} \quad (j = 1, 2, \ldots, [\tfrac{1}{2}m];\ p_j > 1).$$

In each case, S^m generates a cyclic self-conjugate subgroup of order n. The quotient groups are (1.1), (1.2), (1.3), respectively. In (1.4) and (1.6), and in (1.5) when $n = 2$, this cyclic subgroup is central.

When m and n are co-prime, we can exhibit (1.6) as the direct product of (1.3) and the cyclic group of order n. To do this, we select integers μ, ν, such that

$$\mu n - \nu m = 1,$$

and let

$$S' = S^{\mu n} = S^{\nu m + 1}.$$

Since $S^{\nu m}$ belongs to the central, the operators S' and T of (1.6) generate (1.3) (with S' in place of S). Every operator of this subgroup of (1.6), of index n, is permutable with every operator of the subgroup of order n generated by S^m. Hence (1.6) is their direct product.

Similarly, writing $R' = R^{\mu n} = R^{\nu m + 1}$, we see that, when m and n are co-prime, (1.4) is the direct product of (1.1) and the cyclic group of order n. Also (1.5), with m odd and $n = 2$, is the direct product of (1.2) and the group of order 2.

Since, in (1.4), (1.5), and (1.6), respectively,

$$R'^{-1} S' = R^{-1} S \,.\, S^{2\nu m}, \quad S'^{-1} T S' T = S^{-2\nu m} \,.\, S^{-1} T S T = S^{-1} T S T \,.\, S^{-2\nu m},$$

and

$$S' T = S T \,.\, S^{\nu m},$$

all these statements continue to hold when we modify the groups by adjoining the relation

$$(R^{-1} S)^{\frac{1}{2}nq} = 1 \quad \text{to} \quad (1.1) \text{ and } (1.4),$$

$$(S^{-1} T S T)^{\frac{1}{2}nq} = 1 \quad \text{to} \quad (1.2) \text{ and } (1.5),$$

$$(S T)^{nq} = 1 \quad \text{to} \quad (1.3) \text{ and } (1.6),$$

provided that the value of q is such as to render these relations consistent

with the original ones. (*E.g.*, when $p_1 = 3$ while every other $p_j = 2$, we write $q = m-1$.)

2. *The notion of augmenting a group by adjoining the cyclic permutation of its generators.*

If an abstract group G is formally symmetrical in its l generators, we can derive a larger group Γ, in which G is a self-conjugate subgroup of index l, by adjoining the cyclic permutation of the generators; *i.e.*, we introduce a new abstract operator T, such that

$$R_i = T^{-i} R_l T^i, \quad T^l = 1,$$

and so obtain a group generated by R_l and T. The operator T may transform G according to either an inner or an outer automorphism. In the former case, Γ is the direct product of G and the cyclic group of order l. (For, if t is the operator of G that has the same effect as T, then $T^{-1}t$ is permutable with every operator of G.) In the latter case, if T^k (but no lower power of T) transforms G according to an inner automorphism, Γ is the direct product of a subgroup of the holomorph of G and the cyclic group of order l/k. Of course it may happen that $k = l$.

For instance, let G be the group (1.1), which is formally symmetrical between R and S. By writing

$$R = TST, \quad T^2 = 1,$$

we derive the group (1.2), whose order is therefore twice as great. Now, the symmetry between R and S is preserved if we replace R by R^{-1}. Hence the group (1.3) has the same order as (1.2). In fact, (1.2) and (1.3) have a common self-conjugate subgroup of index 2, namely (1.1).

Similarly, (1.5) and (1.6) have the same order, each containing (1.4) as a self-conjugate subgroup of index 2.

These statements continue to hold when we adjoin extra relations as at the end of § 1.

3. *Application to groups generated by reflections.*

In an earlier paper* it was seen that the abstract relations

$$(3.1) \qquad \begin{cases} R_i^2 = 1 \quad (i = 1, 2, \ldots, m), \\ (R_i R_j)^{k_{ij}} = 1 \quad (i = 1, \ldots, m-1;\ j = i+1, \ldots, m;\ k_{ij} > 1) \end{cases}$$

* Coxeter, **3**. (Dr. Du Val has pointed out that Lemmas 1 and 2 are unnecessary. In the proof of Lemma 3, the invariant positive definite form can immediately be used to establish a Euclidean metric.)

are consistent, and that the group defined by them is finite for certain specified values of the integers k_{ij}. We shall call it a g.g.r. ("group generated by reflections"). At present we are concerned with the special case when

$$k_{ij} = p_{j-i}, \quad p_j = p_{m-j},$$

so that the group has the cyclic permutation of the R's as an automorphism. The representative *graph* generally consists of an m-gon with its sides marked p_1, its first diagonals marked p_2, and so on. But in all the interesting cases this will be simplified by most of the p's being equal to 2.

The above mentioned results show that the group

$$(3.2)\quad \begin{cases} R_i^2 = 1 \quad (i = 1, 2, \ldots, m), \\ (R_i R_{i+j})^{p_j} = 1 \quad (i = 1, \ldots, m-1;\ j = 1, \ldots, m-i;\ p_j = p_{m-j} > 1) \end{cases}$$

is finite only when $p_j = 2$ for all $j < \frac{1}{2}m$. When *every* $p_j = 2$, it is $[\,]^m$, the Abelian group of order 2^m and type $(1, 1, \ldots)$; the graph consists merely of m separate dots. When m is even and every $p_j = 2$ except $p_{\frac{1}{2}m} = p$, the group is $[p]^{\frac{1}{2}m}$, of order $(2p)^{\frac{1}{2}m}$, the direct product of $\frac{1}{2}m$ dihedral groups; the graph consists of $\frac{1}{2}m$ pairs of dots, each pair being linked, and marked with the number p.

Let S denote the cyclic permutation of the R's, so that

$$R_i = S^{-i} R_m S^i, \quad S^m = 1.$$

The relations (3.2) become

$$R_m{}^2 = 1,$$

$$(R_m S^{-j} R_m S^j)^{p_j} = 1 \quad (j = 1, 2, \ldots, [\tfrac{1}{2}m];\ p_j > 1).$$

Hence, if we write T in place of R_m, the group derived from (3.2) by adjoining S is

$$(1.3)\qquad S^m = T^2 = (S^{-j} T S^j T)^{p_j} = 1 \quad (j = 1, 2, \ldots, [\tfrac{1}{2}m];\ p_j > 1).$$

This group, and therefore also (1.2), is

(i) *of order* $m \cdot 2^m$ *when every* $p_j = 2$,

(ii) *of order* $m(2p)^{\frac{1}{2}m}$ *when* m *is even and every* $p_j = 2$, *except* $p_{\frac{1}{2}m} = p$,

(iii) *infinite in all other cases**.

The subgroup (1.1) has half this order, in each case; (1.5) and (1.6) have n times this order; and (1.4) has $\frac{1}{2}n$ times this order.

---

* Although (i) and (ii) are trivial, I believe that it would be really difficult to prove (iii) without the earlier paper's appeal to geometry.

4. *Special cases.*

It is perhaps worth while to write down explicitly the finite groups of the form (1.1), (1.2), (1.3) :

$$R^m = S^m = (R^j S^j)^2 = 1 \quad (j = 1, 2, \ldots, [\tfrac{1}{2}m]), \tag{4.1}$$

of order $m \cdot 2^{m-1}$;

$$\left.\begin{aligned} S^m = T^2 = (S^j T)^4 = 1 \qquad (4.2)\\ S^m = T^2 = (S^{-j} T S^j T)^2 = 1 \qquad (4.3)\end{aligned}\right\} \quad (j = 1, 2, \ldots, [\tfrac{1}{2}m]),$$

of order $m \cdot 2^m$;

$$R^m = S^m = (R^j S^j)^2 = (R^{\frac{1}{2}m} S^{\frac{1}{2}m})^p = 1 \tag{4.4}$$

$$(j = 1, 2, \ldots, \tfrac{1}{2}m-1;\ m \text{ even}),$$

of order $\frac{1}{2}m(2p)^{\frac{1}{2}m}$;

$$\left.\begin{aligned} S^m = T^2 = (S^j T)^4 = (S^{\frac{1}{2}m} T)^{2p} = 1 \qquad (4.5)\\ S^m = T^2 = (S^{-j} T S^j T)^2 = (S^{\frac{1}{2}m} T)^{2p} = 1 \qquad (4.6)\end{aligned}\right\} \quad \begin{matrix}(j = 1, 2, \ldots, \tfrac{1}{2}m-1;\\ m \text{ even}),\end{matrix}$$

of order $m(2p)^{\frac{1}{2}m}$.

The group (4.3) has an obvious representation as a permutation group of degree $2m$:

$$S = (a_1 a_2 \ldots a_m)(b_1 b_2 \ldots b_m), \quad T = (a_m b_m).$$

In fact,

$$S^{-j} T S^j T = (a_j b_j)(a_m b_m).$$

The $2m$ symbols a_j, b_j can be regarded as the vertices of the *cross-polytope* β_m, *i.e.*, points at a fixed distance from the origin in both directions along all the axes of an m-dimensional Cartesian frame. S is just the cyclic permutation of the axes.

The operator

$$(ST)^m = (a_1 b_1)(a_2 b_2) \ldots (a_m b_m)$$

generates the central (of order 2). Thus the central quotient group*

$$S^m = T^2 = (ST)^m = (S^{-j} T S^j T)^2 = 1 \quad \{j = 1, 2, \ldots, [\tfrac{1}{2}(m-1)]\} \tag{4.7}$$

is of order $m \cdot 2^{m-1}$. When m is odd, the relations (4.7) are satisfied by the operators S, $T(ST)^m$ of (4.3); hence the group (4.3) is then the direct product of (4.7) and the group of order 2.

---

* It is easily seen that, when m is even, the relation

$$(S^{-\frac{1}{2}m} T S^{\frac{1}{2}m} T)^2 = 1$$

is a consequence of the rest. *Cf.* Coxeter, 4.

When $m = 3$, (4.1) and (4.7) reduce to the tetrahedral group, while (4.2) is octahedral, and (4.3) pyritohedral.

When $m = 4$, (4.4), (4.5), and (4.6) become

$$R^4 = S^4 = (RS)^2 = (R^2 S^2)^p = 1^*, \quad \text{of order } 8p^2,$$

$$\left.\begin{aligned} S^4 = T^2 = (ST)^4 = (S^2 T)^{2p} &= 1\dagger, \\ S^4 = T^2 = (S^{-1} TST)^2 = (S^2 T)^{2p} &= 1\ddagger, \end{aligned}\right\} \quad \text{of order } 16p^2.$$

When $m = 5$, (4.1) and (4.2) become

$$R^5 = S^5 = (RS)^2 = (R^2 S^2)^2 = 1, \quad \text{of order } 80,$$

$$S^5 = T^2 = (ST)^4 = (S^2 T)^4 = 1, \quad \text{of order } 160.$$

The conclusion of § 3 enables us to assert that the groups

$$R^4 = S^4 = (RS)^{p_1} = (R^2 S^2)^{p_2} = 1,$$

$$S^4 = T^2 = (ST)^{2p_1} = (S^2 T)^{2p_2} = 1,$$

$$S^4 = T^2 = (S^{-1} TST)^{p_1} = (S^2 T)^{2p_2} = 1,$$

$$R^5 = S^5 = (RS)^{p_1} = (R^2 S^2)^{p_2} = 1,$$

$$S^5 = T^2 = (ST)^{2p_1} = (S^2 T)^{2p_2} = 1$$

are infinite if $p_1 > 2$ and $p_2 > 1$, and that the last two of these are still infinite if $p_1 = 2$ and $p_2 > 2$.

When $m > 5$, (1.1), and (1.2) involve the relations $(R^3 S^3)^{p_3} = 1$ and $(S^3 T)^{2p_3} = 1$ (respectively), and by omitting them we tacitly make p_3 infinite. Hence the groups§

$$R^m = S^m = (RS)^{p_1} = (R^2 S^2)^{p_2} = 1, \tag{4.8}$$

$$S^m = T^2 = (ST)^{2p_1} = (S^2 T)^{2p_2} = 1 \tag{4.9}$$

are infinite if $m > 5$ (and $p_1, p_2 > 1$).

---

* Burnside, 1; III, $a = b = p$.

† *Ibid.*, $a = 2p$, $b = 0$.

‡ Since

$$(S^2 T)^2 = S^{-1} T . TS^{-1} TS . ST = S^{-1} T . S^{-1} TST . ST = (S^{-1} T)^2 (ST)^2,$$

this is a special case of the group

$$S^4 = T^2 = (S^{-1} TST)^2 = (S^{-1} T)^{2b} (ST)^{2c} = 1,$$

of order $8(b^2 + c^2)$. See Coxeter, 5, 65; Sinkov, 1, 81.

§ When $p_1 = 2$, the expression $R^2 S^2$ can be replaced by a commutator, since

$$R^2 S^2 = RS^{-1} R^{-1} S.$$

5. *A generalization of the symmetric group.*

The most interesting infinite group of the form (1.3) is

$$S^m = T^2 = (S^{-1}TST)^3 = (S^{-j}TS^jT)^2 = 1 \qquad (j = 2, \ldots, [\tfrac{1}{2}m];\ m > 2)^*, \tag{5.1}$$

which is derived from the g.g.r. whose graph is an m-gon. This g.g.r.† can be generated by reflections in the primes

$$x_1 = x_2, \quad x_2 = x_3, \quad \ldots, \quad x_{m-1} = x_m, \quad x_m = x_1 + 1$$

of Euclidean m-space. But the group operates essentially in $m-1$ dimensions, since all the primes are perpendicular to

$$x_1 + x_2 + \ldots + x_m = 0.$$

These generators are transpositions of the coordinates, combined‡ in one case with a translation. Hence all the operators of the g.g.r. are operators of the symmetric group combined with translations, and S is the cyclic permutation $(x_1 x_2 \ldots x_m)$ combined with a translation. Taking T to be the transposition $(x_1 x_2)$, *i.e.* the reflection in $x_1 = x_2$, ST is $(x_2 \ldots x_m)$ combined with a translation, and consequently $(ST)^{m-1}$ is a pure translation§. The operator $(ST)^{m-1}$ and its conjugates generate an infinite Abelian self-conjugate subgroup of (5.1), whose quotient group

$$S^m = T^2 = (ST)^{m-1} = (S^{-1}TST)^3 = (S^{-j}TS^jT)^2 = 1 \qquad (j = 2, \ldots, [\tfrac{1}{2}m];\ m > 2) \tag{5.2}$$

is the symmetric group (of degree m)‖. Since the number of dimensions is $m-1$, the longer translation $(ST)^{n(m-1)}$ and its conjugates generate a subgroup of index n^{m-1} in the above Abelian group, and we have the more

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* De Séguier, **1**, 261 (4). When $m = 2$, (5.1) reduces to the dihedral group [6], of order 12.

† Coxeter, **1**, 162 (§ 17.9).

‡ By "combined with" I first mean "transformed by." But *transformation* by one translation is equivalent to *multiplication* by another.

§ *Cf.* Coxeter, **5**, 72 (Lemma 6.2). $(ST)^{m-1}$ cannot be the identity, since then the groups (5.1) and (5.2) would be the same, whereas we know that they are infinite and finite, respectively.

‖ Moore, **1**, 358. As a g.g.r., this is $[3^{m-2}]$.

general quotient group*

$$(5.3) \qquad S^m = T^2 = (ST)^{n(m-1)} = (S^{-1}TST)^3 = (S^{-j}TS^jT)^2 = 1 \qquad (j = 2, \ldots, [\tfrac{1}{2}m];\ m > 2),$$

of order $n^{m-1}m!$.

This is derivable from the larger group

$$(5.4) \qquad S^{nm} = T^2 = (ST)^{n(m-1)} = S^{-m}TS^mT = (S^{-1}TST)^3 = (S^{-j}TS^jT)^2 = 1$$

as the quotient group of the cyclic subgroup (central) generated by S^m. The group (5.4) is thus of order $n^m m!$. We proceed to prove that it is representable as a permutation group of degree nm in the form

$$(5.5) \qquad \begin{cases} S = (1_1 1_2 \ldots 1_m 2_1 2_2 \ldots 2_m \ldots n_1 n_2 \ldots n_m), \\ T = (1_1 1_2)(2_1 2_2) \ldots (n_1 n_2). \end{cases}$$

These permutations satisfy all the relations, since

$$ST = (1_2 1_3 \ldots 1_m 2_2 2_3 \ldots 2_m \ldots n_2 n_3 \ldots n_m),$$

$$S^{-j}TS^j = (1_{j+1} 1_{j+2})(2_{j+1} 2_{j+2}) \ldots (n_{j+1} n_{j+2}),$$

and $$S^m = s_1 s_2 \ldots s_m,$$

where $$s_i = (1_i 2_i \ldots n_i).$$

We have also $$(ST)^{m-1} = s_2 \ldots s_m,$$

whence

$$(5.6) \qquad S^{-m}(ST)^{m-1} = s_1^{-1}$$

and
$$\begin{aligned} S^{1-m}(ST)^{m-1} &= Ss_1^{-1} \\ &= (1_1 1_2 \ldots 1_m)(2_1 2_2 \ldots 2_m) \ldots (n_1 n_2 \ldots n_m) \\ &= t, \quad \text{say.} \end{aligned}$$

The permutations t and T generate the symmetric group on the m suffixes. Since

$$S = ts_1,$$

the group generated by the permutations S and T coincides with that generated by s_1, t and T; it is therefore of order $n^m m!$, its general operator being expressible in the form

$$(5.7) \qquad (\ )\, s_1^{l_1} s_2^{l_2} \ldots s_m^{l_m} \quad (0 \leqslant l_i \leqslant n-1),$$

* In every numbered abstract definition, until the end of § 6, we shall forgo repeating the statement of the range of j and the restriction on m.

where ( ) stands for any permutation of the suffixes, *i.e.* any operation of the symmetric group generated by t and T. Hence the given permutations generate the whole group (5.4) (and not merely a factor group of it)*.

When $n = 2$, so that $S = (1_1 1_2 \dots 1_m 2_1 2_2 \dots 2_m)$ and $T = (1_1 1_2)(2_1 2_2)$, the symbols 1_j and 2_j can be regarded as opposite vertices of the polytope β_m. The operators t and T permute the Cartesian axes, while s_j reverses the sign of the coordinate x_j. Thus

$$(5.8) \quad S^{2m} = T^2 = (ST)^{2(m-1)} = S^{-m} T S^m T = (S^{-1} TST)^3 = (S^{-j} T S^j T)^2 = 1$$

is the hyper-octahedral group† $[3^{m-2}, 4]$; and

$$(5.9) \qquad S^m = T^2 = (ST)^{2(m-1)} = (S^{-1} TST)^3 = (S^{-j} T S^j T)^2 = 1$$

is the central quotient group $\frac{1}{2}[3^{m-2}, 4]$, which can be regarded as a g.g.r. in elliptic $(m-1)$-space‡.

When $n > 2$, the symbols

$$1_1, 2_1, \dots, n_1; \quad 1_2, 2_2, \dots, n_2; \quad \dots; \quad 1_m, 2_m, \dots, n_m$$

can be regarded as the vertices of m equal, concentric n-gons, lying in absolutely perpendicular planes in Euclidean $2m$-space. The operator s_j is then a rotation through $2\pi/n$ in the j-th plane, while t and T generate the symmetric group on the m planes (preserving the sequence of the vertices of the polygons).

For the sake of closer analogy with the case when $n = 2$, we may work in complex m-dimensional space, letting h_j $(h \leqslant n,\ j \leqslant m)$ denote the point whose j-th coordinate is $e^{2\pi i h/n}$, while the remaining coordinates vanish. (The transformations are now unitary instead of orthogonal.)

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* The group consists, in fact, of all permutations (of the nm symbols) commutative with

$$(1_1 2_1 \dots n_1)(1_2 2_2 \dots n_2) \dots (1_m 2_m \dots n_m).$$

Cf. Miller, Blichfeldt, and Dickson, **1**, 20 (§ 8).

† Young, **1**; Coxeter, **2**, 607. The symbol $[3^{m-2}, 4]$ means (3.1) with

$$k_{12} = k_{23} = \dots = k_{m-2\,m-1} = 3, \quad k_{m-1\,m} = 4, \text{ and every other } k_{ij} = 2.$$

The isomorphism with (5.8) may be verified in a direct manner by writing

$$S = R_m R_{m-1} \dots R_1, \quad T = R_1,$$

whence
$$R_{i+1} = S^{-i} T S^i \quad (i < m-1)$$

and
$$R_m = S R_1 R_2 \dots R_{m-1} = S(TS^{-1})^{m-1} S^{m-1}.$$

For the actual work involved, see Coxeter and Todd, **1**, (7).

‡ Coxeter, **2**, 616.

When m and n are co-prime, the argument used in § 1 shows that (5.4) is the direct product of (5.3) and the cyclic group of order n. Regarded as a subgroup of (5.4), (5.3) is generated by S^{vm+1} and T. These can be written as permutations.

In particular, when $n = 2$ and m is odd, we have

$$S^{m+1} = (1_1\,2_2\,1_3\,2_4 \ldots 1_m)(2_1\,1_2\,2_3\,1_4 \ldots 2_m), \quad T = (1_1\,1_2)(2_1\,2_2),$$

or, by a change of notation,

$$S' = (a_1 a_2 \ldots a_m)(b_1 b_2 \ldots b_m), \quad T = (a_1 b_2)(b_1 a_2).$$

These permutations, when regarded as operating on the vertices of β_m, belong to the g.g.r.* $[3^{m-3,1,1}]$ (the complete symmetry group of the half-measure-polytope $h\gamma_m$). This geometrical group, being of order $2^{m-1}m!$, can thus be identified with (5.3) (m odd, $n = 2$). On comparing this with our previous representation for (5.3), we see that

$$[3^{m-3,1,1}] \sim \tfrac{1}{2}[3^{m-2}, 4] \quad (m \text{ odd}).$$

In § 8 we shall give a more direct proof of this result.

Putting $m = 3$ or 4 in (5.3), we obtain

$$S^3 = T^2 = (ST)^{2n} = (S^{-1}TST)^3 = 1\dagger, \quad \text{of order } 6n^2;$$

$$S^4 = T^2 = (ST)^{3n} = (S^{-1}TST)^3 = (S^2T)^4 = 1, \quad \text{of order } 24n^3.$$

6. *A generalization of the alternating group of odd degree.*

When $n(m-1)$ is even, the operators $TS^{-1}T$ and S of (5.3) generate‡ the group

$$(6.1) \qquad R^m = S^m = (R^{-1}S)^{\frac{1}{2}n(m-1)} = (RS)^3 = (R^jS^j)^2 = 1,$$

of order $\frac{1}{2}n^{m-1}m!$. Introducing a new T so that

$$R = TST, \quad T^2 = 1,$$

we derive the group

$$(6.2) \qquad S^m = T^2 = (S^{-1}TST)^{\frac{1}{2}n(m-1)} = (ST)^6 = (S^jT)^4 = 1,$$

---

* Called $\begin{bmatrix} 3^{m-3} \\ 3 \\ 3 \end{bmatrix}$ in Coxeter, **1**, 151; **2**, 607.

† Sinkov, **1**, 82.

‡ *Cf.* § 2, and Coxeter, **4**.

of order $n^{m-1}m!$. Similarly, (5.4) leads to

$$(6.3)\quad R^{nm} = S^{nm} = (R^{-1}S)^{\frac{1}{2}n(m-1)} = R^m S^m = (RS)^3 = (R^j S^j)^2 = 1,$$

of order $\frac{1}{2}n^m m!$; and

$$(6.4)\quad S^{nm} = T^2 = (S^{-1}TST)^{\frac{1}{2}n(m-1)} = (S^m T)^2 = (ST)^6 = (S^j T)^4 = 1,$$

of order $n^m m!$.

When $n = 1$ (and m is odd), (6.1) is $[3^{m-2}]'$, the alternating group of degree m, generated by the permutations

$$R = (a_1 a_2 a_m a_{m-1} \dots a_3), \quad S = (a_1 a_2 a_3 \dots a_m).$$

The operator T of (6.2) interchanges R and S, and therefore (when $n = 1$) transforms the alternating group according to the automorphism

$$(a_3 a_m)(a_4 a_{m-1}) \dots (a_{\frac{1}{2}(m+1)} a_{\frac{1}{2}(m+5)}),$$

which is an outer or inner automorphism according as $\frac{1}{2}(m+1)$ is odd or even*. Thus (6.2) with $n = 1$ is the symmetric group $[3^{m-2}]$, or the direct product† $[3^{m-2}]' \times [\ ]$, according as $m \equiv 1$ or $3 \pmod 4$. We have shown elsewhere‡ that the relation $(S^{-1}TST)^3 = 1$ may be omitted from (5.2) [though not from (5.1)]. It follows that, when $n = 1$, the relation $(RS)^3 = 1$ may be omitted from (6.1), and $(ST)^6 = 1$ may be omitted from (6.2). We have thus an elegant new definition for the symmetric group of degree $m \equiv 1 \pmod 4$:

$$(6.5)\quad S^m = T^2 = (S^{-1}TST)^{\frac{1}{2}(m-1)} = (S^j T)^4 = 1 \quad [j = 2, \dots, \tfrac{1}{2}(m-1)].$$

The above connection between the groups (6.1), (6.2), and (5.3), when $n = 1$, continues to hold when $n = 2$. In this case (6.1), (6.2), (6.3), (6.4) become

$$(6.6)\quad R^m = S^m = (R^{-1}S)^{m-1} = (RS)^3 = (R^j S^j)^2 = 1,$$

$$(6.7)\quad S^m = T^2 = (S^{-1}TST)^{m-1} = (ST)^6 = (S^j T)^4 = 1,$$

$$(6.8)\quad R^{2m} = S^{2m} = (R^{-1}S)^{m-1} = R^m S^m = (RS)^3 = (R^j S^j)^2 = 1,$$

$$(6.9)\quad S^{2m} = T^2 = (S^{-1}TST)^{m-1} = (S^m T)^2 = (ST)^6 = (S^j T)^4 = 1.$$

When $m = 4$, (6.7) and (5.9) are obviously identical. We proceed to prove the following theorem.

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* I am indebted to Mr. P. Hall for this remark.

† The direct product of the alternating group and the group of order 2.

‡ Coxeter and Todd, 1, (5).

When $m \equiv 0$ *or* 1 (mod 4), *the groups* (6.7) *and* (6.9) *are the same as* (5.9) *and* (5.8) *respectively. When* $m \equiv 2$ *or* 3 (mod 4), (6.7) *is the direct product of* (6.6) *and the group of order two, while* (6.9) *is similarly related to* (6.8).

Writing a_j, b_j for 1_j, 2_j in (5.5), we represent the generators of (5.8) and (6.8) in the form

$$T = (a_1 a_2)(b_1 b_2),$$

$$S = (a_1 a_2 a_3 \dots a_m b_1 b_2 b_3 \dots b_m),$$

$$R = TS^{-1}T = (a_1 a_2 b_m \dots b_3 b_1 b_2 a_m \dots a_3).$$

The T of (6.9) transforms (6.8) according to the automorphism

$$T' = (a_3 b_m)(a_4 b_{m-1}) \dots (a_{m-1} b_4)(a_m b_3)$$

or
$$T'' = (a_1 b_1)(a_2 b_2)(a_3 a_m)(a_4 a_{m-1}) \dots (b_3 b_m)(b_4 b_{m-1}) \dots .$$

Actually, T'' is more convenient than T' for our present purpose, since, in the notation of (5.7),

$$T'' = ( )\, s_1 s_2,$$

where, as a permutation of suffixes,

$$( ) = (3, m)(4, m-1) \dots \begin{cases} (\tfrac{1}{2}m+1,\ \tfrac{1}{2}m+2) & (m \text{ even}). \\ (\tfrac{1}{2}m+\tfrac{1}{2},\ \tfrac{1}{2}m+\tfrac{5}{2}) & (m \text{ odd}). \end{cases}$$

This is an odd permutation if $m \equiv 0$ or 1 (mod 4), and an even permutation if $m \equiv 2$ or 3 (mod 4).

Now, the operators of (6.3) are just those operators of (5.4) whose expression in terms of S and T involves T an even number of times. Since (5.4) contains all permutations of the suffixes, and since every transposition is conjugate to $(1, 2) = T$, (6.3) must contain all the *even* permutations. In particular, (6.8) contains all the even permutations. It also contains $s_1 T^{-1} s_1 T = s_1 s_2$; hence it does or does not contain T'' according as ( ) is an even or odd permutation. In the former case, T'' is an inner automorphism of (6.8), and (6.9) is a direct product. In the latter, T'' is an outer automorphism of (6.8), but still belongs to (5.8).

Our theorem is now proved, so far as (6.9), (6.8), and (5.8) are concerned. Consequently, the central quotient groups (6.7), (6.6), and (5.9) are related in the same manner.

The argument used in § 1 shows that, when m is odd, (6.8) is the direct product of (6.6) and the group of order two, while (6.9) is similarly related to (6.7). Hence, *when* $m \equiv 3 \pmod 4$, (6.7) *and* (6.8) *are the same group.*

The following particular cases of (6.1) and (6.2) seem worthy of explicit mention. (We put $m = 3, 4, 5$, in turn, and write $2q$ for n whenever n has to be even.)

$$R^3 = S^3 = (RS)^3 = (R^{-1}S)^n = 1^*, \text{ of order } 3n^2,$$

$$S^3 = T^2 = (ST)^6 = (S^{-1}TST)^n = 1^*, \text{ of order } 6n^2,$$

$$R^4 = S^4 = (RS)^3 = (R^2S^2)^2 = (R^{-1}S)^{3q} = 1, \text{ of order } 96q^3,$$

$$S^4 = T^2 = (ST)^6 = (S^2T)^4 = (S^{-1}TST)^{3q} = 1, \text{ of order } 192q^3,$$

$$R^5 = S^5 = (RS)^3 = (R^2S^2)^2 = (R^{-1}S)^{2n} = 1, \text{ of order } 60n^4,$$

$$S^5 = T^2 = (ST)^6 = (S^2T)^4 = (S^{-1}TST)^{2n} = 1, \text{ of order } 120n^4.$$

7. *A generalization of the alternating group of even degree.*

When m is even, the following remark can be made about all the groups (1.2), (4.7), (5.3), (6.2): every generating relation involves the generators an even number of times. It follows that the operators of each group fall into two classes, according to the parity of the "degree" of their expressions as products of S's and T's. (*E.g.*, ST is of degree 2, $S^{-2}T$ of degree -1.) The class of operators of even degree constitutes a self-conjugate subgroup of index 2. Since $ST \cdot TS = S^2$, this subgroup is generated by

$$U = TS, \quad V = ST.$$

(It is symmetrical between U and V, the automorphism which interchanges them being effected by T.)

When $m = 4$ or 6, an abstract definition for the subgroup is immediately obtainable by observing that

$$S^2 = VU, \quad (S^2T)^2 = VU^2V, \quad S^3T = VUV, \quad S^{-1}TST = U^{-1}V,$$

$$S^{-2}TS^2T = U^{-1}V^{-1}UV.$$

Taking the four groups in the above order, we have, when $m = 4$:

$$U^{2p_1} = V^{2p_1} = (UV)^2 = (U^2V^2)^{p_2} = 1\dagger,$$

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* Burnside, 1, II, IV; $a = n$, $b = 0$.

† This is merely a special case of (4.8).

of order $8p_2^2$ if $p_1 = 2$, infinite if $p_1 > 2$ (and $p_2 > 1$);

$$U^4 = V^4 = (UV)^2 = (U^{-1}V)^2 = 1^*, \quad \text{of order } 16;$$

$$(7.1) \qquad U^{3n} = V^{3n} = (UV)^2 = (U^{-1}V)^3 = (U^2V^2)^2 = 1,$$

of order $12n^3$;

$$(7.2) \qquad U^6 = V^6 = (UV)^2 = (U^2V^2)^2 = (U^{-1}V)^{3q} = 1\dagger,$$

of order $96q^3$. And when $m = 6$:

$$(7.3) \qquad U^{2p_1} = V^{2p_1} = (UV)^3 = (U^2V^2)^{p_2} = (UV^2)^{2p_3} = 1,$$

of order $24p_3^3$ if $p_1 = p_2 = 2$, infinite for all greater values (so long as $p_3 > 1$);

$$U^6 = V^6 = (UV)^3 = (U^{-1}V)^2 = (U^{-1}V^{-1}UV)^2 = 1\ddagger,$$

of order 96;

$$(7.4) \quad U^{5n} = V^{5n} = (UV)^3 = (U^{-1}V)^3 = (U^{-1}V^{-1}UV)^2 = (UV^2)^4 = 1,$$

of order $360n^5$;

$$(7.5) \qquad U^6 = V^6 = (UV)^3 = (U^2V^2)^2 = (UV^2)^4 = (U^{-1}V)^{5q} = 1,$$

of order $11520q^5$.

When we try to extend these results to the general even value of m, we are faced with the difficulty that, when $j > 3$, the powers of S^jT have no elegant expression in terms of U and V. Thus (1.2) and (6.2) give no interesting results in general; but, by writing $2r$ for m in (1.2), and putting $p_j = \infty$ for all $j > 3$, we see that the group

$$(7.6) \qquad U^{2p_1} = V^{2p_1} = (UV)^r = (U^2V^2)^{p_2} = (UV^2)^{2p_3} = 1$$

is infinite if $r > 3$ (and $p_1, p_2, p_3 > 1$).

Moreover, in the case of (4.7) or (5.3), we can prove by induction that

$$TS^{-j}TS^j = U^{-(j-1)}V^{-1}UV^{j-1}.$$

This is clearly true when $j = 1$. Assuming it for a particular $j \geqslant 1$, we have

$$TS^{-(j+1)}TS^{j+1} = S^{-(j+1)}TS^{j+1}T = S^{-1}T \,.\, TS^{-j}TS^j \,.\, ST$$
$$= U^{-1} \,.\, U^{-(j-1)}V^{-1}UV^{j-1} \,.\, V = U^{-j}V^{-1}UV^j.$$

---

* This is a case of $U^{2p} = V^{2q} = (UV)^2 = (U^{-1}V)^2 = 1$, of order $4pq$, which can be deduced from (1.5) by putting $m = 2$.

† Coxeter, **5**, 72 (Theorem 6).

‡ This is a special case of (7.1) or (7.2), with U replaced by U^{-1}.

Thus (4.7) and (5.3) (with $m = 2r$) lead to the subgroups*

$$(7.7) \qquad U^{2r} = V^{2r} = (UV)^r = (U^{-j} V^{-1} U V^j)^2 = 1 \quad (j = 0, 1, \ldots, r-2),$$

of order $2^{2r-1} r$, and

$$(7.8) \quad U^{n(2r-1)} = V^{n(2r-1)} = (UV)^r = (U^{-1} V)^3 = (U^{-j} V^{-1} U V^j)^2 = 1 \quad (j = 1, 2, \ldots, r-1;\ r > 1),$$

of order $\frac{1}{2} n^{2r-1} (2r)!$.

The analogous subgroup of (5.4) is

$$(7.9) \quad U^{n(2r-1)} = V^{n(2r-1)} = (UV)^{nr} = (U^{-1} V^{-1})^r (UV)^r = (U^{-1} V)^3 = (U^{-j} V^{-1} U V^j)^2 = 1 \quad (j = 1, 2, \ldots, r-1;\ r > 1),$$

of order $\frac{1}{2} n^{2r} (2r)!$. Here $(UV)^r$ generates a cyclic central of order n, whose quotient group is (7.8).

When $n = 1$, (7.8) is $[3^{2r-2}]'$, the alternating group of degree $2r$, generated by the permutations

$$U = (a_1 a_3 a_4 \ldots a_{2r}), \quad V = (a_2 a_3 a_4 \ldots a_{2r}).$$

8. *Symmetry groups of the Cartesian frame.*

The expression (5.7), suitably interpreted, is the general operator of any of the groups (5.4), (6.3), (7.9). When m is even, (5.6) shows that s_1 is of even "degree" in S, T, while t is of odd degree; also t is an odd permutation of the suffixes. Hence (5.7) is the general operator of (7.9), if the symbol ( ) stands for any *even* permutation of the suffixes (and $m = 2r$). When m is odd, s_1 and t involve T an even number of times, and t is an even permutation of the suffixes. Hence the general operator of (6.3) is then (5.7) with the same meaning for ( ). Again, when m and n are even, s_1 and t involve T an odd number of times. Hence, in the general operator of (6.3) with m even, ( ) must stand for an even or odd permutation of the suffixes, according as Σl_i is even or odd.

When $n = 2$, we can make these matters clearer by speaking of permutations of the Cartesian axes, and changes of sign of coordinates. The group of all such permutations and changes of sign, of order $2^m m!$, has been denoted by $[3^{m-2}, 4]$. The following four self-conjugate sub-

* It is, perhaps, not quite obvious that these relations suffice to define the subgroups. The rigorous procedure consists in reconstructing the $\{S, T\}$ groups by adjoining to the $\{U, V\}$ groups the operator T which interchanges U and V. *Cf.* Coxeter, **4**.

groups (three of index 2, and one of index 4) are easily described:

$[(3^{m-2})', 4]$, when only *even* permutations are admitted;

$[3^{m-2}, 4]'$, when the permutations are of the same parity as the number of changes of sign;

$[3^{m-3,1,1}]$, when the number of changes of sign is even;

$[3^{m-3,1,1}]'$, when both are even*.

The symbol $[(3^{m-2})', 4]$ has not been used before, but seems natural, as implying that the symmetric subgroup $[3^{m-2}]$ of $[3^{m-2}, 4]$ is replaced by its alternating subgroup $[3^{m-2}]'$. The group $[3^{m-2}, 4]'$ consists of all positive or *rotational* symmetries of the Cartesian frame. When $m > 4$, $[3^{m-3,1,1}]$ and $[3^{m-3,1,1}]'$ are the complete and rotational groups of the polytope $h\gamma_m$, whose vertices are $\pm(1, 1, \ldots, 1)$ with an even (or odd) number of negative signs. Clearly, $[3^{m-2}, 4]$ contains all of $[(3^{m-2})', 4]$, $[3^{m-2}, 4]'$, $[3^{m-3,1,1}]$ as self-conjugate subgroups of index 2, while all of these contain $[3^{m-3,1,1}]'$.

The central inversion, which changes the sign of all the m coordinates, belongs to $[3^{m-2}, 4]$ and $[(3^{m-2})', 4]$ for all values of m, but to the other three groups only when m is even. Thus, when m is even, we have the five central quotient groups

$$\tfrac{1}{2}[3^{m-2}, 4], \quad \tfrac{1}{2}[(3^{m-2})', 4], \quad \tfrac{1}{2}[3^{m-2}, 4]', \quad \tfrac{1}{2}[3^{m-3,1,1}], \quad \tfrac{1}{2}[3^{m-3,1,1}]',$$

which can be defined geometrically by agreeing that the simultaneous change of sign of all coordinates shall be deemed to have no effect. *E.g.*, when $m = 4$, we have

$$\tfrac{1}{2}[3, 3, 4], \quad \tfrac{1}{2}[(3, 3)', 4], \quad \tfrac{1}{2}[3, 3, 4]', \quad \tfrac{1}{2}[3^{1,1,1}], \quad \tfrac{1}{2}[3^{1,1,1}]',$$

which are Goursat's† XLVII, XLI, XXVII, XLII, XXII, respectively.

But, when m is odd, $[3^{m-2}, 4]$ contains the central inversion while its subgroups $[3^{m-2}, 4]'$ and $[3^{m-3,1,1}]$ do not. Hence

$$[3^{m-2}, 4] \sim [3^{m-2}, 4]' \times [\,] \sim [3^{m-3,1,1}] \times [\,]$$

---

* When $m = 3$, these five groups are:

[3, 4], the extended octahedral group;
[3′, 4], the pyritohedral group;
[3, 4]′, the octahedral group;
[3, 3], the extended tetrahedral group;
[3, 3]′, the tetrahedral group.

† Goursat, 1, 78, 67.

and

$$\tfrac{1}{2}[3^{m-2}, 4] \sim [3^{m-2}, 4]' \sim [3^{m-3,1,1}] \quad (m \text{ odd}). \tag{8.1}$$

For a similar reason,

$$[(3^{m-2})', 4] \sim [3^{m-3,1,1}]' \times [\,]$$

and

$$\tfrac{1}{2}[(3^{m-2})', 4] \sim [3^{m-3,1,1}]' \quad (m \text{ odd}). \tag{8.2}$$

Thus there are only seven groups instead of ten; and of these seven only four are abstractly distinct: $[3^{m-3,1,1}]$, $[3^{m-3,1,1}]'$, and the direct products of these with the group of order 2.

By the remarks at the beginning of this section, (5.8) is $[3^{m-2}, 4]$, (7.9) with $n = 2$ is $[(3^{2r-2})', 4]$, and (6.8) is $[(3^{m-2})', 4]$ or $[3^{m-2}, 4]'$ according as m is odd or even. Consequently, (5.9) is $\frac{1}{2}[3^{m-2}, 4]$, (7.8) with $n = 2$ is $\frac{1}{2}[(3^{2r-2})', 4]$, and (6.6) is $[3^{m-3,1,1}]'$ or $\frac{1}{2}[3^{m-2}, 4]'$ according as m is odd or even. When $m \equiv 0$ or $1 \pmod 4$, (6.9) and (6.7) provide alternative forms for $[3^{m-2}, 4]$ and $\frac{1}{2}[3^{m-2}, 4]$; and when $m \equiv 3 \pmod 4$, (6.7) provides an alternative form for $[(3^{m-2})', 4]$. By (8.1), we have now accounted for all our geometrical groups save $[3^{2r-3,1,1}]$, $\frac{1}{2}[3^{2r-3,1,1}]$, $[3^{2r-3,1,1}]'$, and $\frac{1}{2}[3^{2r-3,1,1}]'$*.

It seems worth while to give these abstract definitions explicitly in the important case when $m = 4$ (or $r = 2$).

$[3, 3, 4]$: $S^8 = T^2 = (ST)^6 = (S^2T)^4 = (S^{-1}TST)^3 = (S^4T)^2 = 1.$

$[(3, 3)', 4]$†:

$$U^6 = V^6 = (UV)^4 = (U^{-1}V)^3 = (U^{-1}V^{-1}UV)^2 = (U^{-1}V^{-1})^2(UV)^2 = 1.$$

$[3, 3, 4]'$: $R^8 = S^8 = (R^{-1}S)^3 = (RS)^3 = (R^2S^2)^2 = R^4S^4 = 1.$

$\frac{1}{2}[3, 3, 4]$: $S^4 = T^2 = (ST)^6 = (S^2T)^4 = (S^{-1}TST)^3 = 1.$

$\frac{1}{2}[(3, 3)', 4]$: $U^6 = V^6 = (UV)^2 = (U^{-1}V)^3 = (U^{-1}V^{-1}UV)^2 = 1.$

$\frac{1}{2}[3, 3, 4]'$: $R^4 = S^4 = (R^{-1}S)^3 = (RS)^3 = (R^2S^2)^2 = 1.$

---

* $[3^{m-3,1,1}]$ may be described as the subgroup of (5.8) whose operators involve S an even number of times; it is therefore generated by S^2, T, and $S^{-1}TS$. $[3^{m-3,1,1}]'$ may be described as the subgroup of (6.8) whose operators are of even "degree" in R, S; it is therefore generated by R^2, S^2, and RS. But no elegant definitions in terms of two generators have been found.

† Actually, this group has also the more elegant definition

$$S^6 = T^2 = (S^2T)^3 = (S^3T)^4 = 1.$$

There are also the four groups $[3^{1,1,1}]$, $[3^{1,1,1}]'$, $\frac{1}{2}[3^{1,1,1}]$, $\frac{1}{2}[3^{1,1,1}]'$. But, of the whole set of ten groups, only nine are abstractly distinct. For, let $\frac{1}{2}[3^{1,1,1}]$ be defined in the usual form*

$$O^2 = N^2 = P^2 = Q^2 = (PQ)^2 = (QN)^2 = (NP)^2$$
$$= (ON)^3 = (OP)^3 = (OQ)^3 = (ONPQ)^3 = 1,$$

and let us write $R = NOP, \quad S = OPQ.$

It is easily verified that

$$R^4 = S^4 = 1,$$

$$RS = NPOQ \quad \text{(which is conjugate to } ONPQ),$$

$$RS^{-1} = QNOQ \quad \text{(which is conjugate to } ON),$$

$$(R^2S^2)^2 = (ONOQOP)^2 = (NOPQ)^3 = 1,$$

$$R^2S = ONQ,$$

whence

$$Q = R^2SRS^{-1}, \qquad OP = SR^2SRS^{-1},$$
$$N = SR^2SRS^{-1}R^{-1}, \qquad QN = R^2SR^{-1}SRS^{-1}R^{-1},$$
$$O = (R^2S)^2R^{-1}SRS^{-1}R^{-1}, \quad ON = (R^2S)^2RS^{-1},$$
$$P = (R^2S)^2RS^{-1}R\dagger.$$

Thus $\frac{1}{2}[3^{1,1,1}]$ is generated by the operators R, S, which satisfy the above abstract definition for $\frac{1}{2}[3, 3, 4]'$. But these groups have the same order. Hence

$$\frac{1}{2}[3, 3, 4]' \sim \frac{1}{2}[3^{1,1,1}]. \tag{8.3}$$

In other words, *Goursat's groups XXVII and XLII are simply isomorphic.*

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* This, without the final relation, is $[3^{1,1,1}]$, by Coxeter, 1, 149 (16.74). The central inversion is $(NOPQ)^3$ or $(ONPQ)^3$, by Coxeter, 2, 607 (iii) $(n = 1)$.

† After some manipulation, these expressions can be simplified; in fact,

$$N = R^{-1}SRS^2 = S^{-1}RSR^2, \quad O = SRS^2R^{-1},$$
$$P = RSR^{-1}S^2 = R^2S^{-1}RS, \quad Q = R^2SRS^{-1} = S^2RSR^{-1}.$$

All these nine groups occur in Dr. Burns's list of groups of degree eight*.

Group.	Order.	No. in Dr. Burns's list.	Her generators in terms of ours.
[3, 3, 4]	384	1	$s_1 = S^2T,\quad s_2 = TS^{-1}$
$\frac{1}{2}$[3, 3, 4]	192	2	$s_1 = ST,\quad s_2 = S$
[(3, 3)′, 4]	192	3	?†
$[3^{1,1,1}]$	192	4	$s_1 = PON,\quad s_2 = NPOQ$
[3, 3, 4]′	192	5	$s_1 = R^{-1},\quad s_2 = S$
$\frac{1}{2}$[3, 3, 4]′ or $\frac{1}{2}[3^{1,1,1}]$	96	6	$s_1 = R,\quad s_2 = S^4_+$
$\frac{1}{2}$[(3, 3)′, 4]	96	7	$s_1 = U,\quad s_2 = V$
$[3^{1,1,1}]'$	96	8	
$\frac{1}{2}[3^{1,1,1}]'$	48	3	$s_1 = NO,\quad s_2 = OP,\quad s_3 = OQ$

9. *Groups involving the k-th power of the symmetric group.*

The importance of the group (5.1) is doubtless connected with the fact that it is "only just infinite", *i.e.*, that its geometrical representation is neither hyperbolic nor spherical, but Euclidean (like the infinite progenitors of the familiar "groups of genus one§"). This property is shared by the more general group

$$(9.1)\qquad S^{mk} = T^2 = (S^{-k}TS^kT)^3 = (S^{-j}TS^jT)^2 = 1$$

$$(j = 1, \ldots, k-1, k+1, \ldots, [\tfrac{1}{2}mk]\|\ ;\ m > 2)\P,$$

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* Burns, 1, 207–209. There are so many misprints that it is advisable to compare her abstract definitions with their restatement on pp. 211–213.

† Dr. Burns's definition in this case is inconsistent with the given expressions for the generators as permutations.

‡ Here the relation $(s_1^2 s_2)^4 = 1$ is not an adequate substitute for $(s_1^{-1} s_2)^3 = 1$. Putting $p_1 = p_2 = p_3 = 2$ in (7.3), we see that the group

$$U^4 = V^4 = (UV)^3 = (U^2V^2)^2 = (UV^2)^4 = 1$$

is of order 192, not 96.

§ Burnside, 1, 410.

‖ Meaning $j = 1, 2, \ldots, k-1, k+1, k+2, \ldots, [\frac{1}{2}mk]$.

¶ We might have written

$$S^m = T^2 = (S^{-k}TS^kT)^3 = (S^{-j}TS^jT)^2 = 1 \quad (j = 1, \ldots, k-1, k+1, \ldots, [\tfrac{1}{2}m];\ m > 2k).$$

But the important case is when m is a multiple of k, since otherwise, on replacing S by a suitable multiple of itself, we should have (m, k) in place of k.

When $m = 2$, (9.1) reduces to (4.6) with $p = 3$, and so is not infinite but of order $2k.6^k$.

which is derived from the g.g.r. (3.2) with $p_k = 3$ and every other $p_j = 2$, so that the representative graph consists of k separate m-gons. When we omit the last k of the mk generators $R_1, R_2, \ldots, R_{mk}$ (*i.e.*, delete one vertex and two adjacent sides from each of the m-gons), this g.g.r. yields the finite subgroup

$$(9.2) \qquad \begin{cases} R_i^2 = 1 & [i = 1, 2, \ldots, (m-1)k], \\ (R_i R_{i+k})^3 = 1 & [i = 1, \ldots, (m-2)k], \\ (R_i R_{i+j})^2 = 1 & (j \neq k), \end{cases}$$

which is $[3^{m-2}]^k$, the direct product of k symmetric groups of degree m. If we regard the i-th symmetric group as operating on the m symbols $a_i, a_{i+k}, \ldots, a_{i+(m-1)k}$, we may write

$$R_i = (a_i a_{i+k}) \quad [1 \leqslant i \leqslant (m-1)k].$$

By introducing the new operator

$$S = (a_1 a_2 \ldots a_{mk}),$$

so that

$$S^{-1} R_i S = R_{i+1}$$

and

$$(9.3) \qquad S^{-k} = R_1 R_2 \ldots R_{(m-1)k},$$

we increase the order from $(m!)^k$ to $k(m!)^k$.

On substituting $S^{-i} T S^i$ for R_i*, (9.2) and (9.3) become

$$T^2 = (S^{-k} T S^k T)^3 = (S^{-j} T S^j T)^2 = 1 \quad (j \neq k),$$

and

$$\begin{aligned} S^k &= S^{-(m-1)k} T S^{(m-1)k} \cdot S^{-(m-1)k+1} T S^{(m-1)k-1} \ldots S^{-2} T S^2 \cdot S^{-1} T S \\ &= S^{-(m-1)k} (TS)^{(m-1)k}, \end{aligned}$$

i.e.

$$S^{mk} = (ST)^{(m-1)k}.$$

Since S is of period mk, we can now exhibit our group of order $k(m!)^k$ as

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* We could obtain the same relations by substituting $S^{-i} T S^i$ for R_{i+1}, and then we should have $T = R_1$ instead of

$$T = S R_1 S^{-1} = S^k R_k S^{-k} = R_{(m-1)k} \ldots R_{2k} R_k R_{2k} \ldots R_{(m-1)k}.$$

a factor group of (9.1), in the form

$$(9.4)\qquad S^{mk} = T^2 = (ST)^{(m-1)k} = (S^{-k}TS^kT)^3 = (S^{-j}TS^jT)^2 = 1$$
$$(j = 1, \ldots, k-1, k+1, \ldots, [\tfrac{1}{2}mk];\ m > 2)^*.$$

Thus (9.4) is generated by the permutations

$$S = (a_1 a_2 \ldots a_{mk}), \quad T = (a_{mk} a_k),$$

or, after a trivial change of notation, by

$$S = (a_1 a_2 \ldots a_{mk}), \quad T = (a_1 a_{k+1})\dagger.$$

E.g., when $m = 4$ and $k = 2$, it is the group of degree 8 and order 1152‡.

When $(m-1)k$ is even, the operators S and $TS^{-1}T$ generate the subgroup

$$(9.5)\S\qquad R^{mk} = S^{mk} = (R^{-1}S)^{\frac{1}{2}(m-1)k} = (R^kS^k)^3 = (R^jS^j)^2 = 1,$$

of order $\frac{1}{2}k(m!)^k$.

The geometrical considerations of §5 can be extended so as to give an $(m-1)k$-dimensional representation for (9.1), in which $(ST)^{(m-1)k}$ appears as a translation. The reflecting primes are now

$$x_i = x_{k+i} \qquad [i = 1, 2, \ldots, (m-1)k],$$
$$x_{(m-1)k+i} = x_i + 1 \quad (i = 1, 2, \ldots, k),$$

and the number of dimensions is reduced from mk to $(m-1)k$ by putting

$$x_i + x_{k+i} + x_{2k+i} + \ldots + x_{(m-1)k+i} = 0 \quad (i = 1, 2, \ldots, k).$$

The longer translation $(ST)^{n(m-1)k}$ and its conjugates generate a subgroup of index $n^{(m-1)k}$ in the subgroup generated by $(ST)^{(m-1)k}$ and *its*

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* For the rest of the paper, we shall forgo repeating this statement of the range of j and the restriction on m.

A consideration of simple cases seems to indicate that the relation $(S^{-k}TS^kT)^3 = 1$ is always superfluous (as we know it is when $k = 1$).

† It is natural to expect that the analogous permutations

$$(a_1 a_2 \ldots a_m), \quad (a_1 a_{k+1}),$$

where now m is prime to k, will generate the symmetric group of degree m in the form

$$S^m = T^2 = (ST)^{(m-k)k} = (S^{-k}TS^kT)^3 = (S^{-j}TS^jT)^2 = 1 \quad (j = 1, \ldots, k-1, k+1, \ldots, [\tfrac{1}{2}m]).$$

The sufficiency of these relations has not been proved.

‡ Burns, 1, 210.

§ When $m = 4$ and $k = 2$, this is Dr. Burns's "Order 576, No. 3" (with a different abstract definition). Actually, this group is the same as

$$P^4 = Q^8 = (PQ)^2 = (P^{-1}Q)^3 = (P^2Q^2)^6 = 1.$$

conjugates. Thus the quotient group

$$(9.6)\quad S^{mk} = T^2 = (ST)^{n(m-1)k} = (S^{-k}TS^kT)^3 = (S^{-j}TS^jT)^2 = 1$$

is of order $kn^{(m-1)k}(m!)^k$.

This is derivable from the larger group

$$(9.7)\quad S^{nmk} = T^2 = (ST)^{n(m-1)k} = S^{-mk}TS^{mk}T = (S^{-k}TS^kT)^3 = (S^{-j}TS^jT)^2 = 1$$

as the quotient group of the cyclic subgroup (central) generated by S^{mk}. The group (9.7) is thus of order $kn^{(m-1)k+1}(m!)^k$. It is easily seen to be representable as a permutation group of degree nmk in the form

$$S = (1_1\, 1_2 \dots 1_{mk}\, 2_1\, 2_2 \dots 2_{mk} \dots n_1\, n_2 \dots n_{mk}),$$

$$T = (1_1\, 1_{k+1})(2_1\, 2_{k+1}) \dots (n_1\, n_{k+1}).$$

It is a subgroup of index n^{k-1} in the group whose general operator is

$$(\,)\, s_1^{l_1} s_2^{l_2} \dots s_{mk}^{l_{mk}},$$

where $(\,)$ denotes a permutation of the suffixes in accordance with the group (9.4), and $s_i = (1_i\, 2_i \dots n_i)$. The index is determined by the fact that the only combinations of the s's alone that belong to (9.7) are the products of conjugates of

$$s_1 s_2 \dots s_k = S^{mk}(ST)^{-(m-1)k},$$

i.e., those combinations for which

$$l_1 + l_{k+1} + l_{2k+1} + \dots \equiv l_2 + l_{k+2} + l_{2k+2} + \dots \equiv \dots \equiv l_k + l_{2k} + \dots + l_{mk} \pmod{n}.$$

The operators $(\,)$ that can occur unaccompanied by s's form a subgroup of index k in (9.4), namely the subgroup generated by T and its conjugates. It seems probable that the whole group of order $kn^{mk}(m!)^k$ cannot be generated by two operators (when $k > 1$ and $n > 1$). *E.g.*, when $n = m = k = 2$, this is the group of degree 8 and order 128*.

As analogous generalizations of (6.1) and (6.2), we have

$$(9.8)\quad R^{mk} = S^{mk} = (R^{-1}S)^{\frac{1}{2}n(m-1)k} = (R^kS^k)^3 = (R^jS^j)^2 = 1,$$

of order $\frac{1}{2}kn^{(m-1)k}(m!)^k$, and

$$(9.9)\quad S^{mk} = T^2 = (S^{-1}TST)^{\frac{1}{2}n(m-1)k} = (S^kT)^6 = (S^jT)^4 = 1,$$

of order $kn^{(m-1)k}(m!)^k$.

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* Burns, 1, 209.

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ON THE GEOMETRY OF THE ELECTROMAGNETIC FIELD IN GENERAL RELATIVITY

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The four-dimensional geometry of the three electromagnetic tensors (namely the six-vector F_{ab}, its dual F^*_{ab}, and the energy tensor E_{ab}) has been studied by Rainich†, and more recently by Synge‡. The former regards the tensor $g^{ac}F_{cb}$ as an operator for the transformation of vectors, and uses a four-dimensional vector algebra to investigate the two-dimensional elements which are transformed into themselves. The latter, who restricts himself to special relativity, pays chief attention to the principal null directions defined by the three tensors, a principal direction of any tensor T_{ab} being defined in the usual way by a vector ξ^a which satisfies the relation

$$(T_{ab}+\Lambda g_{ab})\,\xi^b = 0,$$

where the principal invariant Λ is a root of the determinantal equation

$$|T_{ab}+\Lambda g_{ab}| = 0.$$

The present paper may be regarded as a coordination and simplification of the work of the previous writers. The geometry of the electromagnetic field is shown to be reducible to a familiar configuration of linear complexes and quadrics in a three-dimensional projective space, which, on account of its simplicity and symmetry, provides a more pleasing geometrical representation of the electromagnetic field than any hitherto given. Many of the results obtained by Rainich and Synge, and particularly the latter, are shown to be nearly or completely obvious, as, for example, the result that the three tensors have the same principal null directions.

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† *Trans. American Math. Soc.*, 27 (1925), 106–136; see references on p. 113.

‡ *Univ. of Toronto Studies*, Applied Math. Series, No. 1 (1935).

The present work is based on Synge's in that it gives chief place to the principal null directions, and the notation is to a large extent the same as his. In fact the analytical part of the paper, except for certain new results appearing in § 8 and § 9, is devoted to re-establishing his results and showing how they arise naturally out of the geometry. It is fairly obvious from his paper that those of his formulae which do not relate to special coordinate-systems are extensible to general relativity, and the following work shows that this is actually the case. No attempt is made to state the results in physical terms, since this is done adequately by Synge for special relativity, and the corresponding statement for general relativity is too similar to require separate treatment.

The method adopted below is simple. It consists of considering the hyperplane at infinity in the centred affine tangent-space defined at any given point of space-time by the totality of contravariant vectors at that point†. In that hyperplane, which is a three-dimensional projective space S_3, the fundamental tensor g_{ab} represents a quadric, namely the section of the null cone by the hyperplane. The tensors F_{ab}, F^*_{ab} represent linear complexes which are polar to one another with respect to the quadric, and which in general contain four of its generators—two of each system. This defines upon the quadric a skew quadrilateral whose vertices represent the principal null directions of the electromagnetic tensors, and the energy tensor E_{ab} defines a second quadric passing through the skew quadrilateral. The two quadrics are symmetrically related: they are apolar both ways, and each is its own polar with respect to the other. The interpretation of the six-vectors in terms of linear complexes is clearly related to Rainich's interpretation of them as operators, since with any linear complex is associated a null polarity.

Thus the essential pure mathematical theory of this paper is that of the well-known‡ relations between a quadric g_{ab} and a linear complex F_{ab} in a three-dimensional space. The theory is here developed afresh, partly in order to avoid the tedious necessity of translating earlier writers' work into a tensor form suitable for physical interpretation, and partly because the precise formulae required for present purposes are not all to be found elsewhere§.

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† Veblen and Whitehead, *Foundations of differential geometry* (Camb. Tract No. 29), ch. V.

‡ See, *e.g.*, Weitzenböck, *Journal für Math.*, 137 (1910), 65, and Das Gupta, *Proc. London Math Soc.* (2), 31 (1930), 487.

§ Such geometry as is required for the understanding of this paper may be found in the standard text-books. See, *e.g.*, Sommerville, *Analytical geometry of three dimensions* (Cambridge, 1934), ch. 13, 15, 16.

It is interesting to notice that a consideration of the three-dimensional space S_3 yields results for the four-dimensional space-time. Actually, a little care is needed, since only the ratios of the components of tensors are of significance in S_3, whereas their actual values, except in the case of null vectors, are of significance in space-time.

1. *The space* S_3.

We suppose that the metric of the space-time V_4 is given in terms of a general coordinate-system (x^1, x^2, x^3, x^4) by the formula

$$ds^2 = g_{ab}\, dx^a\, dx^b. \tag{1.1}$$

We take the signature of this quadratic form to be $(- - - +)$, this being now rather more common than the signature $(+ + + -)$ adopted by Synge. Also, it is unnecessary for present purposes, and indeed undesirable, to follow him in making the metric positive definite by the introduction of imaginaries. These remarks account for certain minor differences in sign in the formulae of his paper and the corresponding formulae obtained below.

The components of a contravariant vector X^a at any given point (x^a) of V_4 are the non-homogeneous coordinates of a point of the centred affine tangent-space T_4 at (x^a). The origin O (0, 0, 0, 0) of all coordinate-systems in T_4 is to be regarded as coinciding with the point of contact (x^a). The equation

$$g_{ab}\, X^a\, X^b = 0 \tag{1.2}$$

is that of the null cone in T_4.

If now we introduce homogeneous coordinates into T_4 by replacing X^a by X^a/X^0, where X^0 is the extra coordinate, the equation

$$X^0 = 0 \tag{1.3}$$

represents the threefold plane S_3 at infinity, and equations (1.2), (1.3) taken together represent the section of the cone by S_3, that is, they represent a quadric† in S_3.

Any geometrical figure in S_3, which, as remarked above, is a three-dimensional projective space, corresponds to the figure in T_4 obtained by

† If the quadratic form in (1.2) is taken to define distance in T_4, which then becomes a pseudo-Euclidean space, the quadric in S_3 is the absolute of T_4.

joining the origin O to the points of the figure in S_3. Thus to a system of lines in S_3, such as a linear complex, corresponds in T_4 the system of planes through these lines and the origin O.

2. *Notation and identities.*

Suffixes will be raised and lowered in the usual way by means of the fundamental tensor g_{ab}. Thus

$$F^{ab} = g^{ac} g^{bd} F_{cd}. \tag{2.1}$$

Also we denote by ϵ_{abcd} the covariant tensor having the values $\pm\sqrt{(-g)}$ when *abcd* is an even or odd permutation of 1234, and having the value zero when two or more suffixes are the same; and by ϵ^{abcd} the corresponding contravariant tensor $\{\pm 1/\sqrt{(-g)}, 0\}$. The tensor dual to F_{ab} is then

$$F^{*ab} = \tfrac{1}{2}\epsilon^{abcd} F_{cd}, \tag{2.2}$$

which gives, lowering the suffixes by means of g_{ab},

$$F^{*}_{ab} = -\tfrac{1}{2}\epsilon_{abcd} F^{cd}. \tag{2.3}$$

The appearance of the minus sign on the right is due to the fact that the ϵ-tensors involve $\sqrt{(-g)}$ and not $\sqrt{g}$.

If X^{ab}, Y_{ab} are any pair of skew-symmetric tensors, we denote their inner product by (XY); thus

$$(XY) \equiv X^{ab} Y_{ab} \equiv (YX).$$

With this notation, we have

$$F^{*ab} F_{bc} \equiv -\tfrac{1}{4}(FF^*)\,\delta^a_c \equiv F^{ab} F^{*}_{bc}, \tag{2.4}$$

$$F^{ab} F_{bc} - F^{*ab} F^{*}_{bc} = -\tfrac{1}{2}(FF)\,\delta^a_c, \tag{2.5}$$

and

$$(FF) = -(F^* F^*). \tag{2.6}$$

The formulae (2.4) and (2.5) are slightly disguised forms of well-known identities satisfied by skew-symmetric matrices, and are easily proved.

Further, if ξ^a, η^a are any two vectors, it will be convenient to use the notation $(\xi\eta)$ defined by

$$(\xi\eta) \equiv g_{ab}\,\xi^a \eta^b \equiv \xi^a \eta_a \equiv \xi_a \eta^a. \tag{2.7}$$

3. *Interpretation of* F_{ab} *in* S_3.

We now return to the three-dimensional projective space S_3. If X^a are current coordinates in S_3, a linear equation

$$\phi_a X^a = 0$$

represents a plane. Hence the components of any covariant vector ϕ_a of V_4 are to be regarded as the coordinates of a plane in S_3.

If ϕ_a, ψ_a are any two planes of S_3, the Plücker coordinates of their line of intersection are

$$p_{ab} = \phi_a \psi_b - \psi_a \phi_b ;$$

and if Y^a, Z^a are any two points on the same line, the dual set of Plücker coordinates is

$$p^{*ab} = Y^a Z^b - Z^a Y^b .$$

The two sets are connected by the relation

$$p^{*ab} = \rho \epsilon^{abcd} p_{cd},$$

where ρ is a scalar, and the coordinates satisfy the identity

$$\epsilon^{abcd} p_{ab} p_{cd} \equiv 0, \quad \text{or} \quad p^{*ab} p_{ab} \equiv 0, \tag{3.1}$$

these being different forms of the same relation.

The lines whose Plücker coordinates satisfy the linear equation

$$F_{ab} p^{*ab} = 0 \tag{3.2}$$

constitute a linear complex. The polar plane of any point Y^a with respect to the associated focal system† has coordinates $F_{ab} Y^b$, and the pole of any plane ψ_a with respect to the same focal system has coordinates $F^{*ab} \psi_b$.

Thus the components of the electromagnetic six-vector F_{ab} are the coordinates in S_3 of a linear complex, while those of F^{*ab} are the dual coordinates of the same complex. Further, F^{ab} represents the linear complex polar to F_{ab} with respect to the quadric g_{ab}, and F^*_{ab} the dual coordinates of this polar complex.

The complex F_{ab} is special, that is, it consists of all the lines meeting a given line (the axis of the special complex), if

$$F^{*ab} F_{ab} = 0 ; \tag{3.3}$$

in this case, F^{*ab} and F_{ab} are the Plücker coordinates of the axis.

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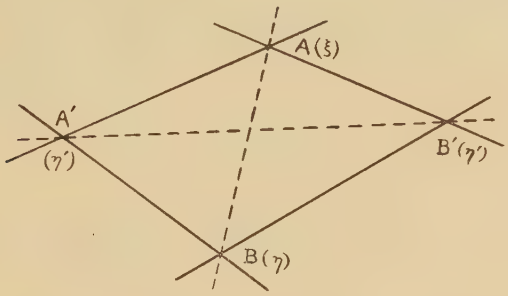
† Baker, *Principles of geometry*, 3 (1923), 61; I use the phrase *focal system* instead of the more common *null system* because the word *null* is used in this paper in another sense.

4. *The principal null directions of the electromagnetic six-vectors.*

We begin with the most general case in which the linear complex F_{ab} has two distinct lines in common† with each regulus (system of generators) of the quadric g_{ab}. Then to F_{ab} belong the sides of a skew quadrilateral $AA'BB'$ lying on the quadric, AA' and BB' being generators of one system



and AB', $A'B$ of the other. Let the coordinates of A, A', B, B' be respectively ξ^a, ξ'^a, η^a, η'^a. Then since each of these points lies on the quadric, we have, in the notation of (2.7),

$$(\xi\xi) = 0 = (\xi'\xi') = (\eta\eta) = (\eta'\eta'). \tag{4.1}$$

Now $\xi_a \equiv g_{ab}\xi^b$ is the polar plane of ξ^a with respect to the quadric. But ξ^a is on the quadric, hence ξ_a is the tangent-plane $AA'B'$ at A. Since A', B' lie on this plane, we have

$$\left.\begin{aligned} (\xi\xi') = 0 = (\xi\eta'), \\ \text{and similarly} \qquad (\xi'\eta) = 0 = (\eta\eta'). \end{aligned}\right\} \tag{4.2}$$

But
$$(\xi\eta) \neq 0, \quad (\xi'\eta') \neq 0. \tag{4.3}$$

In language appropriate to the original Riemannian space, (4.1) means that the vectors ξ, ξ', η, η' are null and self-perpendicular, and (4.2), (4.3) mean that each is perpendicular to two of the others but not to the third.

Now $F_{ab}\xi^b$ is the polar plane of the point ξ^a with respect to the focal system F_{ab}. Since this plane contains all the lines of the complex F_{ab} which pass through ξ^a, it contains AA', AB' ; that is, it is the tangent-

† The well-known fact that a linear complex has in general two lines in common with a regulus is most easily seen by representing the lines of S_3 by the points of a fourfold quadric in a space of five dimensions: see Sommerville, *op. cit.*, 343 *et seq.*

plane $AA'B'$ of the quadric. We have already remarked that this tangent-plane has coordinates $g_{ab}\,\xi^b$, so there exists a number λ such that

$$F_{ab}\,\xi^b = -\lambda g_{ab}\,\xi^b,$$

i.e.

$$(F_{ab}+\lambda g_{ab})\,\xi^b = 0.$$

Hence the null direction ξ^a of the original Riemannian space is a principal null direction of the electromagnetic six-vector, and λ is the principal invariant belonging to ξ^a. Similar remarks are true of ξ', η, η', so we have

$$\left.\begin{aligned}(F_{ab}+\lambda g_{ab})\xi^b &= 0 \quad (a),\\ (F_{ab}+\mu g_{ab})\eta^b &= 0 \quad (b),\\ (F_{ab}+\lambda' g_{ab})\xi'^b &= 0 \quad (c),\\ (F_{ab}+\mu' g_{ab})\eta'^b &= 0 \quad (d),\end{aligned}\right\} \tag{4.4}$$

where, quite obviously, λ, μ, λ', μ' are the roots of the determinantal equation

$$|F_{ab}+\Lambda g_{ab}| = 0. \tag{4.5}$$

There are therefore in general four distinct principal null directions associated with the electromagnetic six-vector, but it will be shown below that only two of these are real.

Multiply (4.4)(a) by η^a and (4.4)(b) by ξ^a, summing for a. Using the fact that F_{ab} is skew, we get, on adding,

$$(\lambda+\mu)(\xi\eta) = 0,$$

whence, using (4.3),

$$\lambda = -\mu,$$

and similarly

$$\lambda' = -\mu'.$$

Also, for the general case at present under consideration, it is evident that

$$\lambda \neq \pm\lambda'. \tag{4.6}$$

There are therefore only two essentially distinct principal invariants for F_{ab}, namely λ and λ', and it will appear presently that only one of these is real. Substituting in (4.4), we now have

$$\left.\begin{aligned}(F_{ab}+\lambda g_{ab})\,\xi^b &= 0 \quad (a),\\ (F_{ab}-\lambda g_{ab})\,\eta^b &= 0 \quad (b),\\ (F_{ab}+\lambda' g_{ab})\,\xi'^b &= 0 \quad (c),\\ (F_{ab}-\lambda' g_{ab})\,\eta'^b &= 0 \quad (d).\end{aligned}\right\} \tag{4.7}$$

Since a generator of a quadric is its own polar with respect to the quadric, it is clear that the complex F^*_{ab} polar to F_{ab} has the same skew quadrilateral in common with the reguli of the quadric. Consequently the principal null directions of the tensor F^*_{ab} are the same as those of the tensor F_{ab}. To find the corresponding principal invariants, multiply (4.7)(a) by F^{*ca} and use (2.4). We get

$$-\tfrac{1}{4}(FF^*)\,\xi^c+\lambda F^{*ca}\,\xi_a=0, \tag{4.8}$$

whence, lowering the suffix c,

$$\left\{F^*_{ca}-\frac{1}{4\lambda}\,(FF^*)\,g_{ca}\right\}\,\xi^a=0, \tag{4.9}$$

showing that the principal invariant for F^*_{ab}, associated with ξ^a, is

$$\lambda^*=-\tfrac{1}{4}(FF^*)/\lambda. \tag{4.10}$$

The other principal invariants for F^*_{ab} are given by similar formulae. In particular, that associated with η^a is $-\lambda^*$.

Now the polar planes of A, B with respect to both the quadric and the focal system F_{ab} are respectively $AA'B'$ and $BA'B'$. Hence the lines AB, $A'B'$ are polar to one another with respect to both g_{ab} and F_{ab}. Let P be a point on AB. Its polar plane with respect to F_{ab} is $PA'B'$, and all the lines through P in this plane belong to the linear complex F_{ab}. Hence the linear congruence having AB, $A'B'$ as directrices, *i.e.* the totality of lines which meet AB, $A'B'$, belongs to the complex F_{ab}. Consequently, if p_{ab}, p'_{ab} are the Plücker coordinates of the lines AB, $A'B'$, there exist coefficients θ, ϕ such that

$$F_{ab}=\theta p'_{ab}+\phi p_{ab}. \tag{4.11}$$

Now $A'B'$ is the intersection of the planes $AA'B'$, $BA'B'$ of coordinates ξ_a, η_a respectively. Hence

$$p'_{ab}=\xi_a\eta_b-\eta_a\xi_b. \tag{4.12}$$

Similarly $$p_{ab}=\xi_a'\eta_b'-\eta_a'\xi_b'.$$

In § 5 it will be shown that ξ', η' are complex if ξ, η are real, so instead of taking p_{ab} in this form we observe that, since it is the line joining the points ξ^a, η^a, its dual coordinates are

$$p^{*ab}=\xi^a\eta^b-\eta^a\xi^b,$$

and p_{ab} is therefore a multiple of $\frac{1}{2}\epsilon_{abcd}\,p^{*cd}$. From this, combined with (4.11) and (4.12), we deduce that coefficients θ, ϕ exist such that

$$F_{ab}=\theta(\xi_a\eta_b-\eta_a\xi_b)+\tfrac{1}{2}\phi\epsilon_{abcd}(\xi^c\eta^d-\eta^c\xi^d). \tag{4.13}$$

If we multiply this by $\frac{1}{2}\epsilon^{klab}$, summing for a and b, we get

$$F^{*kl}=\tfrac{1}{2}\theta\epsilon^{klab}(\xi_a\eta_b-\eta_a\xi_b)+\phi(\xi^k\eta^l-\eta^k\xi^l). \tag{4.14}$$

Multiplying (4.13) by ξ^b and (4.14) by ξ_l, summing for the repeated suffixes, we get, on using (4.7)(a), (4.1), (4.8), and (4.10),

$$-\lambda\xi_a=\theta\xi_a(\xi\eta),\quad -\lambda^*\xi^k=\phi\xi^k(\xi\eta),$$

which give

$$\theta=-\lambda/(\xi\eta),\quad \phi=-\lambda^*/(\xi\eta). \tag{4.15}$$

Substituting in (4.13), we get

$$F_{ab}=-\frac{\lambda}{(\xi\eta)}(\xi_a\eta_b-\eta_a\xi_b)-\frac{\lambda^*}{2(\xi\eta)}\,\epsilon_{abcd}(\xi^c\eta^d-\eta^c\xi^d). \tag{4.16}$$

This expresses† the electromagnetic six-vector in terms of the null vectors ξ, η and the scalars λ, λ^*. It is interesting to notice how the geometry suggests the existence of such a relation.

5. *Reality of the principal null vectors and scalars.*

We know that the components of the six-vector F_{ab} and the tensor g_{ab} are real. Consequently the coefficients of equation (4.5), viz.

$$|F_{ab}+\Lambda g_{ab}|=0, \tag{5.1}$$

are real, and we know that the roots are $\pm\lambda$, $\pm\lambda'$. Hence, if Λ is any one root, $-\Lambda$ is another, and the coefficients of Λ, Λ^3 in (5.1) are therefore zero ; that is, (5.1) is of the form

$$|F_{ab}|+A\Lambda^2+g\Lambda^4=0, \tag{5.2}$$

where $|F_{ab}|$ is the determinant of F_{ab}, g is the determinant $|g_{ab}|$, and A is a function of the F's and g's. $|F_{ab}|\neq 0$, since the complex F_{ab} is non-special by hypothesis. Now if Λ is a root of (5.2), the conjugate complex of Λ is another since the coefficients are real. Combining this with the

† *Cf.* Synge, *loc. cit.*, 44 (9.15). Our $-\lambda$ is Synge's ω.

fact that the roots are known to be $\pm\lambda$, $\pm\lambda'$, we have, in view of (4.6), the following apparent possibilities : (i) λ real, λ' a pure imaginary, or *vice versa*; (ii) λ, λ' both real; (iii) λ, λ' both pure imaginaries; (iv) λ the conjugate complex of λ' or of $-\lambda'$. Now the product of the roots of (5.2) is $\lambda^2\lambda'^2$, whence

$$\lambda^2\lambda'^2 = |F_{ab}|/g.$$

But $|F_{ab}|$ is a real skew-symmetric determinant of even order, so that it is a perfect square and therefore positive. Also $g < 0$, whence $\lambda^2\lambda'^2 < 0$. The only actual possibility is therefore (i), that is, one of λ, λ' is real and the other is a pure imaginary. Suppose that λ is real and λ' imaginary. Then from (4.7) it is evident that ξ^a, η^a are real and ξ'^a, η'^a complex. Finally, by (4.10), λ^* is real.

6. *The electromagnetic energy tensor.*

We return to the quadric g_{ab} and the linear complex F_{ab} in the space S_3, and still suppose that four distinct lines of the complex are generators of the quadric.

The envelope of the polar planes, with respect to the focal system F_{ab}, of the points on the quadric g_{ab} is the quadric whose point-equation is

$$Q_{ab}X^aX^b = 0, \tag{6.1}$$

where

$$Q_{ab} \equiv F_{ac}g^{cd}F_{db}. \tag{6.2}$$

This quadric is also the locus of the poles with respect to F_{ab} of the tangent-planes to the quadric g_{ab}. In other words, the quadric Q_{ab} is the polar of the quadric g_{ab} with respect to the focal system F_{ab}. And since the sides of the skew quadrilateral $AA'BB'$ common to F_{ab} and g_{ab} are self-polar with respect to F_{ab}, this skew quadrilateral also lies on Q_{ab}. Thus the quadrics g_{ab}, Q_{ab} intersect in the skew quadrilateral, the sides of which are generators of both quadrics. Hence the pencil of quadrics

$$Q_{ab} + \sigma g_{ab} \tag{6.3}$$

is of the type [(11)(11)] according to the Segre classification†, and every quadric of the system intersects every other in the skew quadrilateral $AA'BB'$. Now the quadric of the system outpolar‡ to the quadric

† Sommerville, *op. cit.*, 268.

‡ *Ibid.*, 311, or Baker, *op. cit.*, 3, 52.

envelope $g^{ab}\phi_a\phi_b = 0$ is by definition the one for which

$$g^{cd}(Q_{cd}+\sigma g_{cd}) = 0,$$

giving $$\sigma = -\tfrac{1}{4}g^{cd}Q_{cd} = \tfrac{1}{4}(FF) \quad \text{by (6.2).}$$

Substituting in (6.3), we find that the quadric in question is

$$E_{ab}X^aX^b = 0,$$

where $$E_{ab} = F_{ac}g^{cd}F_{db}+\tfrac{1}{4}(FF)g_{ab}, \tag{6.4}$$

and we have $$g^{ab}E_{ab} = 0. \tag{6.5}$$

Lowering the suffix a in (2.5), we get

$$E_{ab} = F_{ac}g^{cd}F_{db}+\tfrac{1}{4}(FF)g_{ab} = F^*_{ac}g^{cd}F^*_{db}-\tfrac{1}{4}(FF)g_{ab}. \tag{6.6}$$

Since, by (2.6), $(FF) = -(F^*F^*)$, it is evident that the same quadric is obtained whether we start with the complex F_{ab} or its polar F^*_{ab} with respect to the quadric g_{ab}. Adding the two equations (6.6), we obtain

$$E_{ab} = \tfrac{1}{2}(F_{ac}g^{cd}F_{db}+F^*_{ac}g^{cd}F^*_{db}). \tag{6.7}$$

In the original Riemannian space, E_{ab} is of course the electromagnetic energy tensor.

Raising the suffixes a, b in (6.6) by means of g^{ab}, we get

$$E^{ab} = F^{ac}g_{cd}F^{db}+\tfrac{1}{4}(FF)g^{ab} = F^{*ac}g_{cd}F^{*db}-\tfrac{1}{4}(FF)g^{ab}, \tag{6.8}$$

whence

$$E^{ac}E_{cb} = \{F^{ak}g_{kl}F^{lc}+\tfrac{1}{4}(FF)g^{ac}\}\{F^*_{cm}g^{mn}F^*_{nb}-\tfrac{1}{4}(FF)g_{cb}\}.$$

Multiplying out, using (2.4) and (2.5), we quickly get

$$E^{ac}E_{cb} = \varpi\delta^a_b, \tag{6.9}$$

where $$\varpi = \tfrac{1}{16}\{(FF^*)^2+(FF)^2\}. \tag{6.10}$$

Hence $\varpi^{-1}(E^{ab})$ is the matrix reciprocal to (E_{ab}), so that $E^{ab}\phi_a\phi_b = 0$ is the tangential equation of the quadric E_{ab}. Since $E^{ab} = g^{ac}g^{bd}E_{cd}$, it follows that the quadric E_{ab} is its own polar with respect to the quadric g_{ab}. Further, raising the suffix b in (6.9), we get $\varpi g^{ab} = E^{ac}E^{bd}g_{cd}$, so that the quadric g_{ab} is its own polar with respect to E_{ab}. Moreover $E^{ab}g_{ab} = 0$, so that the quadric locus g_{ab} is outpolar to the quadric envelope E^{ab}. There is thus a completely symmetrical relationship between the two quadrics : they are apolar both ways and each is self-polar with

respect to the other. It is also true that the complexes F_{ab}, F^*_{ab} are polar to one another with respect to E_{ab} as well as with respect to g_{ab}.

Let Δ denote the determinant $|E_{ab}|$. Lowering the suffix a in (6.9), we have

$$E_{ac}g^{cd}E_{db} = \varpi g_{ab},$$

whence, forming the determinant of both sides,

$$\Delta^2 g^{-1} = \varpi^4 g,$$

which gives

$$\Delta = \pm\varpi^2 g. \tag{6.11}$$

The sign will be determined later.

7. *The principal invariants and null directions of the electromagnetic energy tensor.*

We have seen that the quadrics E_{ab}, g_{ab} have common generators forming the skew quadrilateral $AA'BB'$, the coordinates of whose vertices are ξ^a, ξ'^a, η^a, η'^a. The tangent-plane with respect to either quadric at the point A is therefore $AA'B'$, so that $E_{ab}\xi^b$ and $g_{ab}\xi^b$ are coordinates of the same plane; and similarly for ξ'^a, η^a, η'^a. Therefore, if ζ is any one of the four points ξ, ξ', η, η', there exists a number κ such that

$$E_{ab}\zeta^b = -\kappa g_{ab}\zeta^b.$$

In other words, ξ, ξ', η, η' are the four points on the quadric g_{ab} which satisfy

$$(E_{ab}+\kappa g_{ab})\zeta^b = 0, \tag{7.1}$$

each one of these points corresponding to a root of the equation

$$|E_{ab}+\kappa g_{ab}| = 0. \tag{7.2}$$

The principal null directions determined by the electromagnetic energy tensor are therefore the same as those determined by the six-vectors†. As we have seen, only two, namely ξ, η, are real.

Now (7.2) is the condition that the quadric $E_{ab}+\kappa g_{ab}$ of the pencil determined by E_{ab} and g_{ab}, which is of course the same as the pencil (6.3) determined by Q_{ab} and g_{ab}, should be a cone or a pair of planes. Actually it is obvious that the only degenerate quadrics of the pencil are the pairs of planes $(AA'B', BA'B')$ and $(A'AB, B'AB)$. Now (7.2) may be written

$$\Delta+\Theta\kappa+\Phi\kappa^2+\Theta'\kappa^3+\Delta'\kappa^4 = 0, \tag{7.3}$$

† Synge, *loc. cit.*, 30, 35.

where the coefficients are the well-known relative invariants† of the pair of quadrics E_{ab}, g_{ab}. Since they are apolar, we know that $\Theta = 0$, $\Theta' = 0$. Also $\Delta' = |g_{ab}| = g$, and Δ is the determinant $|E_{ab}|$ considered above. Hence (7.3) becomes

$$\Delta + \Phi\kappa^2 + g\kappa^4 = 0. \tag{7.4}$$

Now we are dealing with the case [(11)(11)], which means that (7.4) is of the form

$$g(k-\kappa)^2(k'-\kappa)^2 = 0,$$

that is, (7.4) has repeated roots k, k'. Multiplying the last equation out and comparing with (7.4), it is at once evident that

$$k' = -k, \tag{7.5}$$

and we get $$gk^4 - 2gk^2\kappa^2 + g\kappa^4 = 0,$$

which is to be identified with (7.4). This shows that

$$\Delta = gk^4. \tag{7.6}$$

There are thus only two principal invariants k, k' associated with null directions of the electromagnetic energy tensor, and one of these is minus the other‡.

Let $+k$ be that root of (7.4) which corresponds to ξ^a. Then

$$\begin{aligned} 0 &= (E_{ab} + kg_{ab})\,\xi^b \\ &= [F_{ac}g^{cd}F_{db} + \{\tfrac{1}{4}(FF) + k\}g_{ab}]\,\xi^b, \quad \text{by} \quad (6.4) \\ &= \{\lambda^2 + \tfrac{1}{4}(FF) + k\}g_{ab}\,\xi^b \end{aligned}$$

by a double application of (4.7)(a). Hence§

$$k = -\{\lambda^2 + \tfrac{1}{4}(FF)\}. \tag{7.7}$$

Again, k is unaltered if λ is replaced by $-\lambda$; and since, by (4.7)(b), $-\lambda$ is the principal invariant associated with F_{ab} and η^a, it follows that k is the principal invariant associated with E_{ab} and η^a. That is,

$$(E_{ab} + kg_{ab})\,\eta^b = 0.$$

† Sommerville, *op. cit.*, ch. 15.
‡ Synge, *loc. cit.*, 15.
§ *Ibid.*, 30, (7.8).

Similarly ξ' and η' both satisfy the equation corresponding to the other repeated root, viz. $-k$, of (7.4). In other words,

$$(E_{ab}-kg_{ab})\zeta^b = 0 \tag{7.8}$$

is satisfied by $\zeta \equiv \xi'$ and $\zeta \equiv \eta'$. Hence, just as above,

$$-k = k' = -\{\lambda'^2+\tfrac{1}{4}(FF)\}, \tag{7.9}$$

λ' being the invariant appearing in (4.7)(*c*) and (*d*). Subtracting (7.9) from (7.7), we get

$$k = \tfrac{1}{2}(\lambda'^2-\lambda^2). \tag{7.10}$$

Now λ is real and λ' is a pure imaginary, so that k is real and negative. Consequently $k^4 > 0$, and, since $g < 0$, it follows by (7.6) that $\Delta < 0$. Therefore in (6.11), since ϖ is real, we must take the plus sign. This gives

$$\Delta = \varpi^2 g, \tag{7.11}$$

whence, comparing with (7.6), we get

$$\begin{aligned} k &= \pm\varpi^{\frac{1}{2}} \\ &= -\tfrac{1}{4}\{(FF^*)^2+(FF)^2\}^{\frac{1}{2}} \end{aligned} \tag{7.12}$$

by (6.10), the minus sign being taken since $k < 0$. From this, (7.7), and (4.10), it quickly follows that†

$$k = -\tfrac{1}{2}(\lambda^2+\lambda^{*2}). \tag{7.13}$$

Also, by (7.10) and (7.13),

$$\lambda' = \pm i\lambda^*. \tag{7.14}$$

Now, as we have seen, $E_{ab}\pm kg_{ab}$ are the degenerate quadrics of the pencil (6.3) from which we started, and the two equations

$$(E_{ab}\pm kg_{ab})\,X^a X^b = 0 \tag{7.15}$$

are therefore those of the pairs of planes $(AA'B', BA'B')$, $(A'AB, B'AB)$. To find which is which, we note that the "tangent-plane" of A', (ξ'^a), with respect to the former pair, is indeterminate since A' lies on the line of intersection of this pair. Now the tangent-planes of ξ'^a with respect to the degenerate quadrics (7.15) have coordinates $(E_{ab}\pm kg_{ab})\,\xi'^b$, and of these that corresponding to the minus sign is indeterminate by (7.8).

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† (7.12) and (7.13) may be compared with equations (9.20) on p. 44 of Synge's paper.

Hence $E_{ab}-kg_{ab}$ represents the pair of planes $(AA'B', BA'B')$, and these have coordinates ξ_a, η_a. It follows that the equations

$$(E_{ab}-kg_{ab})X^aX^b=0, \quad (\xi_aX^a).(\eta_aX^a)=0$$

represent the same locus, so that

$$E_{ab}-kg_{ab}=\chi(\xi_a\eta_b+\eta_a\xi_b), \tag{7.16}$$

where χ is to be determined. Multiplying by g^{ab} and using (6.5), we get

$$-4k=2\chi(\xi\eta).$$

Eliminating χ from this and (7.16), we obtain

$$E_{ab}=kg_{ab}-2k(\xi_a\eta_b+\eta_a\xi_b)/(\xi\eta), \tag{7.17}$$

which expresses† the electromagnetic energy tensor in terms of the principal null vectors ξ, η and the scalar k.

It is worth observing that the null directions ξ, η, ξ', η' are not the only principal directions associated with the energy tensor, although the others, which are infinite in number, are not null. For the coordinates X^a of any point on the diagonals AB, $A'B'$ of the skew quadrilateral define a principal direction of E_{ab}. This is easy to see because any point X^a on these diagonals has the same polar plane with respect to both of the quadrics E_{ab}, g_{ab}, which means that a number ρ exists such that $E_{ab}X^b=-\rho g_{ab}X^b$.

To conclude this section, it may be noted that the two planes through the origin O of the tangent-space T_4 and the diagonals of the skew quadrilateral in the plane S_3 at infinity are the invariable planes discovered by Rainich. For since the diagonals are polar to one another with respect to both the focal system F_{ab} and the quadric g_{ab}, they are self-corresponding with respect to the collineation $g^{ac}F_{cb}$ obtained by polarising first with respect to the focal system and then with respect to the quadric. Hence both lines, and therefore the planes through them and O, are invariant under the transformation $g^{ac}F_{cb}$.

8. *Explicit formulae for the principal null directions.*

The real principal null directions ξ^a, η^a may be determined in terms of the electromagnetic tensors by solving equations (4.7)(a) and (b), or by solving (7.1), with $\kappa=k$, in combination with the condition $(\zeta\zeta)=0$. We now obtain explicit formulae for these directions.

† Synge, *loc. cit.*, 44, (9.19).

Let ϑ_a be any plane in S_3, and let it meet the diagonals AB, $A'B'$ of the skew quadrilateral in P, P' respectively. We assume at present that neither P nor P' coincides with any of the four points A, A', B, B'. Now we saw in § 7 that the equation $(E_{ab}-kg_{ab})X^aX^b=0$ is that of the pair of planes $AA'B'$, $BA'B'$. Reciprocating with respect to g_{ab}, we deduce that $(E^{ab}-kg^{ab})\phi_a\phi_b=0$ is the equation of the pair of points A, B regarded as a degenerate quadric envelope. The point Q of coordinates

$$(E^{ab}-kg^{ab})\vartheta_b \tag{8.1}$$

is therefore the polar of the plane ϑ_a with respect to this pair of points, that is, it is the point on AB harmonically conjugate to P with respect to A and B.

Further, it was remarked in § 4 that the congruence having AB, $A'B'$ as directrices belongs to the linear complex F_{ab}, and that AB, $A'B'$ are polar to one another with respect to the quadric g_{ab}. From the latter fact it follows that the congruence also belongs to the complex F^*_{ab} polar to F_{ab} with respect to g_{ab}, and hence that the line PP', which is one of those meeting both AB and $A'B'$, belongs to both of the complexes F_{ab}, F^*_{ab}. The poles of the plane ϑ_a with respect to these complexes, namely the points of coordinates

$$F^{*ab}\vartheta_b, \quad F^{ab}\vartheta_b, \tag{8.2}$$

therefore both lie on PP'. Thus the plane containing the three points (8.1), (8.2) is $PP'Q$, and this also contains A, B. Consequently the coordinates ξ^a, η^a of A, B are both expressible as linear combinations of the three sets of coordinates (8.1) and (8.2).

Consider ξ^a first. We have found that coefficients α, β, γ exist such that

$$\rho\xi^a=\alpha(E^{ab}-kg^{ab})\vartheta_b+\beta F^{ab}\vartheta_b+\gamma F^{*ab}\vartheta_b. \tag{8.3}$$

Since the point ξ^a determines a null vector, only the ratios of its coordinates are of significance either geometrically or physically, and consequently only the ratios of α, β, γ need, or indeed can, be determined. To find them, we make use of (4.7)(a), which may be written

$$F_{ka}\xi^a+\lambda g_{ka}\xi^a=0. \tag{8.4}$$

Now, by (6.8) and (7.7),

$$E^{ab}-kg^{ab}=F^{*ac}g_{cd}F^{*db}+\lambda^2 g^{ab}.$$

If we substitute this in (8.3) and then multiply (8.3) by F_{ka}, summing for a and using (2.4), we get

$$\rho F_{ka}\xi^a=-\tfrac{1}{4}\alpha(FF^*)F^*_{ka}\vartheta^a+\alpha\lambda^2F_{ka}\vartheta^a+\beta F_{ka}F^{ab}\vartheta_b-\tfrac{1}{4}\gamma(FF^*)\vartheta_k. \tag{8.5}$$

Again, from (8.3), (6.6), and (7.7), we get

$$\rho g_{ka}\xi^a = \gamma F^*_{ka}\vartheta^a + \beta F_{ka}\vartheta^a + \alpha F_{ka}F^{ab}\vartheta_b + \alpha\{\tfrac{1}{2}(FF)+\lambda^2\}\vartheta_k. \quad (8.6)$$

Substituting from (8.5) and (8.6) in (8.4) and equating to zero the coefficients of the various terms, we obtain

$$-\tfrac{1}{4}\alpha(FF^*)+\lambda\gamma = 0,$$

$$\alpha\lambda^2+\lambda\beta = 0,$$

$$\beta+\lambda\alpha = 0,$$

$$-\tfrac{1}{4}\gamma(FF^*)+\lambda\alpha\{\tfrac{1}{2}(FF)+\lambda^2\} = 0.$$

Here are four equations for the determination of the two ratios $\alpha:\beta:\gamma$. Obviously the middle pair are equivalent, and the whole set is consistent. Taking $\alpha = 1$, we get $\beta = -\lambda$ and $\gamma = \tfrac{1}{4}\lambda^{-1}(FF^*) = -\lambda^*$ by (4.10). Substituting in (8.3), we get finally

$$\rho\xi^a = (E^{ab} - kg^{ab} - \lambda F^{ab} - \lambda^* F^{*ab})\vartheta_b, \quad (8.7)$$

where ϑ_a is any plane not passing through ξ^a, η^a, ξ'^a, or η'^a. The factor ρ on the left-hand side indicates that only the ratios of ξ^a are determinate.

Similarly, since the invariants associated with η^a are k, $-\lambda$, $-\lambda^*$, we have

$$\sigma\eta^a = (E^{ab} - kg^{ab} + \lambda F^{ab} + \lambda^* F^{*ab})\vartheta_b. \quad (8.8)$$

The corresponding formulae for ξ'^a and η'^a are the same with $-k$ for k and λ' for λ.

When $\vartheta_a \equiv \xi_a$ it is easy to show that the right-hand side of (8.7) is equal to $-4k\xi_a$; and if ϑ_a is any one of the three planes ξ_a', η_a, η_a', the right-hand side of (8.7) is zero. Therefore (8.7) remains true in these cases if ρ is chosen to be $-4k$ or zero, and a similar remark is true of (8.8). It follows that (8.7) and (8.8) hold for all planes ϑ_a, so, noting that k, λ, λ^* are given by (7.12), (7.7), and (4.10), we have the theorem:

The real principal null directions of the electromagnetic field are given in terms of the electromagnetic six-vectors and the electromagnetic energy tensor by the equations

$$\rho\xi^a = (E^{ab} - kg^{ab} - \lambda F^{ab} - \lambda^* F^{*ab})\vartheta_b. \quad (8.9)$$

$$\sigma\eta^a = (E^{ab} - kg^{ab} + \lambda F^{ab} + \lambda^* F^{*ab})\vartheta_b, \quad (8.10)$$

where the scalars k, λ, and λ^* are defined by

$$k = -\tfrac{1}{4}\{(FF^*)^2+(FF)^2\}^{\frac{1}{2}}, \tag{8.11}$$

$$\lambda = [\tfrac{1}{4}\{(FF^*)^2+(FF)^2\}^{\frac{1}{2}}-\tfrac{1}{4}(FF)]^{\frac{1}{2}}, \tag{8.12}$$

$$\lambda^* = -\tfrac{1}{4}(FF^*)/\lambda, \tag{8.13}$$

and ϑ_a is an arbitrary covariant vector, not necessarily null or real. But for certain choices of ϑ_a, the right-hand sides of (8.9) *and* (8.10) *may vanish.*

9. *Special cases.*

A consideration of the five-dimensional representation referred to in § 4 makes it evident that the following is an exhaustive list of the possible special cases.

I. The linear complex F_{ab} may be special with its axis meeting the quadric g_{ab} in two distinct points. By (3.3), we have in this case,

$$(FF^*) = 0, \tag{9.1}$$

but

$$(FF) \neq 0. \tag{9.2}$$

The components of F_{ab} are now the Plücker coordinates of a line, namely the axis of the special complex, and the linear complex F^*_{ab} polar to F_{ab} with respect to the quadric is also special. In fact, the lines having the Plücker coordinates F_{ab}, F^*_{ab} are the diagonals of the skew quadrilateral $AA'BB'$ formed by those generators of the quadric which belong to the special complexes. Suppose that the former line meets the quadric in the points ξ^a, η^a, and that the latter meets it in ξ'^a, η'^a. Then it is easy to show that ξ, η are real if $(FF) > 0$ and complex if $(FF) < 0$, and that ξ', η' are complex or real according as ξ, η are real or complex.

The analysis in this special case follows much the same lines as in the general case, but with (FF^*) everywhere zero. It is however necessary to observe that, since ξ, η are points on the line F_{ab}, we now have

$$F_{ab}\,\xi^b = 0 = F_{ab}\,\eta^b. \tag{9.3}$$

But [*cf.* (4.7)]

$$(F_{ab}+\lambda' g_{ab})\,\xi'^b = 0 = (F_{ab}-\lambda' g_{ab})\,\eta'^b, \tag{9.4}$$

where $\lambda' \neq 0$. There is therefore only one distinct non-zero principal invariant associated with F_{ab}, namely λ'. Further, it follows from (9.3) that

$$F_{ab}(p\xi^b+q\eta^b) = 0$$

for all values of p and q, so that $p\xi^a+q\eta^a$, which represents geometrically a point on the line AB, defines a principal vector of F_{ab} for all values of p, q. Hence there are an infinite number of non-null principal directions associated with F_{ab}, and similarly with F^*_{ab}.

Physically, the condition (9.1) means that the electric and magnetic forces in the local galileian space of any observer are perpendicular, while (9.2) means that they are not equal in magnitude†.

We omit from consideration the case when $(FF)=0$ but $(FF^*)\neq 0$, since this provides no new points of particular interest either geometrically or physically.

II. We now suppose that the complex F_{ab} is non-special and has in common with the quadric g_{ab} two distinct generators of one system and two *coincident* generators of the other system. It is easy to show that the condition for this is

$$(FF^*)^2+(FF)^2=0,$$

which, by (7.12), means that the invariant k associated with the energy tensor is zero. Geometrically this is a distinct case, but, since the tensor F_{ab} is real, it is physically impossible unless (FF^*) and (FF) are both zero. This brings us to the third special case.

III. In this case $$(FF^*)=0 \tag{9.5}$$

and $$(FF)=0. \tag{9.6}$$

The former of these equations means that the F_{ab} are the coordinates of a line, and the latter that the line touches the quadric g_{ab} at a point A of coordinates ξ^a, say. The line F^*_{ab} polar to F_{ab} with respect to the quadric also touches the quadric at A. This is the special case discussed in detail by Synge, and called by him the *null case*. Physically, (9.5) and (9.6) mean that, in the local galileian space of any observer, the electric and magnetic forces are perpendicular to one another and equal in magnitude.

The generators of the quadric through A are to be regarded as coincident opposite sides of the skew quadrilateral which appeared in the general case. The four vertices of the quadrilateral now coincide in A, so that there is only one null direction, namely that corresponding to ξ^a. Moreover, since A is on the lines F_{ab}, F^*_{ab}, we have

$$F_{ab}\,\xi^b=0=F^*_{ab}\,\xi^b,$$

† For the connection of the present special case with Whittaker's theory of calamoids, see Synge, *loc. cit.*, § 10.

showing that the principal invariants defined by the six-vectors are all zero. We have already seen that the invariant k defined by the energy tensor is zero.

If ζ^a is any point on the line F_{ab} other than ξ^a, we have

$$F_{ab}\,\zeta^b = 0,$$

and ζ^a therefore defines a principal non-null direction of F_{ab}. Moreover, since the point ζ^a lies on the tangent-plane ξ_a to the quadric, we have

$$\xi_a\,\zeta^a = 0. \tag{9.7}$$

Now F^{*ab} are the dual coordinates of the line F_{ab}, which joins the points ξ^a, ζ^a; consequently

$$\rho F^{*ab} = \xi^a\,\zeta^b - \zeta^a\,\xi^b, \tag{9.8}$$

where ρ is a scalar. Hence

$$\begin{aligned}\rho F_{ab} &= \tfrac{1}{2}\rho\epsilon_{abcd}\,F^{*cd}\\ &= \epsilon_{abcd}\,\xi^c\,\zeta^d.\end{aligned} \tag{9.9}$$

This expresses the electromagnetic six-vector in terms of the principal null vector ξ^a and any other principal vector ζ^a.

From (6.6) and (9.6),

$$\begin{aligned}\rho^2 E_{ab} &= \rho^2 F^*_{ac} g^{cd} F^*_{db}\\ &= -(\zeta\zeta)\,\xi_a\,\xi_b\end{aligned}$$

by (4.1), (9.8), and (9.7). The last equation may be written

$$E_{ab} = \mu\xi_a\,\xi_b, \tag{9.10}$$

where μ is a scalar. Geometrically, this means that the quadric E_{ab} consists of the tangent-plane ξ_a taken twice, and physically, it gives a particularly simple expression for the electromagnetic energy tensor in terms of a single null vector ξ_a. This result was apparently unnoticed by Synge.

IV. The last case to consider is that in which one whole system of generators of the quadric g_{ab} belongs to the complex F_{ab}, together with two distinct or coincident generators of the other system. This happens, for example, when F_{ab} is a special complex whose axis is a generator of the quadric. Now if all generators of one system belong to the complex

F_{ab}, the polar plane with respect to the focal system F_{ab} of any point on the quadric passes through one of the generators which intersect in that point, and so touches the quadric. Consequently the quadric Q_{ab} polar to g_{ab} with respect to the null system is the quadric g_{ab} itself. Hence, by (6.2), there exists a scalar m such that

$$F_{ac}g^{cd}F_{db} = mg_{ab}.$$

Raising the suffix a and contracting with b, we at once get $m = -\frac{1}{4}(FF)$, and hence, by (6.6),

$$E_{ab} = 0.$$

In this case, therefore, the electromagnetic energy tensor is zero, though the electromagnetic six-vector F_{ab} is not zero by hypothesis. This case is therefore physically impossible.

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ON FLEXURE OF BEAMS OF TRIANGULAR CROSS-SECTION

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In a recent paper* I have discussed the flexure of a beam whose cross-section is a right-angled isosceles triangle. I now propose to discuss the flexure solutions for the following three cases:

(i) For a load acting through G, the centre of gravity of the section, perpendicular to the axis of symmetry of an isosceles triangle.

(ii) For a load acting through G parallel to the hypotenuse of a right-angled triangle.

(iii) For any triangular cross-section and for any direction of load through G if σ, the Poisson's ratio, is equal to $\frac{1}{2}$.

We shall assume that the value of the torsion function ϕ is known for the types of cross-section under consideration†.

Let $(\bar{x}, \bar{y})$ be the coordinates of G. Then, for loads W_x and $W_x P_{xy}/I_y$ acting through G parallel to the axes of x and y respectively, we can write the three non-vanishing stress-components‡ of Saint-Venant's classical

* Seth, *Proc. London Math. Soc.* (2), 37 (1934), 502–511.

† Besides the two known cases when the cross-sections are (i) an equilateral triangle, and (ii) an isosceles right-angled triangle, I have discussed two new cases when they are (i) a right-angled triangle containing an angle of 60°, and (ii) an isosceles triangle containing an angle of 120°, in *Quart. J. of Math.*, 5 (1934), 161–171.

‡ These can be adapted from those given in Love's *Treatise on the mathematical theory of elasticity* (1927), 330–334.

solution as

$$\widehat{xz} = \mu\tau_1\left(\frac{\partial\phi}{\partial x} - y\right) + \frac{\mu W_x}{EI_y}\left[\frac{\partial\chi'}{\partial x} - \tfrac{1}{2}\sigma(x-\bar{x})^2 - (1-\tfrac{1}{2}\sigma)(y-\bar{y})^2\right], \quad (1.1)$$

$$\widehat{yz} = \mu\tau_1\left(\frac{\partial\phi}{\partial y} + x\right) + \frac{\mu W_x}{EI_y}\left[\frac{\partial\chi'}{\partial y} - (2+\sigma)(x-\bar{x})(y-\bar{y})\right], \quad (1.2)$$

$$\widehat{zz} = -\frac{W_x}{I_y}(x-\bar{x})(l-z), \quad (1.3)$$

where, in the usual notation, $\chi'(x, y)$ is the flexure function, I_y is the moment of inertia of the cross-section about an axis through G parallel to the y-axis, P_{xy} is the product of inertia, τ_1 is a constant to be determined from the condition that the torsional couple exerted on the beam is zero, and l is the length of the beam.

The boundary condition

$$\widehat{xz}\cos(x\nu) + \widehat{yz}\cos(y\nu) = 0 \quad (2)$$

gives

$$\left[\frac{\partial\chi'}{\partial x} - \tfrac{1}{2}\sigma(x-\bar{x})^2 - (1-\tfrac{1}{2}\sigma)(y-\bar{y})^2\right]\cos(x\nu) + \left[\frac{\partial\chi'}{\partial y} - (2+\sigma)(x-\bar{x})(y-\bar{y})\right]\cos(y\nu) = 0; \quad (3)$$

and, if we put

$$\chi' = \chi_1 + (H_1+\sigma\bar{y})\phi + x\{\tfrac{1}{2}\sigma\bar{x}^2 + (1-\tfrac{1}{2}\sigma)\bar{y}^2\} + (2+\sigma)\bar{x}\bar{y}y - 2\bar{y}xy - \tfrac{1}{2}\sigma\bar{x}(x^2-y^2) - \tfrac{1}{3}(1-\tfrac{1}{2}\sigma)(x^3-3xy^2), \quad (4)$$

this becomes

$$\left[\frac{\partial\chi_1}{\partial x} + H_1 y - x^2\right]\cos(x\nu) + \left[\frac{\partial\chi_1}{\partial y} - H_1 x - 2\sigma xy + 2(1+\sigma)\bar{x}y\right]\cos(y\nu) = 0, \quad (5)$$

the boundary condition satisfied by ϕ being

$$\left(\frac{\partial\phi}{\partial x} - y\right)\cos(x\nu) + \left(\frac{\partial\phi}{\partial y} + x\right)\cos(y\nu) = 0, \quad (6)$$

ν denoting the direction of the normal drawn to the boundary, and H_1 an undetermined constant.

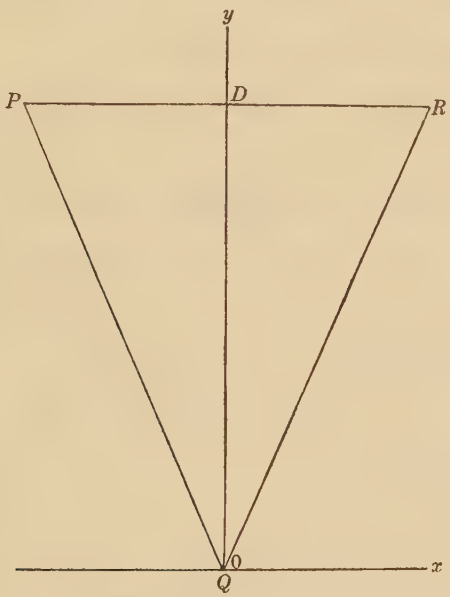
In like manner we have for loads $W_y P_{xy}/I_x$ and W_y, acting through G parallel to the axes of x and y respectively,

$$\chi'' = \chi_2 + (H_2-\sigma\bar{x})\phi + \{(1-\tfrac{1}{2}\sigma)\bar{x}^2 + \tfrac{1}{2}\sigma\bar{y}^2\}y + (2+\sigma)x\bar{x}\bar{y} - 2\bar{x}xy + \tfrac{1}{2}\sigma\bar{y}(x^2-y^2) + \tfrac{1}{3}(1-\tfrac{1}{2}\sigma)(3x^2y-y^3), \quad (7)$$

where χ_2 satisfies the boundary condition

$$\left[\frac{\partial\chi_2}{\partial x}-2\sigma xy+2(1+\sigma)x\bar{y}+H_2 y\right]\cos(x\nu)+\left[\frac{\partial\chi_2}{\partial y}-y^2-H_2 x\right]\cos(y\nu)=0, \tag{8}$$

I_x being the moment of inertia of the cross-section about an axis through G parallel to the x-axis, and H_2 being an undetermined constant. The



$\angle DQR=\beta$, $0D=d$.

Fig. 1.

corresponding values of the three non-vanishing stress-components $\widehat{xz}$, $\widehat{yz}$, and $\widehat{zz}$ can be written down as in (1).

(i) In this case we shall show that

$$\chi+i\psi=A_1 z+A_2 z^2+A_3 z^3+A_4(\phi+i\Psi), \tag{9}$$

where ψ and Ψ are functions conjugate to χ and ϕ respectively, the A's are all constant, and z stands for $x+iy$.

We take χ_1, defined in (4), as

$$\chi_1=A_2 xy+A_3(x^3-3xy^2). \tag{10}$$

The boundary condition (5) gives

$$[(A_2+H_1)y+3A_3(x^2-y^2)-x^2]\cos(xv)$$
$$+[(A_2-H_1)x-6A_3xy-2\sigma xy]\cos(yv)=0. \quad (11)$$

Over the base $y=d$ (see Fig. 1) we have, from (11),

$$A_2-H_1-6A_3d=2\sigma d,$$

and over the sides $x=\pm y\tan\beta$

$$[(A_2+H_1)y+3A_3(\tan^2\beta-1)y^2-y^2\tan^2\beta]$$
$$-y\tan^2\beta[A_2-H_1-6A_3y-2\sigma y]=0,$$

which gives

$$A_2(1-\tan^2\beta)+H_1(1+\tan^2\beta)=0,$$

$$3A_3(3\tan^2\beta-1)=(1-2\sigma)\tan^2\beta.$$

If $\beta=\frac{1}{4}\pi$, $H_1=0$, $A_3=\frac{1}{6}(1-2\sigma)$, $A_2=d$. Hence, for a right-angled isosceles triangle, we have

$$\chi_1=dxy+\tfrac{1}{6}(1-2\sigma)(x^3-3xy^2), \quad (12)$$

and, using (4), we get

$$\chi=\tfrac{2}{9}(2-\sigma)d^2x-\tfrac{1}{3}dxy-\tfrac{1}{6}(1+\sigma)(x^3-3xy^2)+\tfrac{2}{3}d\sigma\phi.$$

If $\beta=\frac{1}{6}\pi$, the triangle is equilateral. H_1 can now be absorbed in A_2 and A_3 [being the coefficient of ϕ in (4)], and we get

$$\chi_1=-\tfrac{1}{6}(x^3-3xy^2), \quad (13)$$

which holds good *only* when $\sigma=\frac{1}{2}$. For all other cases we have

$$\frac{A_2}{1+\tan^2\beta}=-\frac{H_1}{1-\tan^2\beta}=\frac{d\{(1+\sigma)\tan^2\beta-\sigma\}}{3\tan^2\beta-1},$$

$$A_3=\tfrac{1}{3}(1-2\sigma)\frac{\tan^2\beta}{3\tan^2\beta-1},$$

and hence

$$(3\tan^2\beta-1)\chi_1=\{(1+\sigma)\tan^2\beta-\sigma\}(1+\tan^2\beta)dxy$$
$$+\tfrac{1}{3}(1-2\sigma)\tan^2\beta(x^3-3xy^2). \quad (14)$$

For the particular case of $\beta = \frac{1}{3}\pi$ we have

$$\chi_1 = \tfrac{1}{2}(3+2\sigma)\,dxy + \tfrac{1}{8}(1-2\sigma)(x^3-3xy^2). \tag{15}$$

We get the result given in (9) if we combine (14) with (4).

(ii) If we combine the values of $\widehat{xz}$ and $\widehat{yz}$ given in (1) for loads W_x and $W_x P_{xy}/I_y$ with those for loads $W_y P_{xy}/I_x$ and W_y acting through G parallel to the axes of x and y respectively, since the two solutions are additive, we get

$$\widehat{xz} = \mu\tau\left(\frac{\partial\phi}{\partial x} - y\right) + \frac{\mu W_x}{EI_y}\left[\frac{\partial\chi'}{\partial x} - \tfrac{1}{2}\sigma(x-\bar{x})^2 - (1-\tfrac{1}{2}\sigma)(y-\bar{y})^2\right] + \frac{\mu W_y}{EI_x}\left[\frac{\partial\chi''}{\partial x} - (2+\sigma)(x-\bar{x})(y-\bar{y})\right], \tag{15.1}$$

$$\widehat{yz} = \mu\tau\left(\frac{\partial\phi}{\partial y} + x\right) + \frac{\mu W_x}{EI_y}\left[\frac{\partial\chi'}{\partial y} - (2+\sigma)(x-\bar{x})(y-\bar{y})\right] + \frac{\mu W_y}{EI_x}\left[\frac{\partial\chi''}{\partial y} - \tfrac{1}{2}\sigma(y-\bar{y})^2 - (1-\tfrac{1}{2}\sigma)(x-\bar{x})^2\right]. \tag{15.2}$$

Putting $A_1 = \mu W_x/EI_y$, $A_2 = \mu W_y/EI_x$, and $\chi_3 = A_1\chi_1 + A_2\chi_2$, we have

$$\begin{aligned}\chi''' &= A_1\chi' + A_2\chi'' \\ &= \chi_3 + \{A_1(H_1+\sigma\bar{y}) + A_2(H_2-\sigma\bar{x})\}\phi \\ &\quad + A_1[x\{\tfrac{1}{2}\sigma\bar{x}^2 + (1-\tfrac{1}{2}\sigma)\bar{y}^2\} + (2+\sigma)\bar{x}\bar{y}y - 2\bar{y}xy - \tfrac{1}{2}\sigma\bar{x}(x^2-y^2) \\ &\qquad - \tfrac{1}{3}(1-\tfrac{1}{2}\sigma)(x^3-3xy^2)] \\ &\quad + A_2[y\{(1-\tfrac{1}{2}\sigma)\bar{x}^2 + \tfrac{1}{2}\sigma\bar{y}^2\} + (2+\sigma)\bar{x}\bar{y}x - 2\bar{x}xy + \tfrac{1}{2}\sigma\bar{y}(x^2-y^2) \\ &\qquad + \tfrac{1}{3}(1-\tfrac{1}{2}\sigma)(3x^2y-y^3)].\end{aligned} \tag{15.3}$$

The boundary condition (2) now gives

$$\left[\frac{\partial\chi_3}{\partial x} + (A_1H_1+A_2H_2)y - A_1x^2 - 2A_2x\{\sigma y - (1+\sigma)\bar{y}\}\right]\cos(x\nu)$$

$$+\left[\frac{\partial\chi_3}{\partial y} - (A_1H_1+A_2H_2)x - A_2y^2 - 2A_1y\{\sigma x - (1+\sigma)\bar{x}\}\right]\cos(y\nu) = 0. \tag{16}$$

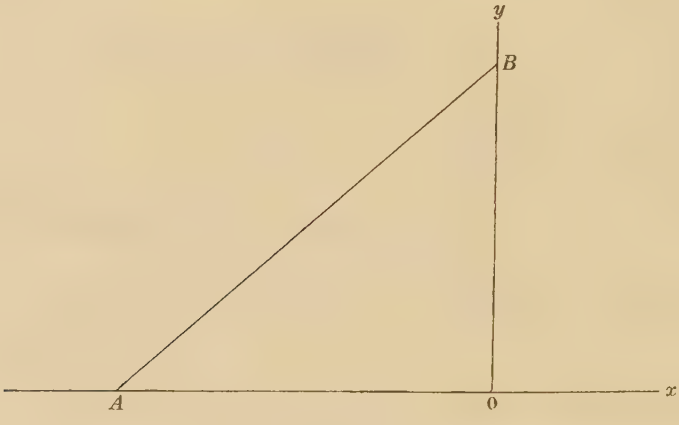
Let us put

$$\chi_3 = \tfrac{1}{2}B_1(x^2-y^2). \tag{17}$$

If we assume that the triangle is bounded by the two axes and the straight line AB (as shown in Fig. 2) whose equation is $y = mx+c$, we easily see that, if $H_1 = H_2 = 0$, (16) is satisfied for the two sides $0A$ and $0B$ for all values of the constant B_1 in χ_3. For the third side AB, (16) gives

$$mx[B_1-A_1x-A_2\{2\sigma(mx+c)-2(1+\sigma)\bar{y}\}]$$
$$+[B_1+A_1\{2\sigma x-2(1+\sigma)\bar{x}\}+A_2(mx+c)](mx+c) = 0.$$

Equating the coefficients of x^2, x, and the constant term to zero, and



tan $\angle BA0 = m$, $0B = c$.

Fig. 2.

assuming that $\sigma \neq \frac{1}{2}$, we get

$$m = A_1/A_2, \quad B_1+(1+\sigma)(A_2\bar{y}-A_1\bar{x})+A_2c = 0,$$

$$B_1-2A_1(1+\sigma)\bar{x}+A_2c = 0.$$

Since $m = -\bar{y}/\bar{x} = A_1/A_2$, these equations are consistent if

$$B_1 = -A_2\{2\bar{y}(1+\sigma)+c\}. \tag{18}$$

If θ is the angle which W, the resultant load, makes with the x-axis we have

$$W\cos\theta = W_x+W_y P_{xy}/I_x,$$

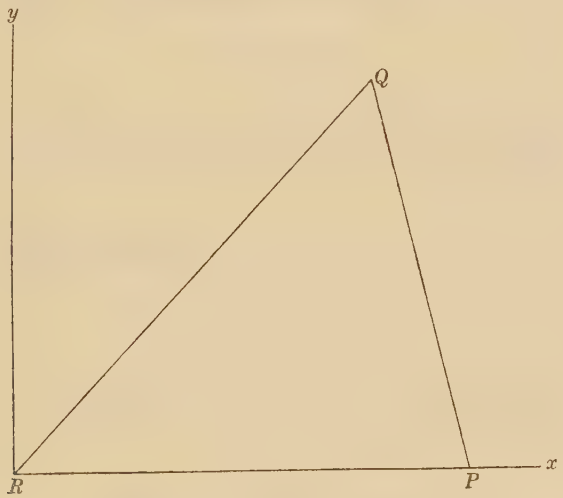
$$W\sin\theta = W_y+W_x P_{xy}/I_y.$$

Also $I_y = \frac{1}{36}c^4/m^3$, $I_x = \frac{1}{36}c^4/m$, $P_{xy} = \frac{1}{72}c^4/m^2$. Hence θ is given by

$$\tan\theta = \frac{I_x + m P_{xy}}{P_{xy} + m I_y} = m,$$

which shows that (17) and (18) give the flexure solution for a load acting through G parallel to the hypotenuse of a right-angled triangle.

(iii) *The case* $\sigma = \frac{1}{2}$. In this case we can show that $\chi + i\psi$ is of the same form as given in (9) for any direction of load.



$\angle RPQ = \theta_1$, $\angle QRP = \theta_2$, $RP = c$.

Fig. 3.

(*a*) For loads W_x and $W_x P_{xy}/I_y$ acting along the axis of x and y respectively, we can assume χ_1, defined in (4), as

$$\chi_1 = A_2(x^2 - y^2) + 2Bxy + A_3(x^3 + 3xy^2), \tag{19}$$

so that (5) becomes

$$[2A_2 x + y(2B + H_1) + 3A_3(x^2 - y^2) - x^2]\cos(x\nu)$$
$$+ [-2A_2 y + x(2B - H_1) - xy(6A_3 + 1) + 3\bar{x}y]\cos(y\nu) = 0. \tag{20}$$

From Fig. 3 we have, for the side $y = 0$, $\cos(x\nu) = 0$, and (20) now gives $B = \frac{1}{2}H_1$. Again, for the side $y = x\tan\theta_2$, (20) becomes

$$[2A_2 y\cot\theta_2 + 2H_1 y + 3A_3 y^2(\cot^2\theta_2 - 1) - y^2\cot^2\theta_2]$$
$$- \cot\theta_2[-2A_2 y - y^2\cot\theta_2(6A_3 + 1) + 3\bar{x}y] = 0,$$

which gives

$$A_2+\tfrac{1}{2}H_1\tan\theta_2=\tfrac{3}{4}\bar{x}\,;\quad A_3(3-\tan^2\theta_2)=0, \tag{21}$$

A_3 and H_1 being both zero if $\theta_2=\frac{1}{2}\pi$.

In like manner the boundary condition for the side $y=(c-x)\tan\theta_1$ gives

$$\left.\begin{aligned} 2H_1\tan\theta_1-3A_3c\tan^2\theta_1-2A_2&=-3\bar{x},\\ 4A_2-2H_1\tan\theta_1+6A_3c(\tan^2\theta_1-1)&=c+3\bar{x},\\ A_3(3-\tan^2\theta_1)&=0,\end{aligned}\right\} \tag{22}$$

H_1 and A_3 being both zero if $\theta_1=\frac{1}{2}\pi$.

For the relations in (21) and (22) to be consistent, we have, if the triangle is not equilateral,

$$A_3=0,\quad A_2=\tfrac{1}{2}c,\quad H_1=-\tfrac{1}{2}c/(\tan\theta_1+\tan\theta_2),$$

$$3\bar{x}=c+c\tan\theta_1/(\tan\theta_1+\tan\theta_2).$$

That this is the value of $\bar{x}$ can be easily seen from Fig. 3.

For the excepted case of the equilateral triangle, H_1 can be absorbed in A_2, B, and A_3, and we thus have

$$A_3=\tfrac{1}{12},\quad A_2=\tfrac{3}{8}c,\quad B=H_1=0.$$

Hence, excepting this case, we have the general result

$$\chi_1=\tfrac{1}{2}c(x^2-y^2)-\tfrac{1}{2}c\,\frac{xy}{\tan\theta_1+\tan\theta_2}. \tag{23}$$

(*b*) For loads $W_y P_{xy}/I_y$ and W_y acting along the x and y axes respectively, we take χ_2, defined by (7), to be of the same form as χ_1 in (19). Using (8) after putting $\sigma=\frac{1}{2}$, we get $B=\frac{1}{2}H_2$, and, if the triangle is not equilateral, proceeding as above, we find that

$$A_3=0,\quad A_2=-\tfrac{3}{2}\bar{y},\quad H_2\tan\theta_1=\tfrac{1}{2}c\tan\theta_1-\tfrac{3}{2}\bar{y},$$

and

$$3\bar{y}=c\tan\theta_1\tan\theta_2/(\tan\theta_1+\tan\theta_2),$$

which is also the value of $\bar{y}$ that we get from Fig. 3.

For the excepted case of the equilateral triangle we find

$$A_3 = -\tfrac{1}{12}\sqrt{3}, \quad A_2 = -\tfrac{1}{8}c\sqrt{3}, \quad B = H_2 = 0;$$

otherwise we have the general result

$$\chi_2 = \tfrac{1}{2}\frac{c\tan\theta_1}{\tan\theta_1+\tan\theta_2}[xy-\tan\theta_2(x^2-y^2)]. \tag{24}$$

Combining the results in (a) and (b), we can get the flexure function for any direction of load.

Thus, in all the three cases (i), (ii), and (iii), we see that $\chi+i\psi$ is of the form given by

$$\chi+i\psi = A_1 z+A_2 z^2+A_3 z^3+A_4(\phi+i\Psi),$$

where A_1, A_2, A_3 are, in general, complex constants, and A_4 is always real.

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INFINITESIMAL DEFORMATIONS OF AN L_m IN AN L_n

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In this note the conditions for parallel deformation of a sub-space V_m of a V_n, which were obtained in an earlier paper*, are given in a form which is at the same time much more tractable and geometrically more significant. Moreover, since the metric does not enter into this part of the deformation-problem, we give the results for the more general case of an L_m in an L_n (L_n denoting an affinely-connected manifold).

1. The L_m is defined by equations

$$x^\alpha = x^\alpha(y^i)$$

$$(\alpha, \beta, \gamma, \ldots = 1, 2, \ldots, n;\ i, j, k, \ldots = 1, 2, \ldots, m),$$

and we shall write $$B_i{}^\alpha \equiv \frac{\partial x^\alpha}{\partial y^i} = \partial_i x^\alpha$$

in place of the $\theta_i{}^\alpha$ of (**2**).

If $\underset{i}{e}{}^k$, $\overset{k}{e}_i$ are the measuring vectors in L_m belonging to the coordinate system y^i, there are m vectors $\underset{i}{B}{}^\alpha \equiv \underset{i}{e}{}^k B_k{}^\alpha$, and the "pseudo-normal X_m^{n-m}" is defined by $n-m$ independent vectors

$$\underset{p}{C}{}^\alpha \quad (p, q, r, \ldots = m+1, m+2, \ldots, n)$$

at each point of L_m, the vectors $\underset{p}{C}{}^\alpha$ being linearly independent of the $\underset{i}{B}{}^\alpha$.

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* Hayden (2), § 7.

The reciprocal systems $\overset{i}{B}_{\alpha}$, $\overset{p}{C}_{\alpha}$ are defined by the equations

$$\overset{i}{B}_{\alpha}\, B^{\alpha}_{k} = \delta_k^{\,i}, \quad \overset{i}{B}_{\alpha}\, C^{\alpha}_{p} = 0, \quad B^{\alpha}_{i}\, \overset{p}{C}_{\alpha} = 0, \quad \overset{p}{C}_{\alpha}\, C^{\alpha}_{q} = \delta_q^{\,p};$$

or by the equivalent set

$$\overset{i}{B}_{\alpha}\, B^{\beta}_{i} + \overset{p}{C}_{\alpha}\, C^{\beta}_{p} = \delta_{\alpha}^{\,\beta}.$$

If e^{q}_{p}, $\overset{q}{e}_{p}$ are the measuring vectors belonging to the system C^{α}_{p}, we have the tensors

$$B_{\alpha}^{\,i} \equiv \overset{k}{B}_{\alpha}\, e^{i}_{k}, \quad C_{p}^{\,\alpha} \equiv C^{\alpha}_{q}\, \overset{q}{e}_{p}, \quad C_{\alpha}^{\,p} \equiv \overset{q}{C}_{\alpha}\, e^{p}_{q}.$$

The connections induced in L_m and X_m^{n-m} are given* by the parameters

$$\Gamma^{i}_{kj} = B_{\alpha}^{\,i}\, \partial_k\, B_j^{\,\alpha} + B^{i\mu\lambda}_{\alpha kj}\, \Gamma^{\alpha}_{\mu\lambda}, \tag{1.1}$$

$$\Gamma^{p}_{kq} = C_{\alpha}^{\,p}\, \partial_k\, C_q^{\,\alpha} + B_k^{\,\mu}\, C^{p\lambda}_{\alpha q}\, \Gamma^{\alpha}_{\mu\lambda}, \tag{1.2}$$

and the first and second Eulerian curvature tensors of L_m in L_n are

$$\Omega_{kj}^{\cdot\cdot\alpha} \equiv D_k\, B_j^{\,\alpha} \quad \text{and} \quad \Phi_{kq}^{\cdot\cdot\alpha} \equiv D_k\, C_q^{\,\alpha}, \tag{1.3}$$

where D_k denotes Lagrange differentiation

(*e.g.* $$\Phi_{kq}^{\cdot\cdot\alpha} = \partial_k\, C_q^{\,\alpha} + B_k^{\,\mu}\, \Gamma^{\alpha}_{\mu\lambda}\, C_q^{\,\lambda} - \Gamma^{p}_{kq}\, C_p^{\,\alpha}).$$

From (1.1) and (1.2),

$$B_{\alpha}^{\,i}\, \Omega_{kj}^{\cdot\cdot\alpha} = 0, \quad C_{\alpha}^{\,p}\, \Phi_{kq}^{\cdot\cdot\alpha} = 0,$$

whence $$\Omega_{kj}^{\cdot\cdot\alpha} = \Omega_{kj}^{\cdot\cdot p}\, C_p^{\,\alpha}, \quad \Phi_{kq}^{\cdot\cdot\alpha} = \Phi_{kq}^{\cdot\cdot i}\, B_i^{\,\alpha}, \tag{1.4}$$

where† $$\Omega_{kj}^{\cdot\cdot p} = C_{\alpha}^{\,p}\, \Omega_{kj}^{\cdot\cdot\alpha}, \quad \Phi_{kq}^{\cdot\cdot i} = B_{\alpha}^{\,i}\, \Phi_{kq}^{\cdot\cdot\alpha}.$$

2. When L_m is subjected to an infinitesimal deformation

$$\delta x^{\alpha} = \epsilon v^{\alpha},$$

the first variation of $B_i^{\,\alpha}$ is‡

$$\delta B_i^{\,\alpha} = \epsilon (D_i v^{\alpha} - 2 S_{\lambda\mu}^{\cdot\cdot\alpha}\, B_i^{\,\lambda} v^{\mu}), \tag{2.1}$$

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* Bortolotti (1), Schouten and van Kampen (3), Struik (5), 49. We follow Schouten and Struik [*vide* (4), 75, foot-note (2)] in writing $\Gamma^{\alpha}_{\mu\lambda}$ for what is ordinarily written as $\Gamma^{\alpha}_{\lambda\mu}$.

† $\Omega_{kj}^{\cdot\cdot p}$ and $\Phi_{kq}^{\cdot\cdot i}$ are often written as $H_{kj}^{\cdot\cdot p}$ and $-L_k^{\cdot\cdot i}{}_q$ respectively.

‡ (2), equation (4). The change in sign before $S_{\lambda\mu}^{\cdot\cdot\alpha}$ is a consequence of our alteration of the order of the suffixes in the connection-parameters.

and necessary and sufficient conditions for a parallel deformation* are that

$$C_\alpha^{\ p}\,\delta B_i^{\ \alpha} = 0. \tag{2.2}$$

If u^α and w^α are the projections of v^α on L_m and X_m^{n-m} respectively, and u^i, w^p are the L_m- and X_m^{n-m}-components of those vectors,

$$v^\alpha = B_i^{\ \alpha}\,u^i + C_p^{\ \alpha}\,w^p;$$

whence $$D_i v^\alpha = B_k^{\ \alpha}\,D_i u^k + \Omega_{ik}^{\ \ \alpha}\,u^k + C_p^{\ \alpha}\,D_i w^p + \Phi_{ip}^{\ \ \alpha}\,w^p,$$

and (2.2) becomes

$$D_i w^p + \Omega_{ik}^{\ \ p}\,u^k - 2C_\alpha^{\ p}\,B_i^{\ \lambda}\,S_{\lambda\mu}^{\ \ \alpha}\,v^\mu = 0,$$

that is $$D_i w^p + (\Omega_{ik}^{\ \ p} - 2S_{ik}^{\ \ p})\,u^k - 2S_{iq}^{\ \ p}\,w^q = 0, \tag{2.3}$$

where $$S_{ik}^{\ \ p} \equiv C_\alpha^{\ p}\,B_{ik}^{\lambda\mu}\,S_{\lambda\mu}^{\ \ \alpha}, \quad S_{iq}^{\ \ p} \equiv C_{\alpha q}^{p\mu}\,B_i^{\ \lambda}\,S_{\lambda\mu}^{\ \ \alpha}.$$

Also $$\Omega_{[ik]}^{\ \ \alpha} = B_{ik}^{\lambda\mu}\,S_{\lambda\mu}^{\ \ \alpha} - B_j^{\ \alpha}\,S_{ik}^{\ \ j},$$

so that $$\Omega_{[ik]}^{\ \ p} = S_{ik}^{\ \ p},$$

and our equation becomes

$$\boxed{D_i w^p + \Omega_{ki}^{\ \ p}\,u^k - 2S_{iq}^{\ \ p}\,w^q = 0}. \tag{2.4}$$

3. Tangential parallel deformations do not exist unless the matrix

$$\left\| \begin{matrix} \Omega_{11}^1 & \Omega_{12}^1 & \dots & \Omega_{1m}^1 & \Omega_{11}^2 & \Omega_{12}^2 & \dots & \Omega_{1m}^2 & \dots & \Omega_{1m}^{n-m} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots \\ \Omega_{m1}^1 & \Omega_{m2}^1 & \dots & \Omega_{mm}^1 & \Omega_{m1}^2 & \Omega_{m2}^2 & \dots & \Omega_{mm}^2 & \dots & \Omega_{mm}^{n-m} \end{matrix} \right\|$$

is of rank less than m. If such a deformation does exist, the displacement at each point is along one of the asymptotic directions of highest order.

The equations for a pseudo-normal parallel deformation are

$$D_i w^k = 2S_{iq}^{\ \ p}\,w^q. \tag{3.1}$$

The integrability conditions are

$$D_{[i}\,D_{j]}\,w^p = \tfrac{1}{2} R_{ijq}^{\ \ \ p}\,w^q - S_{ij}^{\ \ k}\,D_k\,w^p, \tag{3.2}$$

* (2), equation (16).

where $R_{ijq}^{\cdot\cdot\cdot p}$ is the curvature tensor of the connection Γ_{iq}^{p} (the "intrinsic normal curvature tensor of L_m"), viz.*

$$R_{ijq}^{\cdot\cdot\cdot p} = 2\partial_{[i}\Gamma_{j]q}^{p} + 2\Gamma_{[i|r|}^{p}\Gamma_{j]q}^{r},$$

and (3.2) reduces to

$$(R_{ijq}^{\cdot\cdot\cdot p} - 4S_{ij}^{\cdot\cdot k}S_{kq}^{\cdot\cdot p} - 8S_{[j|r|}^{\cdot\cdot p}S_{i]q}^{\cdot\cdot r} - 4D_{[i}S_{j]q}^{\cdot\cdot p})w^q = 0.$$

The conditions for complete integrability are

$$\boxed{R_{ijq}^{\cdot\cdot\cdot p} - 4S_{ij}^{\cdot\cdot k}S_{kq}^{\cdot\cdot p} - 8S_{[j|r|}^{\cdot\cdot p}S_{i]q}^{\cdot\cdot r} - 4D_{[i}S_{j]q}^{\cdot\cdot p} = 0}. \qquad (3.3)$$

If $S_{iq}^{\cdot\cdot p} = 0$ (in particular if the connection of L_n is symmetric) the equations for a pseudo-normal parallel deformation are simply

$$D_i w^p = 0,$$

and L_m admits of a pseudo-normal parallel deformation arbitrarily assigned at any one point if, and only if,

$$R_{ijq}^{\cdot\cdot\cdot p} = 0.$$

If such a deformation exists, the displacement vectors v are equipollent with respect to L_m (though not with respect to L_n).

4. For $m = 1$, equations (2.4) are, of course, always integrable.

For a curve in a Riemannian n-space the equations are

$$\frac{D}{Ds}w^p + \underset{1}{\kappa}\,\underset{2}{\xi^p}\,u = 0, \qquad (4.1)$$

where $\underset{1}{\kappa}$ is the first curvature, $\underset{2}{\xi^a}$ is the first principal normal, and u is the length of the projection of the vector v^a on the tangent to the curve.

If we choose the vectors $\underset{p}{C^a}$ to be the principal normals $\underset{p}{\xi^a}$ for the curve, then $\underset{2}{\xi^p} = \delta_2^p$, and (4.1) become

$$\left.\begin{aligned} &\frac{D}{Ds}w^2 + \underset{1}{\kappa}u = 0, \\ &\frac{D}{Ds}w^p = 0 \qquad (p = 3, 4, \ldots, n). \end{aligned}\right\} \qquad (4.2)$$

* We again follow Schouten and Struik (4), 110, in adopting a change of sign for the curvature tensor.

The connection-parameters $\Gamma_q^{\ p}$ $(\equiv \Gamma_{1q}^{p})$ of V_1^{n-1} for this choice of axes are

$$\Gamma_q^{\ p} = \underset{q}{\kappa}\,\delta_{q+1}^{p} - \underset{q-1}{\kappa}\,\delta_{q-1}^{p} \quad \left(\underset{0}{\kappa} = \underset{n}{\kappa} = 0\right),$$

where $\underset{1}{\kappa}, \underset{2}{\kappa}, \ldots, \underset{n-1}{\kappa}$ are the curvatures of the curve; thus

$$\Gamma_q^{\ p} = 0 \quad \text{unless} \quad |p-q| = 1,$$

$$\Gamma_p^{p+1} = -\Gamma_{p+1}^{p} = \underset{p}{\kappa}.$$

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THE NET OF QUADRIC SURFACES ASSOCIATED WITH A PAIR OF MÖBIUS TETRADS.

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Introduction.

1. Suppose that there are two sets of four points

$$A, \quad B, \quad C, \quad D$$

and

$$A', \quad B', \quad C', \quad D'$$

in three-dimensional space, the letters being arranged thus in an array of two rows and four columns. If any one of the points is chosen, there remain three points in the same row and one point in the same column, these constituting together a set of four points; for example, if C' is chosen the three points D', A', B' are in the same row and C in the same column, and hence there arises the set $D'A'B'C$ of four points. The eight points are presumed to have the property that *each of the eight sets of four points so arising is a set of four coplanar points.* The four points A, B, C, D are then the vertices of a tetrahedron T and the four points A', B', C', D' are the vertices of a tetrahedron T'; the two tetrahedra are such that any face of either of them contains a vertex of the other. Each tetrahedron is both inscribed and circumscribed to the other.

The question whether it is possible for two tetrahedra T and T' to be related in this way was first asked, and answered, by Möbius*, and the two sets of four vertices of the tetrahedra are therefore spoken of as a pair of Möbius tetrads. Let T be taken as tetrahedron of reference for a system of homogeneous coordinates. Then, if we take the four points A', B', C', D' whose coordinates are given by the rows of the matrix

$$\begin{array}{c} A' \\ B' \\ C' \\ D' \end{array} \begin{bmatrix} 0 & -n & m & -l \\ n & 0 & -l & -m \\ -m & l & 0 & -n \\ l & m & n & 0 \end{bmatrix}, \tag{1.0}$$

* *Journal für Math.*, 3 (1828), 273.

where l, m, n, are three arbitrary numbers, it is easily seen that $ABCD$ and $A'B'C'D'$ form a pair of Möbius tetrads. The equations of the faces of T' are

$$\left.\begin{aligned} x' &\equiv \quad -ny+mz-lt=0 \quad \text{for} \quad B'C'D', \\ y' &\equiv nx \quad -lz-mt=0 \quad \text{for} \quad C'D'A', \\ z' &\equiv -mx+ly \quad -nt=0 \quad \text{for} \quad D'A'B', \\ t' &\equiv lx+my+nz \quad =0 \quad \text{for} \quad A'B'C'. \end{aligned}\right\} \qquad (1.1)$$

Since the positions of the vertices of T' depend only on the mutual ratios of l, m, n, we may suppose that $l^2+m^2+n^2=1$. Then x, y, z, t are exactly the same linear functions of x', y', z', t' as x', y', z', t' are of x, y, z, t; (1.1) is still true if the dashed and undashed letters are interchanged.

The four lines $x=x'=0$, $y=y'=0$, $z=z'=0$, $t=t'=0$ we call, respectively, a, b, c, d; these four lines have two transversals, which we call e and f.

The set of coordinates of the vertices of T' given by (1.0) is obtained by Baker*, and will be the basis of our algebra. It is indeed equivalent to the set of barycentric coordinates given by Möbius; the matrix which gives the coordinates of the four points of one of a pair of Möbius tetrads, referred to the other, can always† be taken in the form (1.0).

2. The eight points and planes of the Möbius configuration occur in several well-known figures in projective geometry; they are obtained, for example, by the repeated polarization of a point or a plane with respect to four mutually apolar linear complexes; they also occur in connection with the quartic surface known as the Complex Surface of Plücker. This

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* *Principles of geometry*, 3 (Cambridge, 1923), 138.

† It was shown by W. F. Meyer that the matrix may always be assumed to be skew; see *Math. Zeitschrift*, 2 (1918), 453. But it is possible, by using a linear transformation of coordinates which leaves the tetrahedron of reference unchanged, so to simplify Meyer's matrix that it reduces to (1.0). Take

$$\alpha_1^2 = c_{23}c_{34}c_{42}, \quad \alpha_2^2 = c_{34}c_{41}c_{13}, \quad \alpha_3^2 = c_{41}c_{12}c_{24}, \quad \alpha_4^2 = c_{12}c_{23}c_{31}.$$

Then, if the rows of the skew matrix

$$\begin{bmatrix} 0 & c_{12} & c_{13} & c_{14} \\ -c_{12} & 0 & c_{23} & c_{24} \\ -c_{13} & -c_{23} & 0 & c_{34} \\ -c_{14} & -c_{24} & -c_{34} & 0 \end{bmatrix}$$

are multiplied, respectively, by α_1, $i\alpha_2$, α_3, $i\alpha_4$, and similarly for the columns, the resulting matrix is the product of $(c_{23}c_{31}c_{12}c_{14}c_{24}c_{34})^{\frac{1}{4}}$ and the matrix (1.0), provided that

$$l=-i(c_{23}c_{14})^{\frac{1}{2}}, \quad m=(c_{31}c_{24})^{\frac{1}{2}}, \quad n=-i(c_{12}c_{34})^{\frac{1}{2}}.$$

Here c_{ij} and c_{ji} are the same symbol.

paper, however, is concerned principally with a net of quadric surfaces that is associated with the configuration; the net consists of those quadrics which circumscribe the tetrahedra T and T'. There seems to be practically no work published concerning this net of quadrics, and Staude, in his article in the *Encyklopädie*, does not even allude to it in the section where*, if anywhere, he would have done so.

We may indicate here the main topics with which the paper is concerned. It is divided into four sections. In I it is shown, after giving a simple algebraic confirmation of some known results, that when the quadrics of the net are represented, in Hesse's manner, by the points of a plane, the quartic curve that represents the cones of the net breaks up into two conics. The envelope of those cones represented by the points of either conic is found to be a *Plücker surface*†, and this seems to be a more direct way of obtaining the surface than the usual method of obtaining it from a quadratic complex. A family of Kummer surfaces is also found, each Kummer surface being the envelope of a set of quadrics belonging to the net.

In II the net of quadrics is obtained as a section of a general net of quadrics in [4] by a solid having a certain special relation to this net of quadrics, and a canonical form is incidentally found for the net of quadric surfaces associated with a Möbius configuration.

III is concerned with those four quadrics and four linear complexes which reciprocate the configuration into itself; it is found, among other results, that the harmonic inversion whose base lines are e and f is a product of four polarisations, and leaves every quadric of the net unaltered.

In IV we obtain some results concerning the net of quadrics with a base line.

I.

3. Let the intersections of e with a, b, c, d be E_1, E_2, E_3, E_4; their coordinates are given by the rows of the skew matrix, of rank 2,

$$\begin{array}{c} E_1 \\ E_2 \\ E_3 \\ E_4 \end{array} \begin{bmatrix} 0 & -nl+mp & lm+np & m^2+n^2 \\ nl-mp & 0 & m^2+n^2 & -lm-np \\ -lm-np & -m^2-n^2 & 0 & -nl+mp \\ -m^2-n^2 & lm+np & nl-mp & 0 \end{bmatrix}, \qquad (3.1)$$

where $$l^2+m^2+n^2+p^2=0.$$

* *Encyklopädie der Math. Wissenschaften*, III, C. 2, § 133.

† This surface is studied at length by Plücker in his *Neue Geometrie des Raumes* (Leipzig, 1868).

Writing $-p$ for p, we obtain a second matrix whose rows give the coordinates of the four points in which a, b, c, d are met by their second transversal f.

The equations of the four planes ea, eb, ec, ed are

$$x' = px, \quad y' = py, \quad z' = pz, \quad t' = pt, \tag{3.2}$$

and the four planes fa, fb, fc, fd are obtained from these by changing the sign of p. Thus the two planes which join any one of the four lines a, b, c, d to their two transversals are harmonically conjugate with respect to the faces of T and T' which intersect in that line.

We may always, if we wish, write -1 for p^2, but it is better to retain the symbol p in order to give homogeneity to the algebra.

Again, the four rows of the skew matrix

$$\begin{bmatrix} -p & -n & m & -l \\ n & -p & -l & -m \\ -m & l & -p & -n \\ l & m & n & -p \end{bmatrix} \tag{3.3}$$

give the coordinates of four points on AA', BB', CC', DD' respectively, and all these points satisfy (3.2) and so lie on e. Changing the sign of p, we have a second matrix giving four points on f, one of them on each of the four lines AA', BB', CC', DD'. Hence it follows that the two transversals e and f of a, b, c, d are also the two transversals of AA', BB', CC', DD'; and, further, that the intersections of e and f with any one of these four latter lines are harmonically conjugate with respect to the vertices of T and T' which are joined by that line*.

4. Among the quadrics which pass through the eight points of the configuration are the four plane-pairs

$$xx' = 0, \quad yy' = 0, \quad zz' = 0, \quad tt' = 0.$$

Any three of these are linearly independent, but the four of them are con-

* These results were first obtained by Plücker, *Neue Geometrie des Raumes*, 1 (Leipzig, 1868), 213. Plücker's work is auxiliary to the study of the quadratic complex, but results such as these are properties of the Möbius configuration, and should, therefore, be obtainable from the configuration itself, without reference to any quadratic complex. An elementary proof that the eight lines a, b, c, d, AA', BB', CC', DD' have two common transversals was given by Schröter, *Journal für Math.*, 93 (1882), 143–146.

nected by the identity*

$$xx'+yy'+zz'+tt' \equiv 0, \tag{4}$$

as we see at once on referring to (1.1).

There is thus a *net* of quadrics circumscribing T and T'; these quadrics are all linearly dependent on any three of the four plane-pairs, but the identity (4) suggests that it will be more symmetrical to consider the set of three quadrics

$$Q_1 \equiv yy'+zz' \equiv -xx'-tt' = 0,$$

$$Q_2 \equiv zz'+xx' \equiv -yy'-tt' = 0,$$

$$Q_3 \equiv xx'+yy' \equiv -zz'-tt' = 0.$$

The quadratic forms Q_1, Q_2, Q_3 are linearly independent, and any quadric which circumscribes T and T' must be linearly dependent on the three quadrics $Q_1 = 0$, $Q_2 = 0$, $Q_3 = 0$.

These three quadrics are interesting in themselves†. Take $Q_1 = 0$. It contains the four lines AD, $B'C$, BC', $A'D'$ whose respective equations are $y = z = 0$, $y = z' = 0$, $y' = z = 0$, $y' = z' = 0$; it also contains the four lines BC, $A'D$, AD', $B'C'$ whose respective equations are $x = t = 0$, $x = t' = 0$, $x' = t = 0$, $x' = t' = 0$. Hence $Q_1 = 0$ contains the two skew quadrilaterals $ADA'D'$ and $BCB'C'$. Similarly $Q_2 = 0$ contains the skew quadrilaterals $BDB'D'$ and $CAC'A'$, while $Q_3 = 0$ contains $CDC'D'$ and $ABA'B'$.

5. It has been seen that the planes ae and af are harmonic with respect to the two planes $x = 0$, $x' = 0$ which intersect in the line a; hence e and f

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* This identity is given by Meyer, *loc. cit.*, 458, and, for the dual case of four point-pairs belonging to a tangential net of quadrics, by Baker, *loc. cit.*, 143.

† These three quadrics which arise from the Möbius configuration are really as old as Steiner, see *Gessamelte Werke*, 1, 405–407. This unfinished paper was published in 1832. In Schröter's paper, *loc. cit.*, 144, the eight sides of the two skew quadrilaterals are explicitly written down. The three quadrics were also obtained by Galucci, *Rendiconto dell'Accademia delle Scienze Fis. e Mat. di Napoli* (3), 12 (1906), 77; in the centre of the page $C'A'D'B'$ is a misprint for $C'D'A'B'$. Forms for the equations of the three quadrics are given in a recent note by Turnbull, *Math. Notes*, 29 (Edinburgh, 1935), but the co-ordinate system which Turnbull uses is based not on one of the tetrahedra T, T' but on one of the skew quadrilaterals. His equations are consequently unsymmetrical, nor can the existence of the three sets of eight lines on the quadrics be deduced immediately from his equations.

are polar lines with respect to the plane-pair $xx' = 0$. Similarly they are polar lines with respect to each of the plane-pairs $yy' = 0$, $zz' = 0$, $tt' = 0$. Hence *e and f are polar lines with respect to every quadric of the net.*

6. The quadrics of the net all have equations of the form

$$k_1 Q_1 + k_2 Q_2 + k_3 Q_3 = 0,$$

and so may be represented by the points (k_1, k_2, k_3) of a plane. The three vertices of the triangle of reference represent the quadrics $Q_1 = 0$, $Q_2 = 0$, $Q_3 = 0$, while the four plane-pairs $xx' = 0$, $yy' = 0$, $zz' = 0$, $tt' = 0$ are represented by the four points α, β, γ, δ whose coordinates are $(-1, 1, 1)$, $(1, -1, 1)$, $(1, 1, -1)$, and $(1, 1, 1)$ respectively. It follows, incidentally, that the triangle of reference is self-polar with respect to any conic which circumscribes the quadrangle $\alpha\beta\gamma\delta$; such a conic must have an equation of the form $\kappa_1 k_1^2 + \kappa_2 k_2^2 + \kappa_3 k_3^2 = 0$, where $\kappa_1 + \kappa_2 + \kappa_3 = 0$.

The quadric $k_1 Q_1 + k_2 Q_2 + k_3 Q_3 = 0$ is a cone if

$$\nabla \equiv \begin{vmatrix} 0 & (k_1-k_2)n & (k_3-k_1)m & (k_2+k_3)l \\ (k_1-k_2)n & 0 & (k_2-k_3)l & (k_3+k_1)m \\ (k_3-k_1)m & (k_2-k_3)l & 0 & (k_1+k_2)n \\ (k_2+k_3)l & (k_3+k_1)m & (k_1+k_2)n & 0 \end{vmatrix} = 0.$$

But it may be verified that

$$(m^2+n^2)^2 \nabla \equiv \{(m^2+n^2)^2 k_1^2 + (lm+np)^2 k_2^2 + (nl-mp)^2 k_3^2\} \times \{(m^2+n^2)^2 k_1^2 + (lm-np)^2 k_2^2 + (nl+mp)^2 k_3^2\}.$$

Hence the cones of the net fall into two families; the cones of one family are represented by the conic u whose equation is

$$(m^2+n^2)^2 k_1^2 + (lm+np)^2 k_2^2 + (nl-mp)^2 k_3^2 = 0,$$

while those of the other family are represented by the conic v whose equation is obtained from that of u by changing the sign of p. Since the equations of u and v are both of the proper form to represent conics that circumscribe $\alpha\beta\gamma\delta$, the four plane-pairs of the net are represented by the four intersections of u and v, and so belong to both the families of cones.

It is easy to identify these two families of cones. For any point E of e is common to a pencil of quadrics belonging to the net, and, since f is the polar line of e with respect to all these quadrics, they all have Ef for their tangent plane at E. Hence one quadric of the pencil is a cone with vertex E. Each point of e is therefore the vertex of a cone belonging to the net, so

that there arises a singly-infinite family of cones, the locus of whose vertices is e. This is, in fact, the family of cones represented by the conic u. Similarly the conic v represents a family of cones whose vertices are on f. Since the line of intersection of any one of the four plane-pairs meets both e and f, we see again that the four plane-pairs belong to both families of cones.

The locus of the vertices of cones belonging to a net of quadrics is a curve—the Jacobian curve of the net. It is well-known that, for a general net of quadrics, the Jacobian curve is a sextic; for the special net that we are considering the sextic degenerates into the six lines a, b, c, d, e, f.

The two Plücker surfaces.

7. Of the family of cones whose vertices lie on e there are *two* passing through an arbitrary point of space; for those quadrics of the net which pass through the point are the members of a pencil, and this pencil is represented in the plane by a line having *two* intersections with u. In other words, the cones whose vertices lie on e form a system of index 2. There are points of space such that the two cones of the system which pass through them coincide; the locus of these points is a surface U, the envelope of those cones of the net whose vertices lie on e. If a point lies on U the line which represents the pencil of quadrics passing through the point must touch u. The surface U may therefore be defined as follows: it is the locus of the base curves of those pencils of quadrics that are given by linear relations

$$\lambda_1 k_1+\lambda_2 k_2+\lambda_3 k_3 = 0,$$

where the coefficients $\lambda_1, \lambda_2, \lambda_3$ satisfy the tangential equation of u, namely

$$(m^2+n^2)^{-2}\lambda_1^2+(lm+np)^{-2}\lambda_2^2+(nl-mp)^{-2}\lambda_3^2 = 0.$$

Now the coordinates of any point on the base curve of the pencil satisfy the relations

$$Q_1 : Q_2 : Q_3 = \lambda_1 : \lambda_2 : \lambda_3;$$

hence the equation of U is

$$(m^2+n^2)^{-2}\,Q_1^2+(lm+np)^{-2}\,Q_2^2+(nl-mp)^{-2}\,Q_3^2 = 0.$$

Since the left-hand side of this equation is a homogeneous quadratic function of Q_1, Q_2, Q_3, the surface U must have nodes at the eight points $Q_1 = Q_2 = Q_3 = 0$.

Since the equation of U is

$$(lm+np)^2\,(nl-mp)^2\,(yy'+zz')^2+(nl-mp)^2\,(m^2+n^2)^2\,(zz'+xx')^2$$
$$+(m^2+n^2)^2\,(lm+np)^2\,(xx'+yy')^2 = 0,$$

it may be written in the form

$$(m^2+n^2)\sqrt{(xx')}+(lm+np)\sqrt{(yy')}+(nl-mp)\sqrt{(zz')}=0, \quad (7)$$

since $$(m^2+n^2)^2+(lm+np)^2+(nl-mp)^2=0.$$

This is the standard form for the equation of a quartic surface with fourteen nodes*. But the fourteen-nodal quartic surface is capable of special forms and, since U is the envelope of a singly-infinite family of cones the locus of whose vertices is e, it is natural to expect that e is a nodal line on U, and that U is not a general fourteen-nodal surface but a Plücker surface. This expectation is confirmed when we notice that (7) is equivalent to the standard form of the equation of a Plücker surface†.

The surface U has e as a nodal line, and has the eight isolated nodes $A, B, C, D, A', B', C', D'$; it contains the four lines AA', BB', CC', DD'. The eight planes $x, y, z, t, x', y', z', t'$ are tropes of U, each touching U along a conic‡. Any plane through e meets U in the line e, counted twice, together with a conic; this conic must touch the eight lines in which the plane is met by the tropes of U. Thus not only is any point of e joined to the eight points of the configuration by eight lines which lie on a quadric cone, but any plane through e is met by the eight planes of the configuration in eight lines which touch a conic. The surface U could also be defined as the locus of these conics in the different planes through e.

8. There is a second Plücker surface V associated with the configuration; its equation is found by writing $-p$ for p in the equation of U. Like the

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* Kummer, *Abhandlungen der K. Akad. der Wiss. zu Berlin* (1866), *Math. Abh.*, 86. Hudson, *Kummer's quartic surface* (Cambridge, 1905), 86.

† Take

$$\xi \equiv (l^2+m^2)\,y+(mn-lp)\,z-(nl+mp)\,t,$$
$$\eta \equiv -(l^2+m^2)\,x+(nl+mp)\,z-(mn-lp)\,t,$$
$$\zeta \equiv -(mn-lp)\,x-(nl+mp)\,y-(n^2+p^2)\,t,$$

so that $\xi=0$, $\eta=0$, $\zeta=0$ are the equations of the three planes eAA', eBB', eCC' respectively. Then (7) is the same equation as

$$\sqrt{\{(l^2+p^2)\,lx\xi\}}+\sqrt{\{(lm+np)\,my\eta\}}+\sqrt{\{(nl-mp)\,nz\zeta\}}=0,$$

which is of the well-known standard form. This standard form is found in Hudson, *loc. cit.*, 70, also in Salmon-Rogers, *Geometry of three dimensions*, 2 (Dublin, 1915), 220. In both these books the surface is obtained from a quadratic complex; it is the envelope of those cones of the complex whose vertices lie on an arbitrary line. Baker obtains the surface, with the same standard form of equation, as the section of a quartic locus in [4] by a [3] passing through a double line of the locus; see *Principles of geometry*, 4 (Cambridge, 1925), 142.

‡ Zindler, in his article on line-geometry in Pascal's *Repertorium der höheren Math.*, 2 (1922), Kap. 39, 1015, makes the untrue statement that the four lines a, b, c, d lie on the Plücker surface.

surface U, it has the eight points of the configuration as nodes and the eight planes of the configuration as tropes, and it contains the four lines AA', BB', CC', DD'; but the nodal line of V is f. The surface V is the envelope of those cones of the net whose vertices lie on f, and it may also be obtained as a locus of conics lying in planes through f.

It may be emphasised here that *Plücker surfaces occur in pairs*. Given any Plücker surface, the tangent cone to the surface from any point of its nodal line is (apart from tropes) a quadric cone; the quadric cones so arising from the different points of the nodal line belong to a net of quadrics, and this net of quadrics contains a second family of cones, the locus of whose vertices is another line; the envelope of the cones of this second family is a second Plücker surface. The relation between the two surfaces is mutual; the first may be obtained from the second just as the second has been obtained from the first.

9. When the quadrics of the net are represented by the points of a plane, a line of the plane corresponds to a pencil of the quadrics. If this line touches u, the base curve of the pencil lies on U; while if the line touches v, the base curve of the pencil lies on V. Hence, if the line is a common tangent of u and v, the base curve of the corresponding pencil lies on both U and V. Now u and v have four common tangents; hence there are four pencils of quadrics whose base curves lie on both U and V, and these four base curves make up the whole of the curve of order sixteen common to the two quartic surfaces. It has been seen already that the line AA' lies on both U and V; it must therefore be part of the base curve of one of these four pencils of quadrics. Indeed, those quadrics of the net which contain AA' (a line joining two of the base points of the net) form a pencil whose base curve consists of AA' and the twisted cubic passing through the remaining six base points of the net. The complete curve of intersection of U and V consists therefore of the four lines

$$AA', \quad BB', \quad CC', \quad DD'$$

and the four twisted cubics*

$$BB'CC'DD', \quad CC'DD'AA', \quad DD'AA'BB', \quad AA'BB'CC'.$$

The eight points which are the intersections of e and f with AA', BB', CC', DD' must be double points of the common curve of U and V, since

* The existence of four twisted cubics on a Plücker surface was pointed out by Kleiber, *Zeitschrift für Math. und Phys.*, 33 (1888), 352.

each of the points is a double point on one surface and an ordinary point on the other. Hence the twisted cubic $BB'CC'DD'$ must pass through the intersections of AA' with e and f. It follows that the quadric containing the three lines BB', CC', DD', and hence also e and f, must contain the twisted cubic $BB'CC'DD'$. There are corresponding results for the other three cubics. Now if a quadric contains a twisted cubic one system of its generators consists of chords of the cubic, and their intersections with the cubic are pairs of an involution on the curve. Hence we have the result, already obtained otherwise by Weiss*, that *three pairs of points of a Möbius configuration are in involution on the twisted cubic which passes through them.*

The Kummer surfaces.

10. Take now any conic which circumscribes the quadrangle $\alpha\beta\gamma\delta$; the equation of the conic is of the form

$$\kappa_1 k_1^2+\kappa_2 k_2^2+\kappa_3 k_3^2 = 0,$$

with

$$\kappa_1+\kappa_2+\kappa_3 = 0.$$

The points of this conic represent quadrics of the net forming a singly-infinite system of index 2, and the four plane-pairs of the net all belong to the system; conversely, any singly-infinite system of quadrics, belonging to the net, which contains the four plane-pairs and is of index 2, is represented by a conic through α, β, γ, δ,

Since the tangential equation of the conic is

$$\kappa_2\kappa_3\lambda_1^2+\kappa_3\kappa_1\lambda_2^2+\kappa_1\kappa_2\lambda_3^2 = 0,$$

the envelope of the system of quadrics is the quartic surface

$$\kappa_2\kappa_3 Q_1^2+\kappa_3\kappa_1 Q_2^2+\kappa_1\kappa_2 Q_3^2 = 0,$$

i.e.

$$\kappa_1(\kappa_2+\kappa_3)x^2x'^2+\kappa_2(\kappa_3+\kappa_1)y^2y'^2+\kappa_3(\kappa_1+\kappa_2)z^2z'^2$$
$$+2\kappa_2\kappa_3 yy'zz'+2\kappa_3\kappa_1 zz'xx'+2\kappa_1\kappa_2 xx'yy' = 0.$$

Since $\kappa_1+\kappa_2+\kappa_3 = 0$, this may be written in the form

$$\sqrt{(\kappa_1 xx')}+\sqrt{(\kappa_2 yy')}+\sqrt{(\kappa_3 zz')} = 0.$$

* *Monatshefte für Math. und Phys.*, 38 (1931), 59.

This surface is a *Kummer surface.* Its sixteen nodes consist of the eight points of the configuration and of eight other points, two on each of the lines a, b, c, d; the two nodes on a, for example, are the intersections of a with the quadric $\kappa_2 yy' = \kappa_3 zz'$, and so are given by

$$x = x' = 0, \quad \frac{yy'}{\kappa_3} = \frac{zz'}{\kappa_2} = \frac{tt'}{\kappa_1}.$$

The sixteen tropes of the surface consist of the eight planes of the configuration and of eight other planes, two of these passing through each of the lines AA', BB', CC', DD'. Hence *the envelope of quadrics of the net which form a singly-infinite system of index 2 including all four plane-pairs is a Kummer surface, having the eight points of the configuration among its nodes and the eight planes of the configuration among its tropes*.*

There thus arises, corresponding to the singly-infinite set of values of the ratios $\kappa_1 : \kappa_2 : \kappa_3$ for which $\kappa_1 + \kappa_2 + \kappa_3 = 0$, a singly-infinite set of Kummer surfaces; these include the two Plücker surfaces U and V. Those pairs of nodes of the Kummer surfaces which lie on any one of the lines a, b, c, d form an involution on that line, the double points of the involution being the intersections of the line with e and f. A ninth node of the Kummer surface may be assigned arbitrarily on any one of the lines a, b, c, d, but the remaining seven nodes are thereby determined. There are dual statements concerning the tropes.

II.

11. In order that a general net of quadric surfaces should contain four plane-pairs, it must be subjected to four conditions. If then we regard a net of quadric surfaces in [3] as the section, by a solid, of a general net of quadrics in [4], it should be possible to obtain solids that give by their sections nets of quadric surfaces which include four plane-pairs; for in order that the section should have this particularity the solid must be subjected to four conditions, and it is therefore to be expected that a finite number of solids will answer to the purpose. The existence of such solids has indeed been demonstrated†; there are ten such solids, each of them

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* The problem of finding a singly-infinite set of quadrics having a given Kummer surface as its envelope was considered by Zeuthen in his *Lehrbuch der abzählenden Geometrie* (Leipzig, 1914), 154–159. He found that the quadrics must include four plane-pairs and belong to a net, and he obtained the Jacobian curve of the net.

† Edge, *Acta math.*, 64 (1935), 185–242 (207).

being determined as the join of a pair of conjugate trisecants of the Jacobian curve of the net in [4]. There is a quadric of the net touching the solid at any point of either of the two trisecants, and the polar solid of any point on either trisecant with respect to every quadric of the net passes through the conjugate trisecant*. The net of quadric surfaces which is the section of the net in [4] by this solid is therefore such that there is a cone belonging to the net with vertex at any point of either of the two trisecants, and the polar plane of any point on either trisecant with respect to every quadric surface of the net passes through the other trisecant. Thus the two trisecants are polars of one another with respect to all the quadric surfaces of the net, and give the two lines e and f.

Any quadric of the net in [4] can have its equation written in the form†

$$\xi_1 X_1^2+\xi_2 X_2^2+\xi_3 X_3^2+\zeta Z^2+\eta_1 Y_1^2+\eta_2 Y_2^2+\eta_3 Y_3^2=0,$$

where $$X_1+X_2+X_3\equiv Z\equiv Y_1+Y_2+Y_3.$$

These seven linear forms are fixed, the coefficients varying for different quadrics of the net. The solid $Z=0$ joins a pair of conjugate trisecants of the Jacobian curve; the three solids $X_1=0, X_2=0, X_3=0$ pass through one of these trisecants, and the three solids $Y_1=0, Y_2=0, Y_3=0$ through the other. The equation of the surface which is the section of the quadric by $Z=0$ is

$$\xi_1 X_1^2+\xi_2 X_2^2+\xi_3 X_3^2+\eta_1 Y_1^2+\eta_2 Y_2^2+\eta_3 Y_3^2=0,$$

where $$X_1+X_2+X_3\equiv Y_1+Y_2+Y_3\equiv 0.$$

We may write this in the form

$$aX^2+2hXY+bY^2+cZ^2+2fZW+dW^2=0,$$

where $X=0$ and $Y=0$ are any two planes through one trisecant and $Z=0$ and $W=0$ are any two planes through the other. If then three quadric surfaces circumscribe a pair of Möbius tetrads, their equations may be reduced simultaneously to the form

$$q_1\equiv a_1 X^2+2h_1 XY+b_1 Y^2+c_1 Z^2+2f_1 ZW+d_1 W^2=0,$$

$$q_2\equiv a_2 X^2+2h_2 XY+b_2 Y^2+c_2 Z^2+2f_2 ZW+d_2 W^2=0,$$

$$q_3\equiv a_3 X^2+2h_3 XY+b_3 Y^2+c_3 Z^2+2f_3 ZW+d_3 W^2=0,$$

* Edge, *loc. cit.*, 205.

† *Ibid*, 240.

the line e being $X=Y=0$ and the line f being $Z=W=0$. The discriminant of the quadric $k_1q_1+k_2q_2+k_3q_3=0$ is

$$\begin{vmatrix} a_k & h_k & 0 & 0 \\ h_k & b_k & 0 & 0 \\ 0 & 0 & c_k & f_k \\ 0 & 0 & f_k & d_k \end{vmatrix},$$

where we have written a_k for $k_1a_1+k_2a_2+k_3a_3$, and so on. Hence, when the quadrics of the net are represented by the points (k_1, k_2, k_3) of a plane, the cones belonging to the net are represented by the points of the two conics

$$a_kb_k=h_k^2, \quad c_kd_k=f_k^2.$$

The first conic gives cones whose vertices are on the line $Z=W=0$; the other conic gives cones whose vertices are on the line $X=Y=0$.

If a point satisfies the equations of *both* conics, then its coordinates (k_1, k_2, k_3) cause all the first minors of the discriminant to vanish; hence the four points of intersection of the conics represent four plane-pairs of the net.

III.

12. The set of eight points $A, B, C, D, A', B', C', D'$ may be arranged as a pair of Möbius tetrads in four different ways, namely

$$\left.\begin{matrix} ABCD \\ A'B'C'D' \end{matrix}\right\}, \quad \left.\begin{matrix} AB'C'D \\ A'BCD' \end{matrix}\right\}, \quad \left.\begin{matrix} A'BC'D \\ AB'CD' \end{matrix}\right\}, \quad \left.\begin{matrix} A'B'CD \\ ABC'D' \end{matrix}\right\}.$$

But the eight faces of any one of these four pairs of tetrahedra are always the same eight planes, and the vertices of the tetrahedra are always paired in the same way—AA', BB', CC' and DD'. Let these tetrahedra be named, in the order written,

$$\left.\begin{matrix} T \\ T' \end{matrix}\right\}, \quad \left.\begin{matrix} T_1 \\ T_1' \end{matrix}\right\}, \quad \left.\begin{matrix} T_2 \\ T_2' \end{matrix}\right\}, \quad \left.\begin{matrix} T_3 \\ T_3' \end{matrix}\right\}.$$

On referring to § 1 it is seen that the identity

$$x^2+y^2+z^2+t^2 \equiv x'^2+y'^2+z'^2+t'^2$$

holds, and this establishes at once the existence of four quadrics Ω, Ω_1, Ω_2, Ω_3, each of which has one of the four pairs of tetrahedra as a pair of

self-conjugate tetrahedra. The equations of Ω, Ω_1, Ω_2, Ω_3 are, respectively,

$$\Omega \equiv x^2+y^2+z^2+t^2 \equiv x'^2+y'^2+z'^2+t'^2 = 0,$$

$$\Omega_1 \equiv y^2+z^2-x'^2-t'^2 \equiv y'^2+z'^2-x^2-t^2 = 0,$$

$$\Omega_2 \equiv z^2+x^2-y'^2-t'^2 \equiv z'^2+x'^2-y^2-t^2 = 0,$$

$$\Omega_3 \equiv x^2+y^2-z'^2-t'^2 \equiv x'^2+y'^2-z^2-t^2 = 0.$$

It is seen at once, on referring to (3 . 3), that each of these four quadrics contains the two lines e, f. Also we have the identity

$$p^2\,\Omega^2+l^{-2}\,\Omega_1^{\,2}+m^{-2}\,\Omega_2^{\,2}+n^{-2}\,\Omega_3^{\,2} \equiv 0.$$

Any two of the four quadrics Ω, Ω_1, Ω_2, Ω_3 *are their own polar reciprocals with respect to one another.* This can be shown directly, from their equations, or else may be regarded as a well-known property of the set of four mutually apolar linear complexes (see § 14 below) from which the four quadrics can be obtained.

13. If the figure is reciprocated with respect to any one of the quadrics Ω, Ω_1, Ω_2, Ω_3, the eight points A, B, C, D, A', B', C', D' become the eight planes x, y, z, t, x', y', z', t' (although not necessarily in this order). Thus a quadric which passes through the eight points of the configuration becomes a quadric touching the eight planes of the configuration, and so, if we polarise with respect to any one of the four quadrics Ω, Ω_1, Ω_2, Ω_3, the net of quadrics passing through the eight points becomes the tangential net of quadrics touching the eight planes. It is important to notice, however, that *the three quadrics* Q_1, Q_2, Q_3 *are always reciprocated into themselves.* Take Q_2; it contains the eight lines

$$BD,\quad DB',\quad B'D',\quad D'B,\quad CA,\quad AC',\quad C'A',\quad A'C.$$

The polars of these eight lines with respect to Ω_2, for example, are, respectively,

$$C'A',\quad D'B,\quad CA,\quad DB',\quad B'D',\quad A'C,\quad BD,\quad AC';$$

these are the same eight lines, so that Q_2 is its own polar reciprocal with respect to Ω_2. It is also, in the same way, its own polar reciprocal with respect to Ω, Ω_1, and Ω_3; and so are Q_1 and Q_3 their own polar reciprocals with respect to Ω, Ω_1, Ω_2, Ω_3.

When a quadric Q is its own polar reciprocal with respect to a second quadric, the reciprocation may either interchange the two reguli on Q or leave them unaltered. When any one of Q_1, Q_2, Q_3 is polarised with respect to any one of Ω, Ω_1, Ω_2, Ω_3, the two reguli are always unaltered; a glance at the above two sets of eight lines shows that this is so when Q_2 is polarised with respect to Ω_2.

14. It is possible to reciprocate the figure into itself, so that the eight points become the eight planes, not only by means of the four quadrics Ω, Ω_1, Ω_2, Ω_3 but also by means of four linear complexes Λ, Λ_1, Λ_2, Λ_3. This was shown by Klein*, who also showed that these four linear complexes are mutually apolar; we need then only give the equations of the complexes in our system of coordinates. The complex Λ may be defined as the complex for which the null-plane of any one of the eight points of the configuration is that single face of T or T' which passes through the point; its equation is, if $(L, M, N; L', M', N')$ denote line-coordinates,

$$l(L-L')+m(M-M')+n(N-N')=0,$$

and it is associated with the skew matrix (1 . 0). The complexes Λ_1, Λ_2, Λ_3 are defined similarly by means of the other three pairs of tetrahedra, just as Λ has been defined by means of T and T'; their respective equations are

$$L+L'=0, \quad M+M'=0, \quad N+N'=0.$$

The equations show at once that any two of the four complexes are apolar. Also *e and f belong to all four complexes,* for the six coordinates of e are

$$(-nl-mp, \quad lp-mn, \quad l^2+m^2; \quad nl+mp, \quad mn-lp, \quad p^2+n^2),$$

and the coordinates of f are derived from these by changing the sign of p.

Three linear complexes have in common, in general, the lines of a regulus, and any complex which is apolar to all three must contain the complementary regulus. Hence any three of the four complexes Λ, Λ_1, Λ_2, Λ_3 have in common a regulus to which e and f belong, while the complementary regulus belongs to the fourth complex. Thus there arise four quadric surfaces, each containing the lines e and f. But these quadric surfaces are none other than Ω, Ω_1, Ω_2, Ω_3; for the four reguli, on these quadrics, to which e and f belong have, as may easily be verified, the respec-

* *Math. Annalen,* 7 (1874), 208–211.

tive line equations

$$L+L'=M+M'=N+N'=0,$$

$$M+M'=N+N'=l(L-L')+m(M-M')+n(N-N')=0,$$

$$N+N'=L+L'=l(L-L')+m(M-M')+n(N-N')=0,$$

$$L+L'=M+M'=l(L-L')+m(M-M')+n(N-N')=0.$$

When the figure is polarised with respect to any one of the four complexes $\Lambda, \Lambda_1, \Lambda_2, \Lambda_3$, the eight points of the configuration become the eight planes of the configuration, and the net of quadrics through the eight points becomes the tangential net of quadrics touching the eight planes. And again *the quadrics Q_1, Q_2, Q_3 are reciprocated into themselves*, the reguli on these quadrics being unaltered. For example, the polar lines of the eight lines

$$BD,\quad DB',\quad B'D',\quad D'B,\quad CA,\quad AC',\quad C'A',\quad A'C,$$

which lie on Q_2, with respect to Λ_2, are, respectively,

$$CA,\quad DB',\quad C'A',\quad D'B,\quad BD,\quad AC',\quad B'D',\quad A'C.$$

15. The line e belongs to the four linear complexes $\Lambda, \Lambda_1, \Lambda_2, \Lambda_3$ and is a generator of the four quadrics $\Omega, \Omega_1, \Omega_2, \Omega_3$. If then a cone has its vertex on e and passes through the eight points of the configuration, its polar reciprocal, with respect to any of the four complexes or any of the four quadrics, is a conic whose plane passes through e and which touches the eight planes of the configuration. But the Plücker surface U is both the envelope of these cones and the locus of the conics; hence U is its own polar reciprocal with respect to all four linear complexes and all four quadrics. Similar results hold for the surface V.

It is also easy to prove, by using the result obtained in § 10, that any one of the ∞^1 Kummer surfaces is its own reciprocal with respect to each of the four quadrics and each of the four linear complexes*. Any one, say S, of these Kummer surfaces is the envelope of a singly-infinite system, of index 2, of quadrics passing through the eight points of the configuration. The polar reciprocals of these quadrics give another system of quadrics

* The results of § 10 show at once that the polar of any node of one of the Kummer surfaces with respect to one of the four quadrics or one of the four linear complexes is always a trope of the same surface.

touching the eight planes of the configuration. It is found that this system of quadrics is also of index 2, and it belongs to a second net of quadrics whose base points are the remaining eight nodes of S.

16. It is interesting to give the algebra corresponding to the reciprocation of S into itself.

We take tangential coordinates X, Y, Z, T; then $X=0$, $Y=0$, $Z=0$, $T=0$ are the respective equations of the points A, B, C, D. We take $X'=0$, $Y'=0$, $Z'=0$, $T'=0$ to be the respective equations of the points A', B', C', D', so that X', Y', Z', T' are the same linear functions of X, Y, Z, T as x', y', z', t' are of x, y, z, t. Then

$$XX'+YY'+ZZ'+TT'\equiv 0,$$

and it is easily shown that the tangential equations of the three quadrics $Q_1=0$, $Q_2=0$, $Q_3=0$ are, respectively,

$$YY'+ZZ'=0, \quad ZZ'+XX'=0, \quad XX'+YY'=0.$$

The tangential equation of any quadric touching the eight planes of the configuration is therefore

$$h_1(YY'+ZZ')+h_2(ZZ'+XX')+h_3(XX'+YY')=0.$$

The point equation of this quadric is found to be

$$\begin{aligned} lmn\{&(h_2-h_3)(h_3+h_1)(h_1+h_2)x^2+(h_2+h_3)(h_3-h_1)(h_1+h_2)y^2 \\ &+(h_2+h_3)(h_3+h_1)(h_1-h_2)z^2+(h_2-h_3)(h_3-h_1)(h_1-h_2)t^2\} \\ +l\{&m^2(h_3^2-h_1^2)+n^2(h_1^2-h_2^2)-l^2(h_2^2-h_3^2)\}\{(h_2+h_3)yz-(h_2-h_3)xt\} \\ +m\{&n^2(h_1^2-h_2^2)+l^2(h_2^2-h_3^2)-m^2(h_3^2-h_1^2)\}\{(h_3+h_1)zx-(h_3-h_1)yt\} \\ +n\{&l^2(h_2^2-h_3^2)+m^2(h_3^2-h_1^2)-n^2(h_1^2-h_2^2)\}\{(h_1+h_2)xy-(h_1-h_2)zt\}=0. \end{aligned} \tag{16}$$

This is the point equation, where h_1, h_2, h_3 vary, of the system of quadrics touching the eight planes of the configuration. When $h_2=h_3=0$, for example, it reduces to the point equation of Q_1.

We now wish to select, from this doubly-infinite system of quadrics, a singly-infinite system which includes the four point-pairs and which is such that there are two of its members touching an arbitrary plane. Since the four point-pairs are given by the sets of values (1, 1, 1), (−1, 1, 1),

$(1, -1, 1)$, and $(1, 1, -1)$ for (h_1, h_2, h_3), such a singly-infinite system must be given by a relation of the form

$$\kappa_1 h_1^2+\kappa_2 h_2^2+\kappa_3 h_3^2 = 0,$$

where

$$\kappa_1+\kappa_2+\kappa_3 = 0.$$

Such a relation is satisfied by taking

$$h_1 = \kappa_2\gamma^2+\kappa_3\beta^2, \quad h_2 = \kappa_3\alpha^2+\kappa_1\gamma^2, \quad h_3 = \kappa_1\beta^2+\kappa_2\alpha^2,$$

where

$$\alpha+\beta+\gamma = 0.$$

On substituting these parametric expressions for h_1, h_2, h_3 in (16) it reduces to

$$\begin{aligned} lmn\{(x^2-t^2)\alpha^2\kappa_2\kappa_3+(y^2-t^2)\beta^2\kappa_3\kappa_1+(z^2-t^2)\gamma^2\kappa_1\kappa_2\} \\ +l(m^2\kappa_2+n^2\kappa_3-l^2\kappa_1)\{yz\kappa_1\beta\gamma+xt\alpha(\kappa_2\gamma-\kappa_3\beta)\} \\ +m(n^2\kappa_3+l^2\kappa_1-m^2\kappa_2)\{zx\kappa_2\gamma\alpha+yt\beta(\kappa_3\alpha-\kappa_1\gamma)\} \\ +n(l^2\kappa_1+m^2\kappa_2-n^2\kappa_3)\{xy\kappa_3\alpha\beta+zt\gamma(\kappa_1\beta-\kappa_2\alpha)\} = 0. \end{aligned}$$

Thus the system of quadrics is of index 2, two members of the system passing through any point. The envelope of the system is the locus of points (x, y, z, t) for which the two quadrics of the system coincide; thus the two values of $\alpha:\beta:\gamma$ which satisfy $\alpha+\beta+\gamma = 0$ and give quadrics of the system passing through (x, y, z, t) must coincide. It is then found that the envelope of the system of quadrics is precisely the Kummer surface

$$\kappa_1 x^2 x'^2+\kappa_2 y^2 y'^2+\kappa_3 z^2 z'^2-2\kappa_2\kappa_3 yy' zz'-2\kappa_3\kappa_1 zz' xx'-2\kappa_1\kappa_2 xx' yy' = 0.$$

We have then the following result. Take those quadrics

$$k_1 Q_1+k_2 Q_2+k_3 Q_3 = 0,$$

for which

$$\kappa_1 k_1^2+\kappa_2 k_2^2+\kappa_3 k_3^2 = 0$$

and

$$\kappa_1+\kappa_2+\kappa_3 = 0.$$

We obtain in this way two different systems of quadrics, according as Q_1, Q_2, Q_3 are expressed in point-coordinates or in tangential coordinates. But *the envelope of the two systems is the same Kummer surface.*

Any quadric of either system touches the Kummer surface along an elliptic quartic curve, and the two systems of quadrics give two systems of quartic curves passing through complementary octads of nodes.

17. When two linear complexes are apolar, the two operations of polarisation with respect to them are permutable. Also, when three linear complexes are mutually apolar, the three operations of polarisation with respect to them, when applied in succession, are equivalent to the single operation of polarisation with respect to that quadric surface on which the regulus of lines common to the three complexes lies*. Hence we may write

$$\Omega = \Lambda_1 \Lambda_2 \Lambda_3, \quad \Omega_1 = \Lambda \Lambda_2 \Lambda_3, \quad \Omega_2 = \Lambda \Lambda_3 \Lambda_1, \quad \Omega_3 = \Lambda \Lambda_1 \Lambda_2;$$

the orders of the symbols in the three-term products are immaterial. We also have

$$\Lambda = \Omega_1 \Omega_2 \Omega_3, \quad \Lambda_1 = \Omega \Omega_2 \Omega_3, \quad \Lambda_2 = \Omega \Omega_3 \Omega_1, \quad \Lambda_3 = \Omega \Omega_1 \Omega_2.$$

We need then only concern ourselves either with the four quadrics or with the four linear complexes. Let us retain the quadrics. As for the linear complexes, so for the quadrics; the order of successive polarisations with respect to them in immaterial.

18. When the net of quadrics

$$k_1 Q_1 + k_2 Q_2 + k_3 Q_3 = 0$$

is polarised with respect to any one of Ω, Ω_1, Ω_2, Ω_3, it becomes a tangential net of quadrics; if this tangential net is then, in its turn, polarised with respect to any one of Ω, Ω_1, Ω_2, Ω_3, it becomes the original net of quadrics. The result of two successive polarisations is therefore to transform the net of quadrics into itself. The three quadrics Q_1, Q_2, Q_3 are unaltered by the two polarisations, but the other quadrics of the net will be permuted among themselves. It is easy to determine these permutations; if the two polarisations are with respect to the *same* quadric the resulting permutation is of course the identical permutation, but two successive polarisations with respect to two different quadrics give other permutations.

A pencil of quadrics belonging to the net becomes, after one polarisation, a tangential pencil; after two polarisations it therefore becomes the same or another pencil of quadrics belonging to the net. Hence the permutation, among the quadrics of the net, which results from two polarisations is equivalent to a *linear* transformation of the coordinates (k_1, k_2, k_3) in the plane which represents the net of quadrics. The vertices of the triangle of reference are unchanged by the transformation, and the transformation is therefore determined when one corresponding pair of points, other than

* *Cf.* Hudson, *loc. cit.*, 39–40.

the vertices of the triangle of reference, is given. Consider then the effect of the polarisations on the four plane-pairs on the net. These four plane-pairs are

$$xx'; \quad yy', \quad zz', \quad tt',$$

and the four point-pairs into which they are reciprocated by the quadrics Ω, Ω_1, Ω_2, Ω_3 are

Ω: $AA', \quad BB', \quad CC', \quad DD'$;

Ω_1: $DD', \quad CC', \quad BB', \quad AA'$;

Ω_2: $CC', \quad DD', \quad AA', \quad BB'$;

Ω_3: $BB', \quad AA', \quad DD', \quad CC'$.

Hence the two successive polarisations with respect to Ω_2 and Ω_3, for example, transform the four plane-pairs into

$$tt', \quad zz', \quad yy', \quad xx',$$

respectively, and the quadric

$$k_1 Q_1 + k_2 Q_2 + k_3 Q_3 \equiv k_1(yy'+zz') + k_2(zz'+xx') + k_3(xx'+yy') = 0$$

becomes, by the operation $\Omega_2 \Omega_3$ or $\Omega_3 \Omega_2$,

$$k_1(zz'+yy') + k_2(yy'+tt') + k_3(tt'+zz') \equiv k_1 Q_1 - k_2 Q_2 - k_3 Q_3 = 0.$$

We may therefore write, as regards the linear transformation in the plane,

$$\Omega_2 \Omega_3(k_1, k_2, k_3) = (-k_1, k_2, k_3),$$

the effect being to change the sign of k_1. The transformation is involutory, as we should expect, and the same transformation results from the operation $\Omega\Omega_1$ or $\Omega_1 \Omega$. We obtain in fact, if we include the identical transformation, four transformations in all; we may write them

$$\Omega\Omega(k_1, k_2, k_3) = (k_1, k_2, k_3),$$

$$\Omega\Omega_1(k_1, k_2, k_3) = (-k_1, k_2, k_3),$$

$$\Omega\Omega_2(k_1, k_2, k_3) = (k_1, -k_2, k_3),$$

$$\Omega\Omega_3(k_1, k_2, k_3) = (k_1, k_2, -k_3).$$

These four transformations form a *Vierergruppe*; they consist of the identical transformation and three harmonic inversions; each harmonic inversion has its centre at a vertex of the triangle of reference, the axis of the inversion being the opposite side of the triangle.

All the transformations leave unaltered any conic with regard to which the triangle of reference is self-conjugate.

The transformation $\Omega\Omega_1\Omega_2\Omega_3$ leaves every point (k_1, k_2, k_3) unaltered; thus the effect of four successive polarisations with respect to the four quadrics Ω, Ω_1, Ω_2, Ω_3 is to leave every quadric of the net unaltered. *The collineation* $\Omega\Omega_1\Omega_2\Omega_3$ *in space is, in fact, the harmonic inversion whose base lines are e and f.*

That this harmonic inversion leaves every individual quadric of the net unaltered is obvious on referring to the canonical form for the net of quadrics in § 11; in the system of coordinates there used the harmonic inversion is $(X, Y, Z, W) \rightarrow (X, Y, -Z, -W)$.

IV.

The net of quadric surfaces with a base line.

19. If three quadrics have a common generator g they intersect further, in general, in four distinct points A, B, C, D; they determine a net of quadrics which all pass through g and these four points. The net contains four plane-pairs; a plane-pair consists of a face of the tetrahedron $ABCD$ together with the plane joining g to the opposite vertex.

If we put $p = 0$ in the preceding work, the four points A', B', C', D' are collinear, being the intersections of a line g with the faces of the tetrahedron $ABCD$; thus the net of quadrics circumscribing the two Möbius tetrads reduces, when $p = 0$, to a net of quadrics with a base line. The four planes x', y', z', t' now join g to the four points A, B, C, D; the lines e and f both coincide with g, and the six coordinates of g are $(l, m, n; -l, -m, -n)$.

Each of the quadrics Q_1, Q_2, Q_3 contains a pair of opposite edges of $ABCD$; when the quadrics $k_1 Q_1 + k_2 Q_2 + k_3 Q_3 = 0$ are represented by the points (k_1, k_2, k_3) of a plane, the four plane-pairs are again represented by the points $(-1, 1, 1)$, $(1, -1, 1)$, $(1, 1, -1)$, $(1, 1, 1)$. The discriminant of the quadratic form $k_1 Q_1 + k_2 Q_2 + k_3 Q_3$ is now

$$\nabla \equiv (l^2 k_1^2 + m^2 k_2^2 + n^2 k_3^2)^2,$$

so that the cones of the net are represented by the points of a repeated conic. The vertices of the cones are on g, and each point of g is the vertex of one cone.

The cones of the net form a singly-infinite family, of index 2, and have for their envelope a quartic surface W. Let us first obtain the envelope of any singly-infinite family of quadrics, belonging to the net, which is of

index 2 and which includes the four plane-pairs. Such a family of quadrics is represented by a conic

$$\kappa_1 k_1^2+\kappa_2 k_2^2+\kappa_3 k_3^2 = 0,$$

where

$$\kappa_1+\kappa_2+\kappa_3 = 0,$$

the family of cones arising when

$$\kappa_1 = l^2, \quad \kappa_2 = m^2, \quad \kappa_3 = n^2.$$

It is found, as in § 10, that the envelope of the family of quadrics is the quartic surface K whose equation is

$$\kappa_1^2 x^2 x'^2+\kappa_2^2 y^2 y'^2+\kappa_3^2 z^2 z'^2-2\kappa_2\kappa_3 yzy' z'-2\kappa_3\kappa_1 zxz' x'-2\kappa_1\kappa_2 xyx' y' = 0.$$

This surface has nodes at A, B, C, D and has g as a nodal line.

When a quartic surface has a nodal line there are, in general, eight planes through the nodal line each of which meets the surface, apart from the nodal line, in two lines. If the surface has an isolated node, two of these eight planes coincide with the plane joining this node to the nodal line. Thus the four planes x', y', z', t', which join A, B, C, D to g, will each meet K, apart from g, in two lines. But K has the special property that these two lines, in each of the four planes, coincide; if we put $x' = 0$ in the equation of K, it reduces to $(\kappa_2 yy'-\kappa_3 zz')^2 = 0$.

At a general point of g there are two distinct tangent planes to K; it may be verified that these two planes coincide at each of the four points where g meets a face of the tetrahedron $ABCD$. These are the four pinch-points on the nodal line.

The curve of intersection of any two quadrics of the net consists of g and a twisted cubic Γ; Γ passes through A, B, C, D and has g for a chord. Conversely, any twisted cubic Γ which passes through A, B, C, D and has g for a chord forms, when taken together with g, the base curve of a pencil of quadrics belonging to the net. Any quadric which belongs to the system whose envelope is K, besides passing through the nodal line of K, touches K along a curve Γ, and the curves of contact of any two quadrics of the system with K, when taken with the nodal line, form the complete intersection of K with a quadric of the net. Denote a curve Γ which lies on K by Γ_K; the ∞^1 curves Γ_K include four curves each of which breaks up into a line and a conic, these composite curves being the curves of contact of K with the four plane-pairs.

Through a general point P of K there passes *one* curve Γ_K; it is the unique twisted cubic which passes through the five points P, A, B, C, D and has g for a chord. The curves Γ_K thus form a pencil on K. But there

are *two* curves Γ_K passing through a point of g; they form, with g, the complete intersection of K with the cone of the net whose vertex is at this point. The curves Γ_K thus set up a (2, 2) correspondence on g and, since this correspondence has four united points, four of the curves Γ_K touch g.

20. Any set of values of $\kappa_1, \kappa_2, \kappa_3$ such that $\kappa_1+\kappa_2+\kappa_3 = 0$ gives a surface K, but among these ∞^1 surfaces there is a special surface W, obtained by taking $\kappa_1 = l^2$, $\kappa_2 = m^2$, $\kappa_3 = n^2$. This surface is *the envelope of the cones belonging to the net*, and its equation is

$$l^4x^2x'^2+m^4y^2y'^2+n^4z^2z'^2-2m^2n^2yzy'z'-2n^2l^2zxz'x'-2l^2m^2xyx'y' = 0.$$

If we refer this surface to the tetrahedron $xy'z't$ we find that its equation is

$$m^2n^2\{x(ny'-mz')-t(my'+nz')\}^2$$
$$-2l^2mny'z'\{x(ny'-mz')+t(my'+nz')\}+l^4y'^2z'^2 = 0. \quad (20)$$

Hence the line $y' = z' = 0$ is a *cuspidal line* on the surface, the two tangent planes at any point of the line being coincident. Thus W has g for a cuspidal line and has the four nodes A, B, C, D.

The section of W by a plane through a general point of g has a cusp at this point; if, however, the point is one of A', B', C', D' (the four points in which g is met by the faces of the tetrahedron $ABCD$) the section has a tacnode at the point; A', B', C', D' are *close-points* on the cuspidal line g. The equation (20) shows at once that a plane, through either of the points where $x = 0$ and $t = 0$ meet the cuspidal line $y' = z' = 0$, meets W in a curve with a tacnode.

Any plane through g meets W further in a conic; this conic touches g, and also touches the four lines in which the plane is met by the faces of the tetrahedron $ABCD$; this can easily be verified from the equation of W. Thus W may be defined either as the envelope of cones which pass through the vertices of $ABCD$ and have g as a generator or, dually, as the locus of conics which touch the faces of $ABCD$ and also touch g. The surface W is reciprocated into itself by polarisation with respect to that quadric which has g as a generator and $ABCD$ as a self-conjugate tetrahedron; in the system of coordinates which we are using, the equation of this quadric is $x^2+y^2+z^2+t^2 = 0$.

The two intersections of g with the twisted cubic Γ common to two cones of the net are the vertices of these two cones. Thus any cone of the net touches W along a twisted cubic Γ_W which touches g; there is

one of these curves Γ_W through each point of W, and W may be defined* as *the locus of twisted cubics which circumscribe the tetrahedron $ABCD$ and touch g.* There is a corresponding dual definition of W; it is touched by the planes of all the cubic developables which contain the faces of the tetrahedron $ABCD$ and which have g as a generating line.

21. Since a Plücker surface is obtained from those lines of a general quadratic complex which meet an arbitrary line of space, it is natural to expect that the surface W, which is a specialisation of the Plücker surface, can be obtained in a similar way; and indeed Plücker himself showed† that a surface such as W is obtained from those lines of the quadratic complex which meet, not an arbitrary line of space but, a line of the complex itself. The surface W is also well known as the *singular surface* of a certain type of quadratic complex. If a quadratic complex is of the first kind, *i.e.* a general complex with characteristic [1 1 1 1 1 1], its singular surface is a Kummer surface. When the complex is so particularised that its characteristic is [2 1 1 1 1] its singular surface is a Plücker surface, and when it is further particularised so that its characteristic is [3 1 1 1] its singular surface is a surface such as W. It seems proper to allude to these two methods of obtaining W, since they are well-known and of long standing; but it is simpler to obtain W from a net of quadrics with a base line.

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* This method of generating the surface W is given by Zindler, *Encyclopädie der Math. Wiss.*, III, C. 8, 1105, foot-note 885.

† *Neue Geometrie des Raumes*, 216 *et seq.*

AN INTRODUCTION TO THE THEORY OF OPERATORS: (I) REAL OPERATORS. MODULUS.

By S. W. P. STEEN.

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This summer Dr. Powell asked me whether it was possible to have a theory of operators or transformations in Hilbert space without making any reference to or use of the elements on which the operators or transformations acted. The following pages contain an introduction to the theory of real operators which answers Dr. Powell's question in the affirmative. We consider here a set of entities, which we call real operators, which satisfy six groups of axioms; namely axioms of order, addition, subtraction, multiplication, division, and continuity. These axioms are satisfied by real numbers, and so they do not contradict themselves; further, they differ from the axioms satisfied by real numbers by the omission* of the tricotomy (*i.e.* if α and β are two of the entities, then either $\alpha > \beta$, $\alpha < \beta$, or $\alpha = \beta$). A group of entities which satisfy the axioms are permutable self-adjoint transformations in Hilbert space, more particularly all transformations of the type

$$\big(F(A)f, g\big) = \int_{-\infty}^{\infty} F(\lambda)\, d(E_\lambda f, g),$$

where E_λ is a given resolution of the identity†. The reader acquainted with this theory will find the following representation of the axioms helpful‡. Real operators may be represented by permutable self-adjoint transforma-

* Induction is, of course, used. It could have been introduced as an axiom for integral operators thus: (i) If $E^2 = E$, $E \neq 0$, then E is an integral operator; and, if N is an integral operator, then so is $N+E$. (ii) A set of integral operators which contains all $E^2 = E$, $E \neq 0$, and which contains $N+E$ whenever it contains N, contains all integral operators. This yields a principle of induction, but it seems more useful to use the ordinary integers, and the usual principle of induction.

† For details see Stone, "Linear transformations in Hilbert space", *American Math. Soc. Colloquium Publications*, 15 (1932).

‡ The axioms are also satisfied by certain classes of non-normal transformations in Hilbert space, *e.g.* those associated with a complete set of biorthogonal functions. For transformations acting on functions a change of variable or metric frequently converts a self-adjoint transformation into one of this type.

tions in Hilbert space. $A > O$ corresponds to A being positive definite, *i.e.* $(Af, f) \geqslant 0$ for all f and not always equal to zero. $A+B$, AB mean the closed linear extensions of these transformations. $A = B$ means that A and B are identical. $A^2 > O$ is satisfied if A is self-adjoint. An operator being complete corresponds to a self-adjoint transformation not having zero as a characteristic number. If A_n is a sequence of transformations, and if, for a given resolution of the identity E_λ,

$$(A_n f, g) = \int_{-\infty}^{\infty} F_n(\lambda)\, d(E_\lambda f, g),$$

then l.u.b. A_n is the transformation A defined by

$$(Af, g) = \int_{-\infty}^{\infty} \text{l.u.b.}\, F_n(\lambda)\, d(E_\lambda f, g).$$

The reader who is not acquainted with the theory of linear transformations in Hilbert space will find the following representation of the axioms useful. Real operators may be represented by real measurable functions defined in $(-\infty, \infty)$. $A > O$ corresponds to $F(\lambda) \geqslant 0$ for all λ and $F(\lambda) \neq 0$ in a set of positive measure. If A is represented by $F(\lambda)$ and if B is represented by $G(\lambda)$, then $A+B$, AB are represented by $F(\lambda)+G(\lambda)$, $F(\lambda) \cdot G(\lambda)$ respectively. A being complete corresponds to $F(\lambda) \neq 0$ almost everywhere. l.u.b. A_n corresponds to $F(\lambda) = \text{l.u.b.}\, F_n(\lambda)$. $A = B$ corresponds to $F(\lambda) - G(\lambda) = 0$ almost everywhere.

The position of the axiom $AB = BA$ is curious. Since we are concerned here only with sets of permutable entities, we include it in the list of axioms. But it is doubtful whether it could be omitted from the list without considerable alteration of the remaining axioms. In any case, the possible representations of the abstract entities would be totally different.

In this paper we prove the following theorem: *If A is a real operator, then there exists a sequence of operators E_λ, with the properties $E_\lambda{}^2 = E_\lambda$, $E_\lambda E_\mu = E_\mu$, if $\mu < \lambda$, $E_\lambda \to I$ as $\lambda \to +\infty$, $E_\lambda \to O$ as $\lambda \to -\infty$, $E_{\lambda+\epsilon} \to E_\lambda$ as $\epsilon \to 0_+$ (O is the null element of addition and I the null element of multiplication), and*

$$A = \int_{-\infty}^{\infty} \lambda\, dE_\lambda.$$

Since one of our axioms is an axiom of continuity or completeness, it is possible to define limit and to develop a theory of infinite series, differentiation, integration, etc. To prevent this paper running into too great length we proceed only as far as the above theorem, and only give such properties of integrals, etc., as seem necessary for this purpose.

The connection of the subject matter of this paper with modern algebra* is as follows. An algebra is a set of entities A, B, ... which can be combined together by addition to form sums $A+B$, ..., and by multiplication to form products AB, ..., and which can be multiplied by real numbers c to form products cA, ... Addition is commutative and associative and the inverse operation of subtraction is always possible and yields a unique result. Multiplication is associative and distributative (but not necessarily commutative, except that every real number commutes with every element of the algebra). An algebra so defined necessarily possesses a null element O, but it differs from a field (Körper) in that not every element possesses a unique reciprocal. The theory considered in this paper may be said to be a commutative, semi-ordered, continuous (or complete) algebra with a unit and without nilpotent elements. By an algebra being ordered we mean that there is a relation (denoted by $>$) such that, if A and B are any two elements of the algebra, then either $A > B$, $B > A$, or else $A = B$. By semi-ordered we mean that this relation holds between certain pairs of elements only. An algebra is said to be continuous (or complete) if every bounded set has a least upper bound. If a commutative algebra has a finite basis, a unit, and no nilpotent elements, then any element A can be represented† as $A = \sum_{n=1}^{m} a_n E_n$, where $E_n{}^2 = E_n$, $E_n E_m = 0$, $n \neq m$. If an algebra does not possess a finite basis, this is not in general true (*e.g.* maximal symmetric, but not self-adjoint transformations in Hilbert space). The theorem proved in this paper then allows us to extend this result if the algebra is commutative, semi-ordered, and continuous, with a unit and without nilpotent elements.

A set of entities which satisfy the axioms may be constructed from the class of all equivalent ordered (or positioned) classes of classes of unordered elements (or members), in a similar manner to the construction of real numbers from the class of all equivalent classes of unordered elements. By ordered class we mean that the members of the class have a definite position or place in the class, and two such classes are equivalent if members in corresponding positions can be put into (1, 1) correspondence.

To assist the reader we now give an outline of the proof of the theorem $A = \int \lambda dE_\lambda$. We first define

$$|A| = \text{l.u.b.}(A, O) + \text{l.u.b.}(-A, O) = A_+ + A_-,$$

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* I am indebted to a Referee for pointing out to me this connection with modern algebra.

† Dickson, *Linear algebras* (*Camb. Math. Tracts*, No. 16, 1914), 57.

and we show that $A = A_+ - A_-$, $A_+ A_- = O$. A_+ we could call the positive part of A, and $-A_-$ the negative part of A. If A is represented by $\Sigma x_i \mathbf{e}_i$, where x_i is a real number, and $\mathbf{e}_i$ a set of independent vectors, then $|A| = \Sigma |x_i| \mathbf{e}_i$. We then define majors and minors. If $A_1^2 \geqslant A_1 > O$, we call A_1 a major; and if $A_2^2 \leqslant A_2$, we call A_2 a minor. We then show that, if $A > O$, then $A = A_1 + A_2$, $A_1 A_2 = O$; A_1 is like the part of A that is greater than I, and A_2 is like the part of A that is less than I; but, though $A_2 \leqslant I$ for all A, yet $A_1 \geqslant I$ only if $A \geqslant I$. If $A > O$, we then define $A^{1/n} = \text{l.u.b.}(\mathfrak{B})$, where $\mathfrak{B}$ is the set of all B with $O < B$, $B^n \leqslant A$, and show that $(A^{1/n})^n = A$, and that $A^{1/n}$ is unique. This is like the corresponding definition in the case of real numbers. We then obtain the result that, if $A > O$, $A = A_1 + A_2$, where A_1 is a major and A_2 is a minor, then $A^{1/n} = A_1^{1/n} + A_2^{1/n}$; further, that $\lim_{n\to\infty} A_1^{1/n} = E_1$, where $E_1^2 = E_1$; and that $\lim_{n\to\infty} A_2^{1/n} = E_2$, where $E_2^2 = E_2$. These correspond to: if $F_1(\lambda) \geqslant 1$ or $= 0$, then $\lim_{n\to\infty} \{F_1(\lambda)\}^{1/n} = 1$ or 0; and similarly for $0 \leqslant F_2(\lambda) \leqslant 1$. We then show that $E_1 E_2 = E_1 A_2 = E_2 A_1 = O$; $\lim_{n\to\infty} A^{1/n} = E_1 + E_2 = E$, where $E^2 = E$, $AE = A$. The main part of the proof now follows. [§ XII, Corollary (xi).] We define $E_- = \text{l.u.b.}(\mathfrak{E})$, where $\mathfrak{E}$ is the set of all $E^2 = E$ for which $A_+ E = O$, and show that $E_-^2 = E_-$, $A_+ E_- = O$, $A_- E_- = A_-$. It was in order to prove this last result that it was necessary to introduce major and minors, $A^{1/n}$, etc. We then define E_λ as the E_- of $A - \lambda I$, and show that it has the required properties.

The reader will notice that use has been made of the axioms in Caratheodory's book "Vorlesungen über reele Funktionen", also of the book of Stone already cited, numerous writings of J. v. Neumann* and F. Riesz† on Hilbert space, and a paper by Wecken‡.

Henceforth the notation is as follows: A, B, ..., Z denote real operators; $a, b, \ldots, z$ denote real numbers; in particular k, p, q, n, m denote integers (positive, negative, or zero according to the context). $\mathfrak{A}, \mathfrak{B}, \ldots, \mathfrak{Z}$ denote sets of operators. The axioms will be referred to by their numbers, *e.g.* § I (iii); the corollaries and definitions by the paragraph in which they occur, *e.g.* § II, Corollary (v), § VII, Definition (i), etc. Many of the corollaries are very easy to prove, and in these cases the proof is omitted. No knowledge of Hilbert space is assumed.

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* *Math. Annalen*, 102 (1930), 49–131, in particular.

† *Acta Litt. ac. Sci. Szeged.*, 5 (1930), 23–54, in particular.

‡ *Math. Annalen*, 110 (1934), 722–725.

I.

We consider a set of elements, which we call real operators, and which satisfy the following axioms:

Order. (i) If $A = B$, then $B = A$.

(ii) If $A = B$ and $B = C$, then $A = C$.

(iii) There are operators A and B such that $A \neq B$*.

(iv) If $A > B$, then $B \not> A$.

(v) If $A > B$ and $B = C$, then $A > C$.

(vi) If $A > B$ and $B > C$, then $A > C$†.

Definition (i). If $A > B$, we write $B < A$.

Definition (ii). If $A \geqslant B$, we write $B \leqslant A$.

Thus if $A \not> B$, then $B \not< A$, and conversely.
Also if $A \not\geqslant B$, then $B \not\leqslant A$, and conversely.

Note. $\not<$ does not imply $\geqslant$, $\not\leqslant$ does not imply $>$; also $\not<$ is different from $\not>$, and $\not\leqslant$ is different from $\not\geqslant$.

COROLLARY (i). $A = A$.

COROLLARY (ii). *If* $A_1 = A_2$, $A_2 = A_3$, ..., $A_{n-1} = A_n$, *then* $A_1 = A_n$.

The proof of this and similar corollaries is by induction for the integer n.

COROLLARY (iii). *If* $A_1 = A_2$, ..., $A_{r-1} = A_r$, $A_r \neq A_{r+1}$, $A_{r+1} = A_{r+2}$, ..., $A_{n-1} = A_n$, *then* $A_1 \neq A_n$.

COROLLARY (iv). *The relation* $>$ *is not identical with the relation* $=$.

For if it were, then §I (i) and §I (iv) would contradict each other. [§I (iii) is required so that the relation $=$ cannot mean "identical with or not identical with".]

COROLLARY (v). *If* $A_1 > A_2$, $A_2 > A_3$, ..., $A_{n-1} > A_n$, *then* $A_1 > A_n$.

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* $\neq$ means that $=$ is false, and similarly for $\not>$, $\not\geqslant$, etc.

† These axioms are all independent. They are satisfied among other things by functions defined in $-\infty, \infty$. $=$ may be interpreted as $f(x) = g(x)$ for all x, or else for almost all x. $>$ may be interpreted as $f(x) \geqslant g(x)$ for almost all x and $f(x) > g(x)$ in a set of positive measure, or else it may be interpreted as $f(x) > g(x)$ for $|x| > r_0$ for some r_0.

COROLLARY (vi). *If* $A_1 \geqslant A_2$, $A_2 \geqslant A_3$, ..., $A_{n-1} \geqslant A_n$, *then* $A_1 \geqslant A_n$.

COROLLARY (vii). *If* $A \not\leqslant B$ *and* $B \geqslant C$, *then* $A \not\leqslant C$.

For if $A \leqslant C$, then from $C \leqslant B$, § I (ii), § I (v), and § I (vi) we obtain a contradiction.

COROLLARY (viii). *If* $A \not\geqslant B$ *and* $B \leqslant C$, *then* $A \not\geqslant C$.

COROLLARY (ix). *If* $A_1 \leqslant A_2$, $A_2 \leqslant A_3$, ..., $A_{r-1} \leqslant A_r$, $A_r \not\geqslant A_{r+1}$, $A_{r+1} \leqslant A_{r+2}$, ..., $A_{n-1} \leqslant A_n$, *then* $A_1 \not\geqslant A_n$.

For $A_1 \leqslant A_r$, $A_r \not\geqslant A_{r+1}$, then from § I, Corollary (vii), $A_1 \not\geqslant A_{r+1}$, $A_{r+1} \leqslant A_n$, and so from § I, Corollary (viii), the result follows.

COROLLARY (x). *If* $A \not< B$ *and* $B = C$, *then* $A \not< C$.

II.

Addition. (i) $A+B$ is an operator.

(ii) $A+B = B+A$.

(iii) $(A+B)+C = A+(B+C)$.

(iv) If $A > B$, then $A+C > B+C$.

(v) If $A = B$, then $A+C = B+C$.

(vi) If $A \not\geqslant B$, then $A+C \not\geqslant B+C$.

COROLLARY (i). $A_1+\ldots+A_n$ *is an operator, and the order and bracketing of the terms is immaterial.*

Proof by induction.

COROLLARY (ii). *If* $A \not> B$, *then* $A+C \not> B+C$.

For $\not>$ means $\not\geqslant$ or $=$.

COROLLARY (iii). *If* $A+C > B+C$, *then* $A > B$.

For, if $A \not> B$, we obtain a contradiction with § II, Corollary (ii).

COROLLARY (iv). *If* $A+C = B+C$, *then* $A = B$.

For either $A = B$, $A > B$, or $A \not\geqslant B$, and in the last two cases we obtain contradictions.

COROLLARY (v). *If* $A = A'$ *and* $B = B'$, *then* $A+B = A'+B'$.

COROLLARY (vi). *If* $A > A'$ *and* $B \geqslant B'$, *then* $A+B > A'+B'$.

COROLLARY (vii). *If* $A_1 \geqslant B_1$, $A_2 \geqslant B_2$, ..., $A_n \geqslant B_n$, *then*

$$A_1+\ldots+A_n \geqslant B_1+\ldots+B_n,$$

where the equality sign holds if and only if $A_k = B_k$ *for* $k = 1, 2, \ldots, n$.

Definition (i). $A+\ldots+A = nA$, $A = 1 \,.\, A$.
(n times.)

COROLLARY (viii). $nA+mA = (n+m)A$, $n(mA) = nm \,.\, A$.

COROLLARY (ix). *If* $A_1 \geqslant B_1$, ..., $A_{r-1} \geqslant B_{r-1}$, $A_r \nleqslant B_r$, $A_{r+1} \geqslant B_{r+1}$, ..., $A_n \geqslant B_n$, *then* $A_1+\ldots+A_n \nleqslant B_1+\ldots+B_n$.

III.

Subtraction. (i) If A and B are given, then there is a C such that

$$A = B+C.$$

COROLLARY (i). C *is unique.*

For if $A = B+C$ and $A = B+C'$, then $C = C'$ by § II, Corollary (iv).

COROLLARY (ii). *There exists an operator* O, *such that* $A+O = A$ *for all* A.

In § III (i) take $A = B$, then $A = A+C_A$. Also take $A = B = D$, then $D = D+C_D$; further $D = A+B$ for some unique B, hence

$$A+B = A+B+C_A,$$

i.e. $D = D+C_A$, hence $C_A = C_D$ for all D. We write $C_A = O$.

Definition (i). If $A > O$, we call A positive.

Definition (ii). If $A < O$, we call A negative.

COROLLARY (iii). *If* $B > O$, *then* $A+B > A$,

$B < O$, *then* $A+B < A$,

$B \lessdot O$, *then* $A+B \lessdot A$,

$B \gtrdot O$, *then* $A+B \gtrdot A$.

For, if $B > O$, then $A+B > A+O = A$, etc.

COROLLARY (iv). *If* $A = B+C$, *then* $C > O$, *if* $A > B$,

$C < O$, *if* $A < B$,

$C \not> O$, *if* $A \not> B$,

$C \not< O$, *if* $A \not< B$.

Definition (iii). If $O = B+C$, we write $C = -B$.

COROLLARY (v). $B = -(-B)$.

COROLLARY (vi). $-O = O$.

COROLLARY (vii). *If* $B > O$, *then* $-B < O$,

$B < O$, *then* $-B > O$,

$B \not> O$, *then* $-B \not< O$,

$B \not< O$, *then* $-B \not> O$.

Definition (iv). If $A = B+C$, we write $C = A-B$.

For $C = A+(-B)$.

COROLLARY (viii). $-(A_1+\ldots+A_n) = (-A_1)+(-A_2)+\ldots+(-A_n)$.

This we write as $-A_1-A_2-\ldots-A_n$.

COROLLARY (ix). $(-nA) = (-A)+(-A)+\ldots+(-A) = n(-A)$.
n times.

This we write as $-nA$.

Definition (v). $0 \,.\, A = O$.

COROLLARY (x). $nA+mA = (n+m)A$, $n(mA) = (nm)A$; *these are true whenever n, m are positive integers, negative integers, or zero.*

IV.

Multiplication. (i) AB is an operator, also written as $A \,.\, B$ and $A \times B$.

(ii) $AB = BA$.

(iii) $A(BC) = (AB)C$.

(iv) $A(B+C) = AB+AC$.

(v) If $A > O$ and $B > O$, then $AB \geqslant O$.

(vi) If $A \neq O$, then $A \,.\, A = A^2 > O$.

(vii) If $B = C$, then $AB = AC$.

COROLLARY (i). *$\times$ is different from $+$.*

For if $\times$ were identical with $+$, then $A(B+C)=A+B+C$, and $AB+AC=2A+B+C$; hence $A=O$ for all A, which contradicts §I (iii).

COROLLARY (ii). *$A_1 A_2 \dots A_n$ is an operator, and the order and the bracketing is immaterial.*

COROLLARY (iii).

$$(A_1+\dots+A_n)(B_1+\dots+B_m)=A_1B_1+\dots+A_nB_1+A_1B_2+\dots+A_nB_n.$$

COROLLARY (iv). *If $A_1>O, \dots, A_n>O$, then $A_1 \dots A_n \geqslant O$.*

COROLLARY (v). *If $A_1=B_1, \dots, A_n=B_n$, then $A_1 \dots A_n = B_1 \dots B_m$.*

COROLLARY (vi). *$A.O=O$, for all A.*

For $AB=A(B+O)=AB+AO$, whence $AO=O$.

COROLLARY (vii). *If $C>O$ and $A>B$, then $AC \geqslant BC$,*

$C<O$ and $A>B$, then $AC \leqslant BC$.

For $A=B+D$, where $D>O$, if $A>O$, hence $CD \geqslant O$, and the first result follows.

COROLLARY (viii). *If $A>O$ and $B>O$, then $(-A)B \leqslant O$ and $(-A)(-B)=AB \geqslant O$.*

For $O=O.B=\big(A+(-A)\big)B=AB+(-A)B$, thus

$$(-A)B=-(AB),$$

and the result follows from §III, Corollary (vii). Further,

$$(-A)(-B)=-\big(A(-B)\big)=-\big((-B).A\big)=-(-AB)=AB \geqslant O.$$

COROLLARY (ix). *If $A>O$ and $B<O$, then $AB \leqslant O$,*

$A<O$ and $B<O$, then $AB \geqslant O$.

Definition (i). $A^n = A.A \dots A$ (n factors), $A^1=A$.

COROLLARY (x). $A^n \times A^m = A^{n+m}$, $(A^n)^m = A^{nm}$.

COROLLARY (xi). *If* $A_1 \geqslant B_1 \geqslant O$, ..., $A_n \geqslant B_n \geqslant O$, *then*

$$A_1 \dots A_n \geqslant B_1 \dots B_n \geqslant O.$$

COROLLARY (xii). *If* $A \geqslant B > O$, *then* $AB > O$.

For $AB \geqslant B^2 > O$.

COROLLARY (xiii). *If* $A \neq O$, *then* $A^n \neq O$.

For if $2^m > n$, and if $A^n = O$, then $A^{2^m} = O$; whence

$$A^{2^{m-1}} = A^{2^{m-2}} = \dots = A = O.$$

There are thus no nilpotent operators except O.

V.

Definition (i). If $BD = O$ implies that $D = O$, then B is called a complete operator*.

Division. (i) There is at least one complete operator.

(ii) If B is complete and if A is any operator, then $A = BC$, for some C.

COROLLARY (i). *C is unique.*

For if $A = BC$ and $A = BC'$, then $B(C - C') = O$, whence $C = C'$.

COROLLARY (ii). *There exists an operator I such that $A = I.A$ for all A.*

We first prove the result for complete operators. Let B be complete, then from § V (ii), with $A = B$, we obtain $B = BC_B$, for some unique C_B; further, if B' is another complete operator†, then $B' = B'C_{B'}$, but $B' = AB$ for some A, hence $B' = B'C_B$, and so $B'(C_B - C_{B'}) = O$, *i.e.* $C_{B'} = C_B$ for all B'. Now let A be any operator, then $A = BD$, for some unique D, whence $BD = BDC_B$, *i.e.* $A = AC_B$. We write $I = C_B$.

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* We employ the term usual in the theory of Hilbert space rather than the term non-nilfactor of algebra.

† There is more than one complete operator, for if B is one and if $A > O$ then $B^2 + A$ is another. For if $(B^2 + A)D = O$, then $(B^2 + A)D^2 = O$, but $B^2D^2 \leqslant (B^2 + A)D^2 = O$, *i.e.* $B^2D^2 = O$, whence $BD = O$, and so $D = O$.

COROLLARY (iii). $I > O$.

For $I^2 = I$ and $I^2 > O$, since $I \neq O$; for if $I = O$, then

$$A = A \,.\, I = A \,.\, O = O,$$

i.e. every operator would be O, and this contradicts § I (iii).

COROLLARY (iv). *I is complete.*

For if $I \,.\, C = O$, then $C = O$.

COROLLARY (v). *If $A > O$, then $I + A$ is complete. If $A > B > O$, then A is complete if B is complete. If B is complete, then $-B$ is complete.* (See p. 370, foot-note †.)

COROLLARY (vi). *If B is complete, then B' is complete, where $BB' = I$.*

For if $B' D = O$, then $B^2 B' D = O$, *i.e.* $BD = O$, whence $D = O$.

Definition (ii). We write $B^{-1} = B'$, $(B^{-1})^n = B^{-n}$, $B^0 = I$; here B must be complete.

COROLLARY (vii). $B^{-n} \,.\, B^{-m} = B^{-(n+m)}$, $(B^{-n})^{-m} = B^{nm}$; *here B is complete.*

COROLLARY (viii). *If B is complete, then $B^n \,.\, B^m = B^{n+m}$, $(B^n)^m = B^{nm}$ hold whenever n, m are positive integers, negative integers, or zero.*

COROLLARY (ix). *If C is complete and if $CA = CB$, then $A = B$.*

For $C^{-1} CA = C^{-1} CB$.

COROLLARY (x). *If C is complete and if $C > O$, then $CA > O$ if $A > O$, $CA < O$ if $A < O$, $CA \not> O$ if $A \not> O$, $CA \not< O$ if $A \not< O$.*

For if $A > O$, then $CA \geqslant O$, and $CA = O$ implies $A = O$, hence $CA > O$; if $A \not> O$ and $C > O$, then $C^{-1} = C^{-2} C > O$, hence if $CA \geqslant O$, then $C^{-1} CA \geqslant O$, which is a contradiction.

COROLLARY (xi). *If C is complete and if $C < O$, then $CA < O$ if $A > O$, $CA > O$ if $A < O$, $CA \not< O$ if $A \not> O$, $CA \not> O$ if $A \not< O$.*

COROLLARY (xii). *If C is complete and if $C > O$ and if $A = CB$, then $B > O$ if $A > O$, $B < O$ if $A < O$, $B \not> O$ if $A \not> O$, $B \not< O$ if $A \not< O$.*

COROLLARY (xiii). *If C is complete and if $C < O$ and if $A = CB$, then $B < O$ if $A > O$, $B > O$ if $A < O$, $B \not< O$ if $A \not> O$, $B \not> O$ if $A \not< O$.*

Definition (iii). If B is complete and if $A = BC$, we write $C = A/B$.

COROLLARY (xiv). *If B and B' are both complete, then*

$$\frac{A}{B}+\frac{A'}{B'}=\frac{AB'+A'B}{BB'}, \qquad \frac{A}{B}\times\frac{A'}{B'}=\frac{AA'}{BB'}.$$

Note. $$B^{-1}=\frac{I}{B}, \qquad I=\frac{B}{B}, \qquad \frac{A'}{B'}=\frac{A'B}{B'B}.$$

Definition (iv). If $nA=mC$, we write $C=n/m\,A$, $m\neq O$.

This equation is soluble for C, for it is $nA=mI\,.\,C$, and mI is complete.

COROLLARY (xv).

$$\frac{p}{q}A+\frac{p'}{q'}A=\left(\frac{p}{q}+\frac{p'}{q'}\right)A, \qquad \frac{p}{q}\left(\frac{p'}{q'}A\right)=\frac{p}{q}\,\frac{p'}{q'}\,.A.$$

VI.

Definition (i). If $\mathfrak{A}$ is a set of operators (A) and if $\mathfrak{B}$ is a set of operators (B), then by $\mathfrak{A}+\mathfrak{B}$ we mean the set of operators $A+B$, for all possible combinations. Similarly by $\mathfrak{A}\times\mathfrak{B}$ we mean the set $A\times B$, for all possible combinations. In case $\mathfrak{B}$ consists of a single operator B, we shall write B instead of $\mathfrak{B}$. It is to be observed that $(\mathfrak{A}+\mathfrak{B})^2\neq\mathfrak{A}^2+2\mathfrak{A}\mathfrak{B}+\mathfrak{B}^2$, $(B+C)\,\mathfrak{A}\neq B\mathfrak{A}+C\mathfrak{A}$.

Continuity. If $\mathfrak{A}$ is a set of operators (A) and if $A\leqslant K$ for all A of $\mathfrak{A}$, where K is some fixed operator, then

(i) there exists an operator U, such that $A\leqslant U$, for all A,

(ii) if $V\not\geqslant U$, then there is an A of $\mathfrak{A}$ such that $A\not\leqslant V$.

Definition (ii). K we call an upper bound of the set $\mathfrak{A}$.

Definition (iii). U we call a least upper bound of the set $\mathfrak{A}$, and write

$$U=\text{l.u.b. }(\mathfrak{A}).$$

COROLLARY (i). *If $A_1, \ldots, A_n$ are operators, then* l.u.b. $(A_1, \ldots, A_n)$ *exists.*

It suffices to show that there is an operator B such that $A_i\leqslant B$, for $i=1, 2, \ldots, n$. Since $(A_i-I)^2\geqslant O$, then $A_i\leqslant\frac{1}{2}(I+A_i^2)$. Hence

$$B=\tfrac{1}{2}I+\tfrac{1}{2}\sum_{i=1}^{n}A_i^2$$

satisfies all requirements.

COROLLARY (ii). *$U \leqslant K$, i.e. any* l.u.b. $(\mathcal{A}) \leqslant$ *any upper bound of* $\mathcal{A}$.

Let $-X$ be a l.u.b. $(-U, -K)$; then $X \leqslant U$, $X \leqslant K$ (*i.e.* $-X \geqslant -U$, $-X \geqslant -K$). Further, if $Y \not\leqslant X$ (*i.e.* $-Y \not\geqslant -X$), then either $U \not\geqslant Y$ (*i.e.* $-U \not\leqslant -Y$) or else $K \not\geqslant Y$. Now suppose that $K \not\geqslant U$; then $X \neq U$; for $X = U$ implies that $U \leqslant K$. Since $X \leqslant U$, then $X < U$; then, by § VI (ii), there is an A of the set $\mathcal{A}$ such that $A \not\leqslant X$. Thus we may take $Y = A$, and conclude that either $U \not\geqslant A$ or else $K \not\geqslant A$, and both of these are contradictions. Thus the hypothesis $K \not\geqslant U$ has led to a contradiction; hence $K \geqslant U$.

COROLLARY (iii). *U is unique.*

For suppose that there were two l.u.b.'s U and U'. Then $A \leqslant U$ and $A \leqslant U'$ for all A of $\mathcal{A}$. If we consider $U =$ l.u.b. $(\mathcal{A})$ and U' as an upper bound, we obtain from § VI, Corollary (ii), $U \leqslant U'$; similarly $U' \leqslant U$. If $U \neq U'$, then, by § I (iv), $U \not> U'$, and this contradicts $U' < U$.

COROLLARY (iv). *If $\mathcal{A}$ is a set of operators (A), and if $A \geqslant H$ for all A of $\mathcal{A}$, then there exists an operator L such that*

(i) *$A \geqslant L$, for all A of $\mathcal{A}$,*

(ii) *if $M \not\leqslant L$, then there is an A of $\mathcal{A}$ such that $A \not\geqslant M$.*

For consider the set $(-A)$, then $-A \leqslant -H$ and the results follow from § VI (i) and (ii).

COROLLARY (iv). *$L =$* —l.u.b. $(-\mathcal{A})$, *$L \geqslant H$, and L is unique.*

Definition (iv). H we call a lower bound of the set $\mathcal{A}$.

Definition (v). L we call the greatest lower bound of the set $\mathcal{A}$, and write

$$L = \text{g.l.b. } (\mathcal{A}).$$

COROLLARY (vi). l.u.b. $(\mathcal{A})$+l.u.b. $(\mathcal{A}') =$ l.u.b. $(\mathcal{A}+\mathcal{A}')$.

Let $U =$ l.u.b. $(\mathcal{A})$ and $U' =$ l.u.b. $(\mathcal{A}')$; then $A+A' \leqslant U+U'$ for all A of $\mathcal{A}$ and all A' of $\mathcal{A}'$. Hence, by § VI (i), $U'' =$ l.u.b. $(\mathcal{A}+\mathcal{A}')$ exists, and, by § VI, Corollary (ii), $U'' \leqslant U+U'$. Suppose that $U'' < U+U'$, so that $U''-U < U'$; hence by § VI (ii) there exists an A' of $\mathcal{A}'$ such that $A' \not\leqslant U''-U$, *i.e.* $U''-A' \not\geqslant U$; hence by § VI (ii) there exists an A of $\mathcal{A}$ such that $A \not\leqslant U''-A'$, *i.e.* $A+A' \not\leqslant U''$; this is a contradiction, hence $U'' = U+U'$.

COROLLARY (vii). $\text{l.u.b.}\,(\mathfrak{A}_1)+\ldots+\text{l.u.b.}\,(\mathfrak{A}_n)=\text{l.u.b.}\,(\mathfrak{A}_1+\ldots+\mathfrak{A}_n)$.

COROLLARY (viii). *If B is complete and positive, then*

$$B\,.\,\text{l.u.b.}\,(\mathfrak{A})=\text{l.u.b.}\,(B\mathfrak{A}).$$

Let $U=\text{l.u.b.}\,(\mathfrak{A})$; then $A\leqslant U$, and so $BA\leqslant BU$. Thus, by §VI, $U'=\text{l.u.b.}\,(B\mathfrak{A})$ exists, and $U'\leqslant BU$. Suppose that $U'<BU$; then $U'/B<U$; hence, by §VI (ii), there is an A of $\mathfrak{A}$ such that $A\nleqslant U'/B$, *i.e.* $BA\nleqslant U'$, which is a contradiction; and so $U'=BU$.

COROLLARY (ix). *If $B>O$, then* $B\,.\,\text{l.u.b.}\,(\mathfrak{A})=\text{l.u.b.}\,(B\mathfrak{A})$.

For $I+B$ is complete and positive, further

$$(I+B)\,\text{l.u.b.}\,(\mathfrak{A})=\text{l.u.b.}\Big((I+B)\,\mathfrak{A}\Big)\leqslant\text{l.u.b.}\,(\mathfrak{A})+\text{l.u.b.}\,(B\mathfrak{A})$$
$$\leqslant\text{l.u.b.}\,(\mathfrak{A})\,.\,(I+B),$$

and the result follows.

COROLLARY (x). *If $\mathfrak{A}$ and $\mathfrak{B}$ are both sets of complete positive operators, then* $\text{l.u.b.}\,(\mathfrak{A})\times\text{l.u.b.}\,(\mathfrak{B})=\text{l.u.b.}\,(\mathfrak{A}\times\mathfrak{B})$.

Let $U=\text{l.u.b.}\,(\mathfrak{A})$ and $U'=\text{l.u.b.}\,(\mathfrak{B})$, then $A\leqslant U$, $B\leqslant U'$. Hence $AB\leqslant UU'$, so $U''=\text{l.u.b.}\,(\mathfrak{A}\times\mathfrak{B})\leqslant UU'$. Suppose that $U''<UU'$, so that $U''/U'<U$; thus there exists an A of $\mathfrak{A}$ such that $A\nleqslant U''/U'$, *i.e.* $U''/A\ngeqslant U'$. Hence there exists a B of $\mathfrak{B}$ such that $B\nleqslant U''/A$, *i.e.* $AB\nleqslant U''$, which is a contradiction; thus $U''=UU'$.

COROLLARY (xi). *If $\mathfrak{A}$ and $\mathfrak{B}$ are sets of positive operators, then*

$$\text{l.u.b.}\,(\mathfrak{A})\times\text{l.u.b.}\,(\mathfrak{B})=\text{l.u.b.}\,(\mathfrak{A}\times\mathfrak{B}).$$

For $I+\mathfrak{A}$, $I+\mathfrak{B}$ are sets of positive complete operators $I+A$, $I+B$. Hence*

$$\text{l.u.b.}\,(I+\mathfrak{A})\times\text{l.u.b.}\,(I+\mathfrak{B})=\text{l.u.b.}\,(I+\mathfrak{A})(I+\mathfrak{B})$$
$$\leqslant I+\text{l.u.b.}\,(\mathfrak{A})+\text{l.u.b.}\,(\mathfrak{B})+\text{l.u.b.}\,(\mathfrak{A}\mathfrak{B});$$

hence $\text{l.u.b.}\,(\mathfrak{A})\times\text{l.u.b.}\,(\mathfrak{B})\leqslant\text{l.u.b.}\,(\mathfrak{A}\mathfrak{B})$.

---

* In our notation $(I+\mathfrak{A})(I+\mathfrak{B})\neq I+\mathfrak{A}+\mathfrak{B}+\mathfrak{A}\mathfrak{B}$, for the expression on the left means all operators of the form $(I+A)(I+B)$, and that on the right means all operators of the form $I+A+B+A'B'$. Also it is clear that l.u.b. $(\mathfrak{C})$ is increased or unaltered if we increase the set $\mathfrak{C}$.

Again, $A \leqslant \text{l.u.b.}(\mathfrak{A})$ and $B \leqslant \text{l.u.b.}(\mathfrak{B})$; thus, since A and B are positive, we have $AB \leqslant \text{l.u.b.}(\mathfrak{A}) \times \text{l.u.b.}(\mathfrak{B})$, and hence

$$\text{l.u.b.}(\mathfrak{AB}) \leqslant \text{l.u.b.}(\mathfrak{A}) \times \text{l.u.b.}(\mathfrak{B}).$$

This proves the result.

COROLLARY (xii). *If $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ are sets of positive operators, then*

$$\text{l.u.b.}(\mathfrak{A}_1) \times \ldots \times \text{l.u.b.}(\mathfrak{A}_n) = \text{l.u.b.}(\mathfrak{A}_1 \times \ldots \times \mathfrak{A}_n).$$

COROLLARY (xiii). *If $A > H$ for all A of $\mathfrak{A}$ and if $B > O$, then*

$$B \text{ g.l.b.}(\mathfrak{A}) = \text{g.l.b.}(B\mathfrak{A}).$$

For $$B \text{ l.u.b.}(-\mathfrak{A}) = \text{l.u.b.}(-B\mathfrak{A}),$$

whence the result follows.

COROLLARY (xiv). *If $A_n \leqslant B_n < K$, then* $\underset{n}{\text{l.u.b.}}(A_n) \leqslant \underset{n}{\text{l.u.b.}}(B_n)$.

COROLLARY (xv). *If $H \leqslant A_n \leqslant B_n$, then* $\underset{n}{\text{g.l.b.}}(A_n) \leqslant \underset{n}{\text{g.l.b.}}(B_n)$.

COROLLARY (xvi). *If $A > O$ and if $\mathfrak{A}$ is the set $p/q\,A$ for all $p/q > 0$, then* $\text{g.l.b.}(\mathfrak{A}) = O$.

Suppose that $\text{g.l.b.}(\mathfrak{A}) = L \neq O$, since $p/q\,A > O$, then $L > O$.

Thus $(1/n)A \geqslant L$ for all n, *i.e.* $nL \leqslant A$. The set (nL) is consequently bounded above, and so by § VI the set (nL) has a least upper bound, U say; then $nL \leqslant U$ for all n; further, $U - L < U$. There is consequently an n for which $nL \nleqslant U - L$, *i.e.* $(n+1)L \nleqslant U$, which is a contradiction; hence* $L = O$.

COROLLARY (xvii). $\text{l.u.b.}(\mathfrak{A}) \leqslant \text{l.u.b.}(\mathfrak{A}')$, *if* $\mathfrak{A} \leqslant \mathfrak{A}'$†.

COROLLARY (xviii). $\text{g.l.b.}(\mathfrak{A}) \geqslant \text{g.l.b.}(\mathfrak{A}')$, *if* $\mathfrak{A} \leqslant \mathfrak{A}'$.

COROLLARY (xix). *If $\mathfrak{A}$ and $\mathfrak{A}'$ are sets of positive complete operators, then* $\text{g.l.b.}(\mathfrak{A}) \times \text{g.l.b.}(\mathfrak{A}') = \text{g.l.b.}(\mathfrak{A}\mathfrak{A}')$.

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* Axiom VI not only ensures the inclusion of certain elements but also excludes others, *e.g.* "infinitesimals"—or influences the meanings that can be attributed to $>$. Thus axioms I to V inclusive are satisfied by functions defined in $-\infty, \infty$ with $>$ interpreted as $f(x) - g(x) > 0$ for $|x| > x_0$, for some x_0. We then have $0 < 1/x < p/q$, for all $p/q > 0$. This possibility is excluded by axiom VI, as is shown by § VI, Corollary (xvi).

† $\mathfrak{A} \leqslant \mathfrak{A}'$ means that $\mathfrak{A}$ is part of $\mathfrak{A}'$.

COROLLARY (xx). $\text{g.l.b.}\,(\mathcal{A})+\text{g.l.b.}\,(\mathcal{A}')=\text{g.l.b.}\,(\mathcal{A}+\mathcal{A}')$.

COROLLARY (xxi). $\sum_{i=1}^{n}\text{g.l.b.}\,(\mathcal{A}_i)=\text{g.l.b.}\left(\sum_{i=1}^{n}\mathcal{A}_i\right)$.

COROLLARY (xxii). *If $\mathcal{A}$ and $\mathcal{A}'$ are sets of positive operators, then*

$$\text{g.l.b.}\,(\mathcal{A})\times\text{g.l.b.}\,(\mathcal{A}')=\text{g.l.b.}\,(\mathcal{A}\times\mathcal{A}').$$

COROLLARY (xxiii). *If $\mathcal{A}_i$, $i=1, 2, \ldots, n$, are sets of positive operators, then*

$$\prod_{i=1}^{n}\text{g.l.b.}\,(\mathcal{A}_i)=\text{g.l.b.}\left(\prod_{i=1}^{n}\mathcal{A}_i\right).$$

COROLLARY (xxiv). *If $A_n\leqslant A_{n+1}<K$, $B_n\leqslant B_{n+1}\leqslant K'$, then*

$$\underset{n\geqslant 0}{\text{l.u.b.}}\,(A_n+B_n)=\underset{n\geqslant 0}{\text{l.u.b.}}\,(A_n)+\underset{n\geqslant 0}{\text{l.u.b.}}\,(B_n).$$

COROLLARY (xxv). *If $A_n\geqslant A_{n+1}>H$, $B_n\geqslant B_{n+1}\geqslant H'$, then*

$$\underset{n\geqslant 0}{\text{g.l.b.}}\,(A_n+B_n)=\underset{n\geqslant 0}{\text{g.l.b.}}\,(A_n)+\underset{n\geqslant 0}{\text{g.l.b.}}\,(B_n).$$

VII.

Multiplication by real numbers.

Definition (i). $aI=\underset{p/q\leqslant a}{\text{l.u.b.}}\left(\frac{p}{q}I\right)$.

COROLLARY (i). $0I=O$.

COROLLARY (ii). *aI is complete, if $a\neq 0$.*

COROLLARY (iii). *$aI>O$ if $a>0$, $aI<O$ if $a<0$.*

COROLLARY (iv). $(a+b)I=aI+bI$.

COROLLARY (v). $(ab)I=a(bI)$.

COROLLARY (vi). $-(aI)=(-a)I$.

COROLLARY (vii). *If $aI>O$, then $a>0$; if $aI<O$, then $a<0$.*

COROLLARY (viii). *$(p/q)I$, as defined by* §VII, *Definition* (i), *is equal to $(p/q)I$ as defined by* §V, *Definition* (iv); *it also agrees with* §III, *Definition* (v).

Definition (ii). $aA = aI \,.\, A.$

COROLLARY (ix). $\underset{a>0}{\text{g.l.b.}}\,(aA) = O$, *if* $A > O$.

This follows from § VI, Corollary (xvi).

VIII.

Limits. If A_n is a sequence of operators, we adopt the following definitions:

Definition (i). If $H \leqslant A_n \leqslant K$, then

$$\overline{\lim_{n\to\infty}}\, A_n = \underset{n}{\text{g.l.b.}}\,\{\text{l.u.b.}\,(A_n, A_{n+1}, \ldots)\}.$$

Definition (ii). If $H \leqslant A_n \leqslant K$, then

$$\underline{\lim_{n\to\infty}}\, A_n = \underset{n}{\text{l.u.b.}}\,\{\text{g.l.b.}\,(A_n, A_{n+1}, \ldots)\}^*.$$

Definition (iii). *If* $H \leqslant A_n \leqslant K$, and if $\overline{\lim}\, A_n = \underline{\lim}\, A_n$, then we call their common value $\lim A_n$.

Note. We require the condition $H \leqslant A_n \leqslant K$, since otherwise the bounds in question might not exist.

Definition (iv). If $A_n = \sum_{m=0}^{n} B_m$, and if $\lim A_n$ exists, then we write

$$\lim_{n\to\infty} A_n = \sum_{m=0}^{\infty} B_m.$$

Definition† (v). If $H \leqslant A_\delta \leqslant K$ is a set of operators defined for $0 < \delta \leqslant \delta_0$, then

$$\overline{\lim_{\delta\to 0}}\, A_\delta = \underset{0<\delta\leqslant\delta_0}{\text{g.l.b.}}\left(\underset{0<\delta'\leqslant\delta}{\text{l.u.b.}}\, A_{\delta'}\right),$$

and

$$\underline{\lim_{\delta\to 0}}\, A_\delta = \underset{0<\delta\leqslant\delta_0}{\text{l.u.b.}}\left(\underset{0<\delta'\leqslant\delta}{\text{g.l.b.}}\, A_{\delta'}\right).$$

Definition (vi). If $\overline{\lim}\, A_\delta = \underline{\lim}\, A_\delta$, then we call their common value $\lim A_\delta$.

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* It is clear that $\overline{\lim_{n\to\infty}}\, A_n \geqslant \underline{\lim_{n\to\infty}}\, A_n$.

† It is assumed that δ, λ, μ, Λ are real numbers.

COROLLARY (i). *If* $A_n \leqslant A_{n+1} < K$, *then* $\lim_{n\to\infty} A_n = \text{l.u.b.}_n (A_n)$.

For $\text{l.u.b.}_{p\geqslant 0} (A_{n+p}) = \text{l.u.b.}_{n\geqslant 0} (A_n) = U$, say, also $\text{g.l.b.}_{p\geqslant 0} (A_{n+p}) = A_n$, and the result follows.

COROLLARY (ii). *If* $A_n \geqslant A_{n+1} > H$, *then* $\lim_{n\to\infty} A_n = \text{g.l.b.}_n (A_n)$.

COROLLARY (iii). *If* $A_\delta \leqslant A_{\delta'} < K$, *for* $\delta > \delta' > 0$, *then*

$$\lim_{\delta\to 0} A_\delta = \text{l.u.b.}_{\delta>0} (A_\delta).$$

COROLLARY (iv). *If* $A_\delta \geqslant A_{\delta'} > H$, *for* $\delta > \delta' > 0$, *then*

$$\lim_{\delta\to 0} A_\delta = \text{g.l.b.}_{\delta>0} (A_\delta).$$

COROLLARY (v). *If* $A_n > O$ *and if* $A_n \to A$ *and if* $B > O$, *then*

$$B \lim_{n\to\infty} A_n = \lim_{n\to\infty} BA_n.$$

Integrals. If A_x is defined for $a \leqslant x \leqslant b$, $-\infty \leqslant a < b \leqslant \infty$, and if $A_{x_1} \leqslant A_{x_2}$ for $x_1 < x_2$, and if B_x is defined for $a \leqslant x \leqslant b$, and if

$$H_{\Delta_k} \leqslant B_x \leqslant K_{\Delta_k}$$

for x in Δ_k, where Δ_k is the interval (x_k, x_{k+1}), $x_k < x_{k+1}$*; if, further,

$$H_\delta = \sum_{k=-\infty}^{\infty} H_k \Delta_k A_x \quad \text{and} \quad K_\delta = \sum_{k=-\infty}^{\infty} K_k \Delta_k A_x$$

both exist, where

$$H_k = \text{g.l.b.}_{x \text{ in } \Delta_k} (B_x), \quad K_k = \text{l.u.b.}_{x \text{ in } \Delta_k} (B_x);$$

finally, if

$$\lim_{\delta\to 0} H_\delta = \lim_{\delta\to 0} K_\delta,$$

then we call their common value

$$\int_a^b B_x \, dA_x.$$

In particular, if $B_x = f(x) \,.\, I$, we write

$$\int_a^b f(x) \, dA_x.$$

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* For a finite interval we need only take a finite number of points of subdivision. $\Delta_k A_r = A_{r_{k+1}} - A_{r_k}$.

IX.

Projections.

Definition (i). If $E^2 = E$, then we call E a projection*.

COROLLARY (i). *O and I are projections.*

COROLLARY (ii). *$E > O$, if $E \neq O$.*

For $E = E^2 > O$, if $E \neq O$.

COROLLARY (iii). *$I-E$ is a projection, if E is one. Thus $E \leqslant I$.*

For $(I-E)^2 = I-2E+E^2 = I-E$.

COROLLARY (iv). *E_1+E_2 is a projection, if $E_1 E_2 = O$.*

For $(E_1+E_2)^2 = E_1^2+2E_1E_2+E_2^2 = E_1+E_2$, if $E_1E_2 = O$.

COROLLARY (v). *$E_1 E_2$ is a projection.*

COROLLARY (vi). *E_1-E_2 is a projection, if $E_1 > E_2$.*

For if $E_1 > E_2$, then $E_1E_2 = E_2$; this follows since $E_1-E_2 > O$ and $E_2 > O$ give $E_1E_2 \geqslant E_2$; but $E_1 \leqslant I$, and so $E_1E_2 \leqslant I.E_2 = E_2$. Thus $E_1E_2 = E_2$. Now $(E_1-E_2)^2 = E_1^2-2E_1E_2+E_2^2 = E_1-E_2$.

COROLLARY (vii). *If $E_\lambda > E_\mu$ for $\lambda > \mu$, then* $\underset{\lambda<\Lambda}{\text{l.u.b.}}\,(E_\lambda)$ *and* $\underset{\lambda>\Lambda}{\text{g.l.b.}}\,(E_\lambda)$ *exist, and are projections*†.

Since $O \leqslant E_\lambda \leqslant I$, then the bounds exist. Write $E = \underset{\lambda<\Lambda}{\text{l.u.b.}}\,(E_\lambda)$, then $E \leqslant I$. Further,

$$E^2 = \left\{\underset{\lambda<\Lambda}{\text{l.u.b.}}\,(E_\lambda)\right\}^2 = \underset{\lambda,\,\mu<\Lambda}{\text{l.u.b.}}\,(E_\lambda E_\mu) = \underset{\nu<\Lambda}{\text{l.u.b.}}\,(E_\nu) = E \quad [\nu = \min(\lambda, \mu)],$$

hence the first result. The second follows in a similar manner, or by dealing with $\underset{\lambda>\Lambda}{\text{l.u.b.}}\,(1-E_\lambda)$.

COROLLARY (viii). *If $BE = O$ for all E, then $B = O$.*

For $B(I-E) = O$, *i.e.* $B = BE = O$.

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* Here we follow the language employed in the theory of transformations in Hilbert space, rather than the term idempotent which is employed in modern algebra.

† Λ may be $+\infty$ in the first and $-\infty$ in the second.

COROLLARY (ix). *If $A > O$ and if $F = \text{l.u.b.}(\mathfrak{E})$, where $\mathfrak{E}$ is the set of projections for which $AE = O$, then F is a projection and $AF = O$, $FE = E$.*

The set $\mathfrak{E}$ is not empty, for it contains O. Further, $E \leqslant I$, hence by §VI, F exists, and $F \geqslant E$, thus by §IX, Corollary (vi), $FE = E$, if F is a projection. Now

$$F^2 = \{\text{l.u.b.}(\mathfrak{E})\}^2 = \text{l.u.b.}\left((E_1 E_2)\right) \geqslant \text{l.u.b.}\left((E^2)\right)$$

$$= \text{l.u.b.}(E) = F.$$

On the other hand $O \leqslant F \leqslant I$, thus $F^2 \leqslant F$. We thus have $F^2 = F$, and so F is a projection. Finally,

$$AF = \text{l.u.b.}(A\mathfrak{E}) = \text{l.u.b.}(O) = O.$$

Definition (ii). A set E_λ of projections is called a resolution of the identity if

(*a*) $E_\lambda > E_\mu$, for $\lambda > \mu$,

(*b*) $\text{l.u.b.}\left((E_\lambda)\right) = I$,

(*c*) $\text{g.l.b.}\left((E_\lambda)\right) = O$,

(*d*) $\underset{\lambda > \mu}{\text{g.l.b.}}\left((E_\lambda)\right) = E_\mu$.

X.

Modulus.

Definition (i). $|A| = \text{l.u.b.}(A, O) + \text{l.u.b.}(-A, O) = A_+ + A_-$.

COROLLARY (i). $|A| \geqslant A$.

For l.u.b. $(A, O) \geqslant A$, and l.u.b. $(-A, O) \geqslant O$.

COROLLARY (ii). $|A| \geqslant -A$.

For l.u.b. $(A, O) \geqslant O$, and l.u.b. $(-A, O) \geqslant -A$.

COROLLARY (iii). $A_+ \geqslant O$, $A_- \geqslant O$.

COROLLARY (iv).

$$A = \frac{|A| + A}{2} - \frac{|A| - A}{2} = A_+ - A_-.$$

For $$|A|+A = \text{l.u.b.}\,(A, O)+\text{l.u.b.}\,(-A, O)+A$$
$$= \text{l.u.b.}\,(A, O)+\text{l.u.b.}\,(O, A) = 2A_+,$$
$$|A|-A = \text{l.u.b.}\,(A, O)+\text{l.u.b.}\,(-A, O)-A$$
$$= \text{l.u.b.}\,(O, -A)+\text{l.u.b.}\,(-A, O) = -2A_-.$$

COROLLARY (v). $|A|^2-A^2 = 4A_+A_- = O.$

For $$|A|^2-A^2 = (A_++A_-)^2-(A_+-A_-)^2 = 4A_+A_-.$$

Further
$$(|A|+A_+)(|A|+A_-) = |A|^2+|A|(A_++A_-)+A_+A_-$$
$$= 2|A|^2+A_+A_- \geqslant 2|A|^2.$$

Again,
$$(|A|+A_+).(|A|+A_-) = \text{l.u.b.}\,(A+|A|, |A|)\times\text{l.u.b.}\,(-A+|A|, |A|)$$
$$= \text{l.u.b.}\,(|A|^2-A^2, |A|^2-A|A|, |A|^2+A|A|, |A|^2)$$
$$\leqslant 2|A|^2.$$

Thus $A_+A_- = O.$

COROLLARY (vi). *If* $A = B-C$, $B \geqslant O$, $C \geqslant O$, $BC = O$, *then* B *and* C *are unique.*

Suppose that $A = B-C = B'-C'$, where $BC = B'C' = O$ and $B \geqslant O$, $C \geqslant O$, $B' \geqslant O$, $C' \geqslant O$. Then $B-B' = C-C'$, hence
$$(C-C')^2 = CB-CB'-C'B+C'B' = -(CB'+C'B),$$
thus $(C-C')^2 \leqslant O$, and so $C-C' = O$, hence the result.

COROLLARY (vii). *If* $A \nleqslant B$, $O < A$, $O < B$, *then* $A^n \nleqslant B^n$.

Let $B-A = C_+-C_-$, where $C_+ \geqslant O$, $C_- \geqslant O$, $C_+C_- = O$, then $C_- > O$; for if $C_- = O$, then $B-A \geqslant O$, contrary to hypothesis. If $A^n \leqslant B^n$, then $O \leqslant (B-A)(B^{n-1}+B^{n-2}A+\ldots+A^{n-1})$, hence
$$C_-(B-A)(B^{n-1}+B^{n-2}A+\ldots+A^{n-1}) \geqslant O,$$
i.e.
$$-C_-^2(B^{n-1}+B^{n-2}A+\ldots+A^{n-1}) \geqslant O,$$

thus $C_-^2(B^{n-1}+\ldots+A^{n-1})=O$; since all the terms are positive or zero we see that $C_-^2 B^i A^j = O$, if $i+j=n-1$.

Hence $C_-^2(B-A)^{n-1}=O$, *i.e.* $C_-^{n+1}=O$, thus $C_-=O$, § IV, Corollary (xiii). This is a contradiction, and hence $A^n \not\leqslant B^n$.

COROLLARY (viii). *If* $A>B>O$, *then* $A^n>B^n$.

For then $A \not\leqslant B$, and hence $A^n \not\leqslant B^n$; on the other hand, from $A>B$ we obtain $A^n \geqslant B^n$, hence the result.

COROLLARY (ix). *If* $B^n \leqslant A$, $O<A$, $O<B$, *then* $B \leqslant I+A$.

If $B \not\leqslant I+A$, then $B^n \not\leqslant (I+A)^n$, but $B^n \leqslant A \leqslant (I+A)^n$.

COROLLARY (x). $|A+B| \leqslant |A|+|B|$.

For $|A+B|$ is l.u.b. $(A+B, O)$+l.u.b. $(-A-B, O)$, and

$$\text{l.u.b. } (A+B, O) \leqslant \text{l.u.b. } (A+B, A, B, O)$$
$$\leqslant \text{l.u.b. } (A, O)+\text{l.u.b. } (B, O),$$

and the result follows.

COROLLARY (xi). $|AB| = |A| . |B|$.

For $$AB = A_+B_+ + A_-B_- - (A_+B_- + A_-B_+);$$

also $$(A_+B_+ + A_-B_-)(A_+B_- + A_-B_+) = O,$$

and each of the factors is greater than or equal to O.

COROLLARY (xii). *If* $|A| = O$, *then* $A = O$.

COROLLARY (xiii). $|aA| = |a| . |A|$.

For $|aA| = |a|A_+ + |a|A_- = |a| . |A|$.

XI.

Majors and Minors.

Definition (i). If $B>O$ and if $B^2 \geqslant B$, we call B a major.

Definition (ii). If $B>O$ and if $B^2 \leqslant B$, we call B a minor.

COROLLARY (i). *If* $B>O$, *then* $B=B_1+B_2$, *where* B_1 *is a major and* B_2 *is a minor, and where* $B_1 B_2 = O$.

Let $B_1 = \underset{n\geqslant 1}{\text{g.l.b.}}\,(B^n)$, $B_2 = B - B_1$.

Since $B > O$, then $B_1 \geqslant O$. Further,

$$B_1^2 = \underset{n,\,m\geqslant 1}{\text{g.l.b.}}\,(B^{n+m}) = \underset{n\geqslant 2}{\text{g.l.b.}}\,(B^n) \geqslant \underset{n\geqslant 1}{\text{g.l.b.}}\,(B^n) = B_1.$$

$$B_1 B_2 = B \underset{n\geqslant 1}{\text{g.l.b.}}\,(B^n) - \underset{n\geqslant 2}{\text{g.l.b.}}\,(B^n) = O.$$

$$B_2^2 = B^2 - 2BB_1 + B_1^2 = B^2 - B_1^2.$$

Now $B^{n+1} - B^2 \geqslant B^n - B$, for $n \geqslant 1$, since

$$O \leqslant B(B-I)^2 (B^{n-2} + B^{n-3} + \ldots + I) = B^{n+1} - B^2 - B^n + B,$$

thus $$\underset{n\geqslant 2}{\text{g.l.b.}}\,(B^n - B^2) \geqslant \underset{n\geqslant 1}{\text{g.l.b.}}\,(B^n - B),$$

and so $$B_1^2 - B^2 \geqslant B_1 - B, \quad i.e. \quad B^2 - B_1^2 \leqslant B - B_1,$$

thus $$O \leqslant B_2^2 \leqslant B_2.$$

COROLLARY (ii). *If* $A > O$, $n \leqslant m$, *then* $A^n \leqslant I + A^m$.

Let $A = A_1 + A_2$, where A_1 is a major and A_2 is a minor. Then $A^n = A_1^n + A_2^n \leqslant I + A_1^m + A_2^m$. For $A_2 \leqslant I$, since

$$O \leqslant I - 2A_2 + A_2^2 \leqslant I - A_2, \quad \text{and} \quad A_1 \leqslant A_1^2 \leqslant A_1^3 \leqslant \ldots \leqslant A_1^n.$$

COROLLARY (iii). *If* $A > O$, *then* $A^n \leqslant A + A^{n+1}$.

Definition (iii). If in § XI, Definition (i) and (ii), we omit the equality signs, then we shall speak of strict majors and strict minors.

XII.

n-th root.

Definition (i). If $A > O$, then $A^{1/n} = \text{l.u.b.}\,(\mathfrak{B})$, where $\mathfrak{B}$ is the set

$$O < B, \quad B^n \leqslant A.$$

Clearly $A^{1/n}$ exists; for $B \leqslant I + A$, and since $\left(A/(I+A)\right)^n \leqslant A$, the set $\mathfrak{B}$ is not empty.

COROLLARY (i). $(A^{1/n})^n = A$.

$(A^{1/n})^n = \text{l.u.b.}\left((B_1 B_2 \ldots B_n)\right)$, where $O < B_i$, $B_i^n \leqslant A$. Now, if $B_1 B_2 \ldots B_n \nleqslant A$, then $B_1^n B_2^n \ldots B_n^n \nleqslant A^n$; but $B_1^n B_2^n \ldots B_n^n \leqslant A^n$, and so $B_1 B_2 \ldots B_n \leqslant A$. Thus $(A^{1/n})^n \leqslant A$.

Suppose that $(A^{1/n})^n < A$. Let $b = \max_{0 \leqslant r \leqslant n} \frac{n!}{r!(n-r)!}$, take

$$K = \frac{A-(A^{1/n})^n}{nb\left(I+(A^{1/n})^n+A\right)} \quad [I+(A^{1/n})^n+A \text{ is complete}];$$

since $nb > 1$, we have $O < K < I$; thus $K^m \leqslant K$, for $m \geqslant 1$. Consequently

$$nK(A^{1/n})^{n-1}+\frac{n(n-1)}{2!}K^2(A^{1/n})^{n-2}+\ldots+K^n \leqslant bnK\left(I+(A^{1/n})^n\right) \leqslant A-(A^{1/n})^n.$$

Thus $$\left((A^{1/n})+K\right)^n \leqslant (A^{1/n})^n+A-(A^{1/n})^n = A;$$

hence l.u.b. ($\mathfrak{B}$), where $\mathfrak{B}$ is the set $O < B$, $B^n \leqslant A$, is greater than or equal to $A^{1/n}+K$, this contradicts the properties of $A^{1/n}$; hence $(A^{1/n})^n = A$.

COROLLARY (ii). *$A^{1/n}$ is unique, i.e. if $B^n = A$ and $O < B$, then $B = A^{1/n}$.*

Suppose that $B^n = A$, then $B \leqslant A^{1/n}$, since $A^{1/n}$ is the l.u.b. of such B's. If $B < A^{1/n}$, then $A = B^n < (A^{1/n})^n = A$. Thus $B = A^{1/n}$.

COROLLARY (iii). *If $O < A < B$, then $O < A^{1/n} < B^{1/n}$.*

Clearly $O \leqslant A^{1/n}$; if $O = A^{1/n}$, then $O = (A^{1/n})^n = A$, thus $A^{1/n} > O$. Further, $A^{1/n} \leqslant B^{1/n}$, for $A^{1/n} = \text{l.u.b.}\ (\mathfrak{B}_A)$, $B^{1/n} = \text{l.u.b.}\ (\mathfrak{B}_B)$, and $\mathfrak{B}_A \leqslant \mathfrak{B}_B$. If $A^{1/n} = B^{1/n}$, then $A = (A^{1/n})^n = (B^{1/n})^n = B$.

COROLLARY (iv). *If $O < B < B^2$, then $B^m < B^n$, for $n > m \geqslant 1$, if $B > B^2$, then $B^m > B^n$, for $n > m \geqslant 1$.*

If $O < B < B^2$, then $B \,.\, B^{n-1} \leqslant B^2 \,.\, B^{n-1}$, *i.e.* $B^n \leqslant B^{n+1}$; if for some $n = n_0$, $B^{n_0} = B^{n_0+1}$, then $B^{n_0+m} = B^{n_0+m+1}$, for all $m \geqslant 0$; thus $B^n = B^m$, for all $n \geqslant m \geqslant n_0$. In particular, $B^{2n} = B^n$, $n > n_0$, hence $B^2 = B$, for $B^2 = (B^{2n})^{1/n} = (B^n)^{1/n} = B$. Thus $B^m < B^n$, for $n > m \geqslant 1$. Similarly for the second case.

COROLLARY (v). *If $B = B_1+B_2$, $B_1 \geqslant O$, $B_2 \geqslant O$, $B_1 B_2 = O$, then*

$$B^{1/n} = B_1^{1/n}+B_2^{1/n}.$$

For $B_1^{1/n} B_2^{1/n} = O$, since it is l.u.b.$\big((C_1 C_2)\big)$, $O < C_1$, $O < C_2$, $C_1{}^n \leqslant B_1$, $C_2{}^n \leqslant B_2$, and so $O \leqslant C_1{}^n C_2{}^n \leqslant B_1 B_2 = O$, thus $C_1 C_2 = O$, and hence $B_1^{1/n} B_2^{1/n} = O$. Thus $(B_1^{1/n} + B_2^{1/n})^n = B_1 + B_2 = B$, and so

$$B^{1/n} = B_1^{1/n} + B_2^{1/n}.$$

COROLLARY (vi). *If A_1 is a major, then* $\lim_{n\to\infty} A_1^{1/n} = E_1$, $E_1{}^2 = E_1$.

For $A_1{}^n \geqslant A_1{}^m$, if $n \geqslant m$; thus $A_1^{1/m} \geqslant A_1^{1/n}$, on taking the (nm)-th root. The sequence $A_1^{1/n}$ is thus decreasing and consists of positive terms, and so $\lim_{n\to\infty} A_1^{1/n} = \text{g.l.b.}_{n\geqslant 1} (A_1^{1/n}) = E_1$, say. Now

$$E_1{}^2 = \text{g.l.b.}_{n,\,m\geqslant 1} (A_1^{1/n} A_1^{1/m}) = \text{g.l.b.}_{n\geqslant 1} (A_1^{2/n}) = \text{g.l.b.}_{n\geqslant 1} (A_1^{1/n}) = E_1,$$

for $A_1^{1/n} A_1^{1/m} \geqslant A_1^{2/n}$, if $n \geqslant m$, $A_1^{2/n} \geqslant A_1^{1/n}$ and $A_1^{2/2n} = A_1^{1/n}$.

Clearly a g.l.b. is unaltered if we cast out terms which are greater than one of the terms that is retained.

COROLLARY (vii). *If A_2 is a minor, then* $\lim_{n\to\infty} A_2^{1/n} = E_2$, $E_2{}^2 = E_2$.

For $A_2{}^n \leqslant A_2{}^m$, if $n \geqslant m$; hence $A_2^{1/m} \leqslant A_2^{1/n}$. The sequence $A_2^{1/n}$ is thus an increasing one and every term is less than or equal to I. Thus

$$\lim_{n\to\infty} A_2^{1/n} = \text{l.u.b.}_{n\geqslant 1} (A_2^{1/n}) = E_2,$$

say. Now

$$E_2{}^2 = \text{l.u.b.}_{n,\,m\geqslant 1} (A_2^{1/n} A_2^{1/m}) = \text{l.u.b.}_{n\geqslant 1} (A_2^{2/n}) = \text{l.u.b.}_{n\geqslant 1} (A_2^{2/2n}) = E_2.$$

COROLLARY (viii). *If $A = A_1 + A_2$, where A_1 is a major and A_2 is a minor, and if E_1 and E_2 are the projections of* § XII, *Corollary* (vi) *and* (vii), *then $E_1 E_2 = E_1 A_2 = E_2 A_1 = O$.*

For

$$\begin{aligned} E_1(I - E_2) &= \text{g.l.b.}_{n\geqslant 1} (A_1^{1/n}) \times \text{g.l.b.}_{n\geqslant 1} (I - A_2^{1/n}) \\ &= \text{g.l.b.}_{n,\,m\geqslant 1} (A_1^{1/n} - A_1^{1/n} A_2^{1/m}) = \text{g.l.b.}_{n\geqslant 1} (A_1^{1/n}) = E_1, \end{aligned}$$

since $$A_1^{1/n} A_2^{1/m} = (A_1{}^m A_2{}^n)^{1/nm} = O.$$

Similarly $$E_1 A_2 = E_2 A_1 = O.$$

COROLLARY (ix). $E = \lim_{n\to\infty} A^{1/n} = E_1+E_2$, *where* $A = A_1+A_2$, *the notation being the same as in* §XII, *Corollary* (viii).

For

$$\text{g.l.b.}_{n\geqslant 1}\ \text{l.u.b.}_{p\geqslant 0}\,(A_1^{1/(n+p)}+A_2^{1/(n+p)}) \leqslant \text{g.l.b.}_{n\geqslant 1}\,(A_1^{1/n}+E_2) = E_1+E_2,$$

and

$$\text{l.u.b.}_{n\geqslant 1}\ \text{g.l.b.}_{p\geqslant 0}\,(A_1^{1/(n+p)}+A_2^{1/(n+p)}) \geqslant \text{l.u.b.}_{n\geqslant 1}\,(E_1+A_2^{1/n}) = E_1+E_2.$$

Thus $$E_1+E_2 \leqslant \underline{\lim}_{n\to\infty} A^{1/n} \leqslant \overline{\lim}_{n\to\infty} A^{1/n} \leqslant E_1+E_2,$$

and the result follows.

COROLLARY (x). *If* $A > O$, *then* $A = \lim_{n\to\infty} A^{1+(1/n)}$, *i.e.* $A = AE$, $E = \lim_{n\to\infty} A^{1/n}$.

Now $AE = A_1E_1+A_2E_2$. We now show that $A_1E_1 = A_1$, and that $A_2E_2 = A_2$. Now $A_2E_2 \leqslant A_2$; if $A_2E_2 < A_2$, then

$$A_2 = A_2E_2+A_2(I-E_2);$$

thus $A_2^{1/n} = (A_2E_2)^{1/n}+\left(A_2(I-E_2)\right)^{1/n}$, since $A_2E_2 . A_2(I-E_2) = O$. From §VI, Corollary (xxiv), we see that $E_2 = E_2+E_2'$, $E_2' > O$, where $E_2' = \text{l.u.b.}_{n\geqslant 1}\left(A_2(I-E_2)\right)^{1/n}$, which is positive since the sequence is increasing and $A_2(I-E_2) > O$ by supposition. Also $E_2^{1/n} = E_2$, and so

$$\text{l.u.b.}_{n\geqslant 0}\,(A_2E_2)^{1/n} = E_2E_2 = E_2.$$

We thus have a contradiction, and so $A_2E_2 = A_2$. Secondly, $A_1E_1 \geqslant A_1$, for $A_1E_1 = \text{g.l.b.}_{n\geqslant 1} A_1^{1+(1/n)}$, and $A_1^{1+(1/n)} \geqslant A_1$; but $A_1E_1 \leqslant A_1$, and so $A_1E_1 = A_1$.

COROLLARY (xi). *Let* $E_- = \text{l.u.b.}\,(\mathfrak{E})$, *where* $\mathfrak{E}$ *is the set of all* E *for which* $A_+E = O$, $E^2 = E$, *where* $A = A_+-A_-$. *Then*

$$A_+E_- = O, \quad A_-E_- = A_-, \quad \textit{and} \quad E_-{}^2 = E_-.$$

From §IX, Corollary (ix), $E_-{}^2 = E_-$, and $A_+E_- = O$ follow.

Let $E = \lim_{n\to\infty} A_-^{1/n}$; then $A_- E = A_-$, $E^2 = E$; further,

$$A_+ E = \lim_{n\to\infty} A_+ A_-^{1/n} = O.$$

Thus E is in $\mathfrak{E}$; hence $E \leqslant E_-$. Thus $E_- \geqslant E$, *i.e.* $E_- E = E$; hence

$$A_- E_- = A_- E E_- = A_- E = A_-.$$

COROLLARY (xii). *If $E^2 = E$ and if $AE = O$, then $A_+ E = A_- E = O$ and $E_- E = E$.*

Since $AE = O$, then $A_+^2 E = A_+^2 E^2 = O$, and so $A_+ E = O$; similarly $A_- E = O$. Thus $E \leqslant E_-$, *i.e.* $E_- E = E$.

COROLLARY (xiii). *If $AB = O$, and if $E_0' = \text{l.u.b.}\,(\mathfrak{E}_0)$*, where $\mathfrak{E}_0$ is the set of all $E^2 = E$, $AE = O$, then $BE_0' = B$, $BE_- = B$.*

Let $A = A_+ - A_-$, $B = B_+ - B_-$, then $A_\pm B_\pm = O$. For

$$(A_+ - A_-)(B_+ - B_-) = O;$$

multiply by $A_+ B_+$ and we obtain $A_+^2 B_+^2 = O$, *i.e.* $A_+ B_+ = O$; similarly in the other cases. Thus $|A|.|B| = O$. If $AE = O$, then

$$A_+ E = A_- E = O,$$

and so $|A|E = O$; and conversely, if $|A|E = O$, then $A_+ E = A_- E = O$, and so $AE = O$. Thus $AE = O$ and $|A|E = O$, $E^2 = E$, yield the same set $\mathfrak{E}_0$. Now consider $C = |A| - |B| = C_+ - C_-$. Then the E_- for C is equal to E_0', thus $|B|E_0' = |B|$, *i.e.* $B_+^2 E_0'^2 = B_+^2$, and so $B_+ E_0' = B_+$; similarly $B_- E_0' = B_-$, and hence $BE_0' = B$. Lastly, if $AE = O$, then $A_+ E = O$, and so $E \leqslant E_-$; thus $BE_- = BE_0' E_- = BE_0' = B$.

XIII.

Spectral resolution.

Definition (i). E_λ is the E_- of $A - \lambda I$.

COROLLARY (i). *$E_\lambda E_\mu = E_\mu$, if $\lambda \geqslant \mu$, i.e. $E_\lambda \geqslant E_\mu$.*

Write $F = E_\mu(I - E_\lambda)$, then $E_\mu F = (I - E_\lambda) F = F$, $F \geqslant O$.

* We write E_0' to distinguish from E_0 later on.

Further, $$E_\mu(A-\mu I) = -(A-\mu I)_- \leqslant O,$$

$$(I-E_\lambda)(A-\lambda I) = (A-\lambda I)_+ \geqslant O.$$

Thus $$E_\mu(A-\mu I)\,F = (A-\mu I)\,F \leqslant O,$$

$$(I-E_\lambda)(A-\lambda I)\,F = (A-\lambda I)\,F \geqslant O.$$

Hence $(\lambda-\mu)\,F \leqslant O$, and so, since $(\lambda-\mu)\,I$ is complete, $F \leqslant O$; thus $F = O$, *i.e.* $E_\lambda E_\mu = E_\mu$. Thus $E_\lambda = E_\lambda E_\mu + E_\lambda(I-E_\mu) \geqslant E_\mu$.

COROLLARY (ii). g.l.b. $(E_\lambda) = O$.

For $$E_\lambda = \frac{1}{\lambda} E_\lambda A - \frac{1}{\lambda} E_\lambda(A-\lambda I) \leqslant \frac{1}{\lambda} E_\lambda A, \quad \text{if } \lambda < 0.$$

Thus $$E_\lambda \leqslant \frac{1}{\lambda^2} E_\lambda A^2 \leqslant \frac{1}{\lambda^2} A^2.$$

Hence $$\text{g.l.b.}_\lambda (E_\lambda) \leqslant \text{g.l.b.}_\lambda \left(\frac{1}{\lambda^2} A^2\right) = O.$$

COROLLARY (iii). l.u.b. $(E_\lambda) = I$.

For

$$I-E_\lambda = \frac{1}{\lambda}(I-E_\lambda)A - \frac{1}{\lambda}(I-E_\lambda)(A-\lambda I) \leqslant \frac{1}{\lambda}(I-E_\lambda)A, \quad \text{if } \lambda > 0,$$

and the result follows as in § XIII, Corollary (ii), since

$$\text{g.l.b.}\,(I-E_\lambda) = I - \text{l.u.b.}\,(E_\lambda).$$

COROLLARY (iv). $\text{g.l.b.}_{\lambda>\mu}(E_\lambda) = E_\mu$.

Since $E_\lambda \geqslant E_\mu$, for $\lambda \geqslant \mu$, then $\text{g.l.b.}_{\lambda>\mu}(E_\lambda)$ exists; call it $E_{\mu+0}$. Then $E_{\mu+0}$ is a projection by § IX, Corollary (vii), and $E_{\mu+0} \geqslant E_\mu$. Write $\Delta = \Delta_{\lambda,\mu} = E_\lambda - E_\mu$, $\Delta_0 = E_{\mu+0} - E_\mu$. Then $\Delta^2 = \Delta$, $\Delta_0 = \text{g.l.b.}_{\lambda>\mu}(\Delta)$, ${\Delta_0}^2 = \Delta_0$. We wish to show that $\Delta_0 = O$. Now $E_\lambda \Delta = \Delta$, $E_\mu \Delta = O$. Further,

$$E_\lambda(A-\lambda I)\Delta \leqslant O, \quad (I-E_\mu)(A-\mu I)\Delta \geqslant O;$$

thus $$(A-\lambda I)\Delta \leqslant O \quad \text{and} \quad (A-\mu I)\Delta \geqslant O,$$

and hence $$\mu\Delta \leqslant A\Delta \leqslant \lambda\Delta.$$

Thus $$O \leqslant (A-\mu I)\Delta \leqslant (\lambda-\mu)\Delta \leqslant (\lambda-\mu) I,$$

$$O \leqslant (\lambda I-A)\Delta \leqslant (\lambda-\mu)\Delta \leqslant (\lambda-\mu) I,$$

and $\underset{\lambda>\mu}{\text{g.l.b.}}\,(A-\mu I)\Delta = O$ by § VII, Corollary (ix).

Now $$(A-\mu I)_- E_\mu = (A-\mu I)_-,$$

also $$(A-\mu I)_- E_\lambda = (A-\mu I)_- E_\mu E_\lambda = (A-\mu I)_- E_\mu = (A-\mu I)_-.$$

Thus $$(A-\mu I)_-\Delta = O,$$

and hence $$(A-\mu I)\Delta = (A-\mu I)_+\Delta.$$

Thus $$\underset{\lambda>\mu}{\text{g.l.b.}}\,(A-\mu I)\Delta = \underset{\lambda>\mu}{\text{g.l.b.}}\,(A-\mu I)_+\Delta = (A-\mu I)_+\Delta_0,$$

also $$(A-\mu I)_-\Delta_0 = \underset{\lambda>\mu}{\text{g.l.b.}}\,(A-\mu I)_-\Delta = O,$$

and hence $$\underset{\lambda>\mu}{\text{g.l.b.}}\,(A-\mu I)\Delta = (A-\mu I)\Delta_0 = O.$$

Thus, by § XII, Corollary (xii), $E_\mu\Delta_0 = \Delta_0$. But $E_\mu\Delta_0 = \underset{\lambda>\mu}{\text{g.l.b.}}\,E_\mu\Delta = O$. Thus $\Delta_0 = O$.

COROLLARY (v). *If* $A_{m,n} = \sum_{k=-n}^{m} A\Delta_k$, *then* $\lim_{m\to\infty,\, n\to\infty} A_{m,n} = A$, *where* $\Delta_k = E_{\lambda_{k+1}} - E_{\lambda_k}$, $\lambda_{k+1} > \lambda_k$, $\lambda_k \to +\infty$ *as* $k \to \infty$, $\lambda_k \to -\infty$ *as* $k \to -\infty$, $\lambda_k < 0$ *if* $k < 0$, $\lambda_0 = 0$.

For $$A_{m,n} = A(E_{\lambda_{m+1}} - E_{\lambda_{-n}}),$$

also $$AE = A_+ E_\lambda - A_- \quad \text{if} \quad \lambda > 0,$$

$$AE_\lambda = -A_- E_\lambda \quad \text{if} \quad \lambda < 0.$$

For, if $\lambda > 0$, $$A_- E_\lambda = A_- E_0 E_\lambda = A_- E_0 = A_-$$

(E_0 is the E_- of A), and if $\lambda < 0$,

$$A_+ E_\lambda = A_+(I-E_0) E_\lambda = A_+(E_\lambda - E_\lambda) = O.$$

Thus $$A_{m,n} = A_+ E_{\lambda_{m+1}} - A_- + A_- E_{\lambda_{-n}}.$$

Hence $$\underset{p\geqslant 0,\ q\geqslant 0}{\text{l.u.b.}}\, A_{m+p,n+q} = A_+ - A_- + A_- E_{\lambda_{-n}},$$

$$\underset{m\geqslant 0,\ n\geqslant 0}{\text{g.l.b.}}\ \underset{p\geqslant 0,\ q\geqslant 0}{\text{l.u.b.}}\, A_{m+p,n+q} = A_+ - A_- = A.$$

Again, $$\underset{p\geqslant 0,\ q\geqslant 0}{\text{g.l.b.}}\, A_{m+p,n+q} = A_+ E_{\lambda_{m+1}} - A_-,$$

and $$\underset{m\geqslant 0,\ n\geqslant 0}{\text{l.u.b.}}\ \underset{p\geqslant 0,\ q\geqslant 0}{\text{g.l.b.}}\, A_{m+p,n+q} = A_+ - A_- = A.$$

COROLLARY (vi).

$$A-\delta I \leqslant \sum_{-\infty}^{\infty} \lambda_k \Delta_k \leqslant A \leqslant \sum_{-\infty}^{\infty} \lambda_{k+1} \Delta_k \leqslant A+\delta I,$$

where the notation is the same as in § XIII, *Corollary* (v), *and where*

$$\lambda_{k+1}-\lambda_k \leqslant \delta,$$

for all k.

From § XIII, Corollary (iv), we have $\lambda_k \Delta_k \leqslant A\Delta_k \leqslant \lambda_{k+1}\Delta_k$; hence $(\lambda_k+\delta)\Delta_k \leqslant A\Delta_k+\delta\Delta_k$, but $\lambda_{k+1}\Delta_k \leqslant (\lambda_k+\delta)\Delta_k$. Thus

$$A\Delta_k-\delta\Delta_k \leqslant \lambda_k\Delta_k \leqslant A\Delta_k \leqslant \lambda_{k+1}\Delta_k \leqslant A\Delta_k+\delta\Delta_k,$$

and hence

$$(A-\delta)(E_{\lambda_{m+1}}-E_{\lambda_{-n}}) \leqslant \sum_{-n}^{m} \lambda_k \Delta_k \leqslant A(E_{\lambda_{m+1}}-E_{\lambda_{-n}})$$

$$\leqslant \sum_{-n}^{m} \lambda_{k+1}\Delta_k \leqslant (A+\delta)(E_{\lambda_{m+1}}-E_{\lambda_{-n}}).$$

Thus $\sum_{-n}^{m} \lambda_k \Delta_k$ and $\sum_{-n}^{m} \lambda_{k+1}\Delta_k$ are bounded above and below by $A_+ + \delta I$ and $-A_- - \delta I$.

Further, l.u.b. $\sum_{-n-q}^{m+p} \lambda_k \Delta_k = \sum_{-n}^{\infty} \lambda_k \Delta_k$, § VIII, Definition (iv), and § VIII, Corollary (i), and

$$\text{g.l.b.}_{p,\,q\geqslant 0} \sum_{-n-q}^{m+p} \lambda_k \Delta_k = \sum_{-\infty}^{m} \lambda_k \Delta_k.$$

Thus $\lim_{n,\,m\to\infty} \sum_{-n}^{m} \lambda_k \Delta_k$ exists and equals $\sum_{-\infty}^{\infty} \lambda_k \Delta_k$ by § VIII, Definition (iv).

Further, $$\text{l.u.b.}_{p,\,q\geqslant 0} \sum_{-n-q}^{m+p} \lambda_k \Delta_k \leqslant \text{l.u.b.}_{p,\,q} A(E_{\lambda_{m+p+1}}-E_{\lambda_{-n-q}})$$

$$= A_+ - A_- + A_- E_{\lambda_{-n}},$$

thus $$\lim_{n,\,m\to\infty} \sum_{-n}^{m} \lambda_k \Delta_k \leqslant \text{g.l.b.}_{n}\,(A_+ - A_- + A_- E_{\lambda_{-n}})$$

$$= A_+ - A_- = A.$$

The rest of the corollary is proved in a similar manner.

COROLLARY (vii).

$$\lim_{\delta \to 0} \sum_{-\infty}^{\infty} \lambda_k \Delta_k = \lim_{\delta \to 0} \sum_{-\infty}^{\infty} \lambda_{k+1} \Delta_k = A.$$

We have

$$\text{g.l.b.}_{0<\delta} \text{ l.u.b.}_{0<\delta'<\delta} \sum_{-\infty}^{\infty} \lambda_k \Delta_k \leqslant A,$$

$$\text{l.u.b.}_{0<\delta} \text{ g.l.b.}_{0<\delta'<\delta} \sum_{-\infty}^{\infty} \lambda_k \Delta_k \geqslant \text{l.u.b.}_{\delta} (A - \delta I) = A,$$

whence

$$\lim_{\delta \to 0} \sum_{-\infty}^{\infty} \lambda_k \Delta_k = A.$$

The other result follows similarly.

THEOREM.

$$A = \int_{-\infty}^{\infty} \lambda dE_\lambda.$$

For the integral, if it exists, is the common limit of the two sums of § XIII, Corollary (vii). Since these two sums have A as their common limit, the theorem follows.

XIV.

Metric.

Definition (i). $\|A\| = \text{g.l.b.}(a)$, where $|A| \leqslant aI$.

Note. $\|A\|$ may be $+\infty$.

COROLLARY (i). $\|A+B\| \leqslant \|A\| + \|B\|$,

For $|A+B| \leqslant |A| + |B|$.

COROLLARY (ii). $\|AB\| \leqslant \|A\| . \|B\|$,

For $|AB| = |A| . |B|$.

COROLLARY (iii). $\|aA\| = |a| . \|A\|$.

COROLLARY (iv). *If* $\|A\| = O$, *then* $A = O$.

For if $|A| \leqslant aI$ for all $a > 0$, then $|A| = O$, whence $A = O$.

Definition (ii). A set of elements which satisfy § XIV, Corollary (i)–(iv) inclusive, is called a linear metric multiplicative space, *i.e.* it is a set of

elements A, B, ... for which $A+B$, AB, aA are elements of the same kind, and where there exists an element O, such that $A+O=A$ for all A, and to each element A there is attached a positive number called $\|A\|$ with the properties of § XIV, Corollary (i)–(iv) inclusive. It is now possible to consider topological properties of this space, to study the linear functionals in it and the space that they form, also to consider linear transformation of this space on to other spaces.

We may also define norm (A) and spur (A), when they exist, as follows:

Definition (iii). $\text{norm}\,(A) = \text{l.u.b.}\left[\sum_{n=1}^{m} \|AE_n\|^2\right]^{\frac{1}{2}}$,

where $\sum_{n=1}^{m} E_n = I$, and $E_n E_m = O$, if $n \neq m$, $E_n^2 = E_n$.

Definition (iv).

$$\text{spur}\,(A) = \text{l.u.b.} \sum_{n=1}^{m} \|A_+ E_n\| - \text{l.u.b.} \sum_{n=1}^{m} \|A_- E_n\|,$$

where E_n is as in Definition (iii).

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ON A CLASS OF NON-VANISHING FUNCTIONS

By NORMAN LEVINSON.

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If a function $f(x) \in L(-\pi, \pi)$ and has a Fourier series

$$f(x) \sim \sum_0^\infty a_n e^{inx},$$

that is, if

$$a_{-n} = 0 \quad (n = 1, 2, \ldots), \tag{1.0}$$

then, obviously, $f(x)$ is the boundary function of a function analytic in the unit circle and, being such a boundary function, $f(x)$ cannot be equivalent to zero over any interval, no matter how small, unless $f(x)$ is equivalent to zero in the whole interval $(-\pi, \pi)$.

This result can be immediately generalized to the case where

$$f(x) \sim \sum_{-\infty}^\infty a_n e^{inx},$$

with

$$a_{-n} = O(e^{-\delta n}) \quad (n = 1, 2, \ldots), \tag{1.1}$$

for some fixed $\delta > 0$. For in this case $f(x)$ is the boundary function of

$$g(w) = \sum_{-\infty}^\infty a_n w^n,$$

which is clearly analytic in the ring $e^{-\delta} < |w| < 1$, and therefore again $f(x)$ cannot be equivalent to zero over any interval without being entirely equivalent to zero.

Suppose that we have a function $f(x)$ whose Fourier coefficients (for $n > 0$ as well as $n < 0$) do not satisfy any condition of the form (1.1). Then $f(x)$ is no longer the boundary function of an analytic function. The question now arises whether or not we can find conditions under which such $f(x)$ have the property of vanishing completely if they vanish over any interval. Conditions (1.0) and (1.1) suggest two possibilities: The non-

vanishing property holds (1) if $|a_{-n}|$ $(n>0)$ becomes small [in some weaker sense than (1.1)] as $n\to\infty$; or (2) if most of the set $\{|a_{-n}|\}$ $(n>0)$ are zero.

Both possibilities do, in fact, give positive results. As regards the first one we shall prove

THEOREM I. *Let $f(x)\in L(-\pi,\pi)$ and have the Fourier series*

$$f(x)\sim\sum_{-\infty}^{\infty}a_n e^{inx}.$$

If

$$a_{-n}=O(e^{-\theta(n)})\quad(n>0) \tag{1.2}$$

for some non-decreasing $\theta(n)$ such that

$$\sum_{1}^{\infty}\frac{\theta(n)}{n^2}=\infty,$$

then $f(x)$ cannot be equivalent to zero over any interval without being equivalent to zero over $(-\pi,\pi)$.

The methods which prove this result can be used to prove a non-vanishing theorem if most of the a_{-n} $(n>0)$ are zero. This theorem is, however, a very weak one compared with the following result obtained by somewhat different methods.

THEOREM* A. *Let $f(x)\in L(-\pi,\pi)$ and have the Fourier series*

$$f(x)\sim\sum_{0}^{\infty}a_n e^{inx}+\sum_{1}^{\infty}a_{-n}e^{-i\lambda_n x}.$$

Let $\Lambda(u)$ be the number of $\lambda_n<u$ and let the minimal density

$$d=\lim_{\xi\to 1-0}\varliminf_{u\to\infty}\frac{\Lambda(u)-\Lambda(u\xi)}{u(1-\xi)}=0.$$

Then $f(x)$ cannot be equivalent to zero over any interval unless it is equivalent to zero over $(-\pi,\pi)$.

In particular, the theorem is true if $\varliminf n/\lambda_n=0$ as $n\to\infty$.

That Theorem I is a best possible result will become evident subsequently, while for Theorem A it is immediately obvious.

If we go over to Fourier transforms instead of series, Theorem I goes over intact but Theorem A is replaced by a much weaker result.

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* N. Levinson, *Proc. Camb. Phil. Soc.*, 31 (1935), 336, Theorem II. Owing to a typographical error, d is not properly defined in the original statement of this theorem as given there.

The analogue to Theorem I is

THEOREM II. *Let* $F(x) \epsilon L^p(-\infty, \infty)$, *for some* p, $1 \leqslant p \leqslant 2$. *Suppose that its Fourier transform*

$$G(u) \sim (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} F(x)\, e^{iux}\, dx$$

satisfies

$$G(u) = O(e^{-\theta(u)}) \quad (u > 1), \tag{1.3}$$

where $\theta(u)$ *is a non-decreasing, positive function such that*

$$\int_1^{\infty} \frac{\theta(u)}{u^2}\, du = \infty. \tag{1.4}$$

Then $F(x)$ *cannot be equivalent to zero in any interval unless it is equivalent to zero over* $(-\infty, \infty)$.

In particular, if $G(u)$ vanishes along a half-line, then $F(x)$ is the boundary function of a function analytic in a half-plane. In this way the discussion for series preceding Theorem I can be carried over to the case of the transform.

An immediate corollary of Theorem II is

THEOREM* III. *The conclusion of Theorem II remains valid if* (1.3) *and* (1.4) *are replaced by*

$$\int_1^{\infty} \frac{du}{u^2} \log \int_u^{\infty} |G(v)|\, dv = -\infty. \tag{1.5}$$

The theorem which leads to a weak analogue of Theorem A is

THEOREM IV. *Theorem II remains valid if* (1.3) *and* (1.4) *are replaced by*

$$\int_1^{\infty} \frac{du}{u^2} \log \int_u^{\infty} |G(v)|\, e^{-\phi(v)}\, dv = -\infty, \tag{1.6}$$

where $\phi(v)$ *is a positive, non-decreasing function such that*

$$\int_1^{\infty} \frac{\phi(v)}{v^2}\, dv < \infty. \tag{1.7}$$

---

* This theorem was presented to the American Mathematical Society at Williamstown, September, 1934. It is somewhat more general than Theorem II. However, since Theorem II is simpler to state and understand (although, unfortunately, essentially no simpler to prove), I am indebted to the Referee for pointing out to me that one could derive Theorem III from Theorem II very simply, thus replacing Theorem III by Theorem II as the central theorem of this paper.

2. We require several lemmas, which we shall now prove. Throughout, we shall use A to denote any finite positive constant.

LEMMA 1. *Let $F(x)$ belong to L^p, $1 \leqslant p \leqslant 2$, for some p, and let its Fourier transform be $G(u)$. If $F(x)$ vanishes almost everywhere over some interval, then there exists a function $F_1(x)$, not equivalent to zero unless $F(x)$ is, which vanishes in an interval with midpoint at $x = 0$ and which satisfies*

$$|F_1(x)| < \frac{A}{1+x^2}, \tag{2.00}$$

while $G_1(u)$, the Fourier transform of $F_1(x)$, satisfies the following conditions

$$|G_1(u)| \leqslant \frac{A}{1+u^2} \int_{u-1}^{u+1} |G(v)|\, dv < \frac{A}{1+u^2} \tag{2.01}$$

and

$$\left|\frac{dG_1(u)}{du}\right| < \frac{A}{1+u^2}. \tag{2.02}$$

For, let the width of the interval over which $F(x)$ vanishes be $4l$. If

$$F_0(x) = (2\pi)^{-\frac{1}{2}} \int_{-l}^{l} F(x+\xi)\left(1-\frac{|\xi|}{l}\right) d\xi, \tag{2.03}$$

then, clearly, $F_0(x)$ vanishes over an interval of width $2l$. But

$$(2\pi)^{-\frac{1}{2}} \int_{-l}^{l} e^{-iux}\left(1-\frac{|x|}{l}\right) dx = \frac{2}{l}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{\sin^2 \frac{1}{2}ul}{u^2}. \tag{2.04}$$

Thus the Fourier transform of $F_0(x)$ is

$$G_0(u) = \frac{2}{l}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} G(u)\, \frac{\sin^2 \frac{1}{2}ul}{u^2}. \tag{2.05}$$

Let

$$F_1(x) = F_0(x)\, \frac{4 \sin^2 \frac{1}{2}x}{x^2}; \tag{2.06}$$

then the Fourier transform of $F_1(x)$ is

$$G_1(u) = \int_{u-1}^{u+1} G_0(v)\{1-|u-v|\}\, dv. \tag{2.07}$$

Now, by (2.03) and (2.06),

$$|F_1(x)| \leqslant \frac{A}{1+x^2}.$$

By (2.05) and (2.07), if $q = p/(p-1)$ and $p > 1$, then

$$|G_1(u)| \leqslant \frac{A}{1+u^2}\int_{u-1}^{u+1}|G(v)|\,dv < \frac{A}{1+u^2}\left[\int_{u-1}^{u+1}|G(v)|^q dv\right]^{1/q} < \frac{A}{1+u^2},$$

and, similarly,

$$\left|\frac{dG_1(u)}{du}\right| < \frac{A}{1+u^2}.$$

For $p = 1$, the result follows by the uniform boundedness of $G(u)$.

Since $F_0(x)$ vanishes over an interval of length $2l$, so does $F_1(x)$. Thus $F_1(x)$ satisfies all the conditions of the lemma except that it does not necessarily vanish in an interval with midpoint zero. But, by simply translating $F_1(x)$, this requirement is fulfilled and $G_1(u)$ is merely multiplied by a factor e^{iuL} and therefore it still satisfies (2.01) and (2.02). This proves the lemma.

LEMMA 2. *If $M(u)$ is a continuous positive, non-increasing function, and if*

$$\int_1^\infty \frac{du}{u^2}\log[e^{-u}+M(u)] > -\infty, \tag{2.08}$$

then

$$\int_1^\infty \frac{du}{u^2}\log M(u) > -\infty. \tag{2.09}$$

For suppose (2.08) to be true and that

$$\int_1^\infty \frac{du}{u^2}\log M(u) = -\infty. \tag{2.10}$$

Let the set of open intervals over which $M(u) < e^{-u}$ be denoted by E. Then, by (2.08),

$$\int_E \frac{du}{u^2}\log e^{-u} + \int_{C(E)} \frac{du}{u^2}\log M(u) > -\infty. \tag{2.11}$$

Therefore

$$\int_E \frac{du}{u} < \infty. \tag{2.12}$$

By (2.10) and (2.11) we have

$$\int_E \frac{du}{u^2}\log M(u) = -\infty. \tag{2.13}$$

Now, let $\{a_n, b_n\}$ denote the set of intervals E, ordered so that

$$\frac{b_1}{a_1} \geqslant \frac{b_2}{a_2} \geqslant \ldots.$$

This is possible because, by (2.12), $b_n/a_n \to 1$. Thus (2.12) can be written as

$$\sum_1^\infty \log \frac{b_n}{a_n} < \infty, \tag{2.14}$$

while, from (2.13), we have

$$\sum_1^\infty \int_{a_n}^{b_n} \frac{du}{u^2} \log e^{-b_n} = -\infty,$$

i.e.

$$\sum_1^\infty \left(\frac{b_n}{a_n} - 1\right) = \infty.$$

But this contradicts (2.14) and thus (2.10) is impossible; this contradiction proves the lemma.

In proving the next lemma we require a result of Carleman.

THEOREM*. *If $f(z)$ is analytic for $y = \Im z > 0$ and is continuous for $y \geqslant 0$, then there exists a $\rho > 0$ such that, for any R,*

$$0 \leqslant \frac{1}{2\pi} \int_\rho^R \log|f(x)f(-x)| \left(\frac{1}{x^2} - \frac{1}{R^2}\right) dx + \frac{1}{\pi R} \int_0^\pi \log|f(Re^{i\theta})| \sin\theta \, d\theta + A.$$

LEMMA† 3. *Let $F(x)$ vanish for $x > x_0$, and let*

$$|F(x)| < \frac{A}{1+x^2}. \tag{2.15}$$

If $G(u)$ is the transform of $F(x)$, then

$$\int_1^\infty \frac{du}{u^2} \log|G(u)| > -\infty.$$

We have

$$G(u+iv) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{x_0} e^{-ix(u+iv)} F(x)\, dx.$$

Clearly, this function is analytic for $v > 0$. Moreover, from (2.15), it is obviously continuous for $v \geqslant 0$. Again, from (2.15), for $v \geqslant 0$,

$$|G(u+iv)| \leqslant A \int_{-\infty}^{x_0} \frac{e^{vx}}{1+x^2}\, dx < Ae^{v|x_0|}. \tag{2.16}$$

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* Titchmarsh, *Theory of functions* (1932), 130. The statement above is not identical with that in Titchmarsh, but obviously follows from it at once.

† This lemma is really part of a more general theorem of Paley and Wiener. Paley and Wiener, "Fourier transforms in the complex domain", *American Math. Soc. Coll. Pub.* 19 (1934), Theorem XII. We prove the lemma here because it contains the basic criterion which appears in all our non-vanishing theorems.

Now, applying Carleman's theorem to $G(u+iv)$, we have, using (2.16),

$$0 \leqslant \frac{1}{2\pi}\left[\int_{-R}^{-\rho}+\int_{\rho}^{R}\right]\log^{-}|G(u)|\left(\frac{1}{u^2}-\frac{1}{R^2}\right)du+A,$$

i.e.

$$\int_{\rho}^{\frac{1}{2}R}\log^{-}|G(u)|\left(\frac{1}{u^2}-\frac{1}{R^2}\right)du \geqslant -A.$$

Thus

$$\int_{\rho}^{\frac{1}{2}R}\log^{-}|G(u)|\frac{du}{u^2} \geqslant -A,$$

and, since this is independent of R, the lemma is proved.

Proof of Theorem II. Let $F(x)$ belong to L^p over $(-\infty, \infty)$ and vanish almost everywhere over some interval. Let $G(u)$ be its Fourier transform. Then, by Lemma I, there exists a function $F_1(x)$, not equivalent to zero, vanishing in an interval about $x=0$, and having a Fourier transform $G_1(u)$ satisfying (2.01) and (2.02).

Let

$$F_2(x) = F_1\left(\log\frac{e^x}{1+e^x}\right)\frac{1}{(1+e^x)^{\frac{1}{2}}}. \tag{2.17}$$

Then, for some x_0, $F_2(x)$ vanishes for $x > x_0$. For $-\infty < x < x_0$ we have, from (2.00) and (2.17),

$$|F_2(x)| \leqslant \frac{A}{1+x^2}.$$

Therefore, by Lemma 3, if $G_2(u)$ is the transform of $F_2(x)$,

$$\int_1^{\infty}\frac{du}{u^2}\log|G_2(u)| > -\infty. \tag{2.18}$$

If we express $G_2(u)$ in terms of $F_1(x)$, we have

$$\begin{aligned} G_2(u) &= (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty}F_1\left(\log\frac{e^x}{1+e^x}\right)\frac{e^{iux}}{(1+e^x)^{\frac{1}{2}}}dx \\ &= (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{0}F_1(y)\,e^{iuy}(1-e^y)^{-iu-\frac{1}{2}}dy \\ &= \lim_{\epsilon\to 0}(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{0}F_1(y)\,e^{(iu+\epsilon)y}(1-e^y)^{-iu-\frac{1}{2}}dy. \end{aligned}$$

If we express $F_1(y)$ in terms of its transform, we have

$$G_2(u) = (2\pi)^{-1}\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}G_1(v)\,dv\int_{-\infty}^{0}e^{i(u-v)y+\epsilon y}(1-e^y)^{-iu-\frac{1}{2}}dy.$$

But if we write $x = e^y$,

$$\int_{-\infty}^{0} e^{i(u-v)y+\epsilon y}(1-e^y)^{-iu-\frac{1}{2}}\,dy = \int_0^1 x^{i(u-v)+\epsilon-1}(1-x)^{-iu-\frac{1}{2}}\,dx$$

$$= \frac{\Gamma\{i(u-v)+\epsilon\}\,\Gamma(-iu+\frac{1}{2})}{\Gamma(-iv+\frac{1}{2}+\epsilon)}.$$

Thus

$$(2.19)\qquad G_2(u) = \frac{1}{2\pi}\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} G_1(v)\,\frac{\Gamma\{i(u-v)+\epsilon\}\,\Gamma(-iu+\frac{1}{2})}{\Gamma(-iv+\frac{1}{2}+\epsilon)}\,dv.$$

We now use the well-known result that, as $|y|\to\infty$,

$$(2.20)\qquad |\Gamma(iy+a)| \sim e^{-\frac{1}{2}\pi|y|}|y|^{a-\frac{1}{2}}(2\pi)^{\frac{1}{2}},$$

and that, for small $|w|$,

$$(2.21)\qquad \Gamma(w) = \frac{1}{w} + B + O(|w|),$$

where B is a constant. If the interval of integration in (2.19) is split up into $(-\infty, \frac{1}{2}u)$, $(\frac{1}{2}u, u-\delta)$, $(u-\delta, u+\delta)$, $(u+\delta, \infty)$, where $\delta > 0$, then it follows immediately, by (2.20) and (2.21), that, for $u > 1$,

$$(2.22)\qquad |G_2(u)| \leqslant A\left\{e^{-\frac{1}{2}\pi u} + \delta^{-1}\int_{\frac{1}{2}u}^{\infty}|G_1(v)|\,dv\right\}$$

$$+ \frac{1}{2\pi}\lim_{\epsilon\to 0}\left|\int_{u-\delta}^{u+\delta} dv\, G_1(v)\,\frac{\Gamma\{i(u-v)+\epsilon\}\,\Gamma(-iu+\frac{1}{2})}{\Gamma(-iv+\frac{1}{2}+\epsilon)}\right|.$$

By (2.02),

$$G_1(v) = G_1(u) + O(|v-u|).$$

Using this and (2.21) in (2.22), we get

$$|G_2(u)| \leqslant A\left[e^{-\frac{1}{2}\pi u} + \delta^{-1}\int_{\frac{1}{2}u}^{\infty}|G_1(v)|\,dv + |G_1(u)| + \delta\right].$$

If we now write $$\delta = \left[\int_{\frac{1}{2}u}^{\infty}|G_1(v)|\,dv\right]^{\frac{1}{2}},$$

then $$|G_2(u)| \leqslant A\left\{e^{-\frac{1}{2}\pi u} + |G_1(u)| + \left[\int_{\frac{1}{2}u}^{\infty}|G_1(v)|\,dv\right]^{\frac{1}{2}}\right\}.$$

If we now use (1.3) and (2.01), we have

$$|G_2(u)| \leqslant A(e^{-\frac{1}{2}\pi u} + e^{-\frac{1}{2}\theta(\frac{1}{2}u)}).$$

Hence, from (2.18), it follows that

$$\int_1^\infty \frac{du}{u^2} \log\left(e^{-\frac{1}{2}\pi u} + e^{-\frac{1}{2}\theta(\frac{1}{2}u)}\right) > -\infty,$$

and so, by Lemma 2,

$$\int_1^\infty \frac{\theta(u)}{u^2}\, du < \infty.$$

This completes the proof of Theorem II.

3. *Proof of Theorem I.* We extend the definition of $f(x)$ to $(-\infty, \infty)$ by giving it the period 2π. Let

$$F(x) = f(x)\, \frac{\sin^2 \frac{1}{4}x}{x^2}.$$

Then $F(x) \in L(-\infty, \infty)$ and

$$G(u) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{iux} f(x)\, \frac{\sin^2 \frac{1}{4}x}{x^2}\, dx.$$

Here we can replace $f(x)$ by its Fourier series and integrate term by term. Thus

$$G(u) = (2\pi)^{-\frac{1}{2}} \sum_{-\infty}^{\infty} a_n \int_{-\infty}^{\infty} e^{i(u+n)x}\, \frac{\sin^2 \frac{1}{4}x}{x^2}\, dx.$$

Since the transform of $\sin^2 \frac{1}{4}x/x^2$ vanishes outside $(-\frac{1}{2}, \frac{1}{2})$, we have

$$|G(u)| \leqslant (2\pi)^{-\frac{1}{2}} |a_{-n}| \int_{-\infty}^{\infty} \frac{\sin \frac{1}{4}x}{x^2}\, dx,$$

where $|u-n| < \frac{1}{2}$, and, using (1.2),

$$|G(u)| \leqslant A e^{-\theta([u-1])} \quad (u > 2).$$

Theorem I now follows at once from Theorem II.

Proof of Theorem III. Suppose that $F(x)$ vanishes in an interval but not identically over $(-\infty, \infty)$. Then, by Lemma 1, there exists a function $F_1(x)$ not equivalent to zero, but vanishing in an interval about $x = 0$, and with the transform $G_1(u)$ satisfying (2.01). By integrating the right-hand side of

$$F_1(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} G_1(u)\, e^{-iux}\, du$$

by parts, we get, since $F_1(0) = 0$,

$$\frac{F_1(x)}{-ix} = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} du\, e^{-iux} \int_u^\infty G_1(v)\, dv.$$

That is, $F_1(x)/(-ix)$ vanishes identically in an interval and has for its transform

$$G_2(u) = \int_u^\infty G_1(v)\,dv.$$

By (2.01), $$|G_2(u)| < A\int_{u-1}^\infty |G(v)|\,dv.$$

If we put $$\theta(u) = -\log \int_{u-1}^\infty |G(v)|\,dv,$$

then, by (1.5), $$\int_2^\infty \frac{\theta(u)}{u^2}\,du = \infty.$$

But $|G_2(u)| < Ae^{-\theta(u)}$, $u > 2$. Thus, by Theorem II, $F_1(x)/(-ix)$, and therefore $F_1(x)$, vanishes identically. This proves Theorem III.

In proving Theorem IV we require the following lemma.

LEMMA* 4. *Let $\phi(u)$ be an even, positive, non-decreasing function of $|u|$. If*

$$\int_1^\infty \frac{\phi(u)}{u^2}\,du < \infty, \tag{3.0}$$

then there exists a function $G(u)$ such that

$$|G(u)| \leqslant \frac{e^{-\phi(u)}}{1+u^2}$$

and such that

$$F(x) = (2\pi)^{-\frac{1}{2}}\int_{-\infty}^\infty G(u)e^{iux}\,du$$

vanishes outside a finite interval but is not equivalent to zero.

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* This result is very similar to one obtained by Ingham, *Journal London Math. Soc.*, 9 (1934), 29. Ingham uses $u\epsilon(u)$ in place of $\phi(u)$, where $\epsilon(u)$ is positive and non-increasing, and

$$\int_1^\infty \frac{\epsilon(u)}{u}\,du = \infty. \tag{*}$$

That the above lemma leads to Ingham's result is clear, since, if, given $\epsilon(u)$, we define

$$\phi(u) = \int_1^u \epsilon(y)\,dy,$$

then $\phi(u) \geqslant (u-1)\,\epsilon(u)$ and

$$\int_1^\infty \frac{\phi(u)}{u^2}\,du = \int_1^\infty \frac{\epsilon(y)}{y}\,dy < \infty.$$

On the other hand, it is easy to give functions $\phi(u)$ satisfying (3.0) such that there exists no non-increasing $\epsilon(u)$ satisfying (*) and $u\epsilon(u) \geqslant \phi(u)$ simultaneously.

This lemma proves, incidentally, that Theorems I and II give best possible results.

To prove the lemma we use a result of Wiener and Paley* which states that, *if a positive function* $\Phi(u) \epsilon L^2(-\infty, \infty)$, *and if*

$$\int_{-\infty}^{\infty} \frac{|\log|\Phi(u)||}{1+u^2}\, du < \infty,$$

then there exists a function $g(u)$ *such that* $|g(u)| = \Phi(u)$ *and such that* $f(x)$, *the transform of* $g(u)$, *vanishes for* $x > x_0$.

Let us put $\psi(u) = e^{-\phi(u)}$. By the theorem of Wiener and Paley, there exists a function $g(u)$ such that

$$|g(u)| = \{2\pi\psi(0)\}^{-\frac{1}{2}} \frac{\psi(2u)}{1+4u^2} \tag{3.1}$$

and such that

$$f(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} g(u) e^{iux}\, du$$

vanishes over the half-line $x > x_0 > 0$ but, clearly, is not equivalent to zero. There must therefore exist some point $x = a < x_0$ such that $f(a) \neq 0$. Moreover, we can assume that $a > 0$ because, if we translate $f(x)$, then $|g(u)|$ remains unchanged.

$$G(u) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} g(v-u)\, g(v)\, e^{2ia(v-u)}\, dv$$

is the Fourier transform of $F(x) = f(x)f(2a-x)$. But $F(x)$ is not equivalent to zero because $f(x)$, by (3.1), is continuous and $F(a) = f^2(a) \neq 0$. Moreover, $F(x)$ vanishes for $|x| > 3x_0$.

Clearly,

$$\begin{aligned} |G(u)| &\leqslant \frac{1}{2\pi\psi(0)} \int_{-\infty}^{\infty} \frac{\psi(2v)}{1+4v^2} \frac{\psi\{2(v-u)\}}{1+4(v-u)^2} dv \\ &= \frac{1}{2\pi\psi(0)} \left[\int_{\frac{1}{2}u}^{\infty} + \int_{-\infty}^{\frac{1}{2}u} \right] \frac{\psi(2v)}{1+4v^2} \frac{\psi\{2(v-u)\}}{1+4(v-u)^2} dv \\ &= \frac{1}{\pi\psi(0)} \int_{\frac{1}{2}u}^{\infty} \frac{\psi(2v)}{1+4v^2} \frac{\psi\{2(v-u)\}}{1+4(v-u)^2} dv \leqslant \frac{e^{-\phi(u)}}{1+u^2} \end{aligned}$$

for $u > 0$, and similarly for $u < 0$, and the lemma is proved. Note that

* Wiener and Paley, *loc. cit.*

$x_0 > 0$ can be taken as close to zero as we please and that $F(x)$ can be taken to be bounded and continuous.

Proof of Theorem IV. Let $F(x)$ satisfy the requirements of Theorem IV and vanish almost everywhere over some interval of length $4l$, but not be equivalent to zero. By Lemma 4, there exists a bounded function $f(x)$ that vanishes outside $(-l, l)$ and such that its transform

$$|g(u)| < e^{-\phi(u)},$$

where $\phi(u)$ satisfies the requirements of Theorem IV.

Let us consider

$$F_1(x) = (2\pi)^{-\frac{1}{2}} \int_{-l}^{l} F(x-\xi) f(\xi)\, d\xi.$$

$F_1(x)$ vanishes over an interval of length $2l$. The transform of $F_1(x)$ is $G_1(u) = g(u)G(u)$. [Since $g(u)$ is an integral function and has therefore isolated zeros, it follows that $G_1(u)$ is not equivalent to zero if $G(u)$ is not.] By Theorem III, it follows that

$$\int_1^\infty \frac{du}{u^2} \log \int_u^\infty |G_1(v)|\, dv > -\infty.$$

This proves Theorem IV.

In case $G(v)$, for $v > 0$, vanishes except over sparse intervals, it is easily shown that Theorem IV holds while the others do not. For example, suppose that $\lambda_n = n^n$ and that outside the intervals $(\lambda_n - D, \lambda_n + D)$, $n > 0$, $G(u)$ vanishes. D is some fixed constant. Since $G(u)$ is the transform of a function in L^p, $1 \leqslant p \leqslant 2$,

$$\int_{\lambda_n - D}^{\lambda_n + D} |G(u)|\, du < A.$$

Therefore

$$\int_u^\infty e^{-\phi(v)} |G(v)|\, dv < A \sum_{\lambda_n > u - D} e^{-\phi(\lambda_n - D)}. \tag{3.2}$$

Let $\phi(u) = n^{n-2}$ for $\lambda_n + D < u \leqslant \lambda_{n+1} + D$. Then

$$\int_1^\infty \frac{\phi(u)}{u^2}\, du < \sum_1^\infty n^{n-2} \int_{\lambda_n + D}^\infty \frac{du}{u^2} \leqslant \sum_1^\infty \frac{1}{n^2} < \infty.$$

Thus (1.7) is satisfied. By (3.2), for $\lambda_n + D < u \leqslant \lambda_{n+1} + D$,

$$\log \int_u^\infty e^{-\phi(v)} |G(v)|\, dv \leqslant A - (n+1)^{n-1}.$$

Thus, if $n_0 > D$,

$$\int_1^\infty \frac{du}{u^2} \log \int_u^\infty e^{-\phi(v)} |G(v)|\, dv \leqslant A - \sum_{n_0}^{\infty} \int_{\lambda_n + D}^{\lambda_{n+1}+D} \frac{(n+1)^{n-1}}{u^2}\, du$$

$$\leqslant A - \tfrac{1}{2} \sum_{0}^{\infty} \frac{(n+1)^{n-1}}{(n^n + D)} = -\infty.$$

Thus (1.6) is satisfied and the transform of $G(u)$ is a non-vanishing function. In this sense we have an analogue to Theorem A. However, instead of the mild gap condition there, here we have $\lambda_{n+1}/\lambda_n \to \infty$.

4. Let
$$\phi(z) = \sum_0^\infty b_n z^n$$

be analytic in the unit circle and let

$$\int_0^{2\pi} |\phi(re^{ix})|\, dx < A \quad (r < 1).$$

Then it is well known that there exists a boundary function $\phi(e^{ix}) \epsilon L$ such that

$$\phi(e^{ix}) \sim \sum_0^\infty b_n e^{inx}.$$

It is now obvious that in Theorem I the conclusion can be replaced by:

If $f(x) = \phi(e^{ix})$ almost everywhere over some interval, then $f(x)$ is equivalent to $\phi(e^{ix})$ over $(0, 2\pi)$.

Miss Cartwright suggested to me that it would be of some interest to have a result of this type for the class of analytic functions $H(z)$ for which

$$\int_0^{2\pi} \log^+ |H(re^{ix})|\, dx < A \quad (r < 1). \tag{4.0}$$

In this case we prove

THEOREM V. *Let $H(z)$ be analytic for $|z| < 1$ and satisfy (4.0). Let $f(x)$ satisfy the requirements of Theorem I and, moreover, let*

$$\sum_1^\infty e^{-\theta(n)} < \infty. \tag{4.1}$$

Then, if $f(x) = H(e^{ix})$ almost everywhere over some interval, $f(x)$ is equivalent to $H(e^{ix})$ over $(0, 2\pi)$.

Proof of Theorem V. In order to prove Theorem V it is necessary to determine a property of $H(z)$ which results from (4.0). We use the following theorem due to F. and R. Nevanlinna*.

THEOREM. *A necessary and sufficient condition for a holomorphic function in a domain D to be the quotient of two bounded functions in this domain is that the sequence of harmonic functions u_n, which take the values $\log^+|f(z)|$ on the contours C_n of a nested sequence of domains with limit D, is bounded at some point of D.*

When we take $z=0$ as the point of D and the sequence of circles with increasing radii tending to unity as C_n, it is clear, by (4.0), that

$$u_n(0,0)=\frac{1}{2\pi}\int_0^{2\pi}\log^+|H(r_n e^{i\theta})|\,d\theta$$

is uniformly bounded. Thus

$$H(z)=\frac{h_1(z)}{h_2(z)},$$

where $$|h_1(z)|<A, \quad |h_2(z)|<A, \quad |z|<1.$$

Now let us consider

$$F(x)=\{H(e^{ix})-f(x)\}h_2(e^{ix})\frac{\sin^2\frac{1}{2}x}{x^2} \tag{4.2}$$

$$=h_1(e^{ix})\frac{\sin^2\frac{1}{2}x}{x^2}-f(x)h_2(e^{ix})\frac{\sin^2\frac{1}{2}x}{x^2}.$$

Let $$g_1(u)=(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty}h_1(e^{ix})\frac{\sin^2\frac{1}{2}x}{x^2}\,dx.$$

Since $h(e^{iw})$ is bounded for $\mathfrak{I}(w)\geqq 0$, it follows by bounded convergence that

$$g_1(u)=\lim_{\lambda\to 0}(2\pi)^{-\frac{1}{2}}\int_{-\infty+i\lambda}^{\infty+i\lambda}h_1(e^{iw})\frac{\sin^2\frac{1}{2}w}{w^2}e^{iuw}\,dw.$$

Using Cauchy's theorem, we get, for any fixed $a>0$,

$$g_1(u)=\frac{e^{-au}}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty}h_1(e^{ix-a})\frac{\sin^2\frac{1}{2}(x+ia)}{(x+ia)^2}e^{iux}\,dx.$$

Thus, for $u>0$,

$$|g_1(u)|<Ae^{-au}. \tag{4.3}$$

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* "Über die Eigenschaften analytischer Funktionen in der Umgebung einer singulären Stelle oder Linie", *Acta Societatis Scientiarum Fennicae*, 50 (1922), No. 5, 1–46.

If $g_2(u)$ is similarly related to $h_2(e^{ix})$, then, clearly, $g_2(u)$ is bounded and, for $u > 0$,

$$|g_2(u)| < Ae^{-au}. \tag{4.4}$$

Let
$$G_1(u) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} F(x)\, e^{iux}\, dx,$$

where $F(x)$ is defined by (4.2). Then, for $u > 0$, it follows from (4.3) that

$$|G_1(u)| \leqslant Ae^{-au} + \sum_{-\infty}^{\infty} |a_n|\,|g_2(u+n)|,$$

where $\{a_k\}$ is the set of Fourier coefficients of $f(x)$. By (4.1) and (4.4),

$$|G_1(u)| \leqslant A\left\{e^{-\frac{1}{2}au} + \left(\sum_{[\frac{1}{2}u]}^{\infty} |a_{-n}|^2\right)^{\frac{1}{2}}\right\} \leqslant A\left(e^{-\frac{1}{2}au} + e^{-\frac{1}{2}\theta([\frac{1}{2}u])}\right).$$

If we define $\theta_1(u)$ by

$$e^{-\theta_1(u)} = e^{-\frac{1}{2}au} + e^{-\frac{1}{2}\theta([\frac{1}{2}u])},$$

then
$$|G_1(u)| \leqslant Ae^{-\theta_1(u)},$$

where $\theta_1(u)$ is non-decreasing. Moreover, by Lemma 2,

$$\int_1^{\infty} \frac{\theta_1(u)}{u^2}\, du = \infty. \tag{4.5}$$

Now, let us suppose that $H(e^{ix}) = f(x)$ almost everywhere over some interval. Then, on applying Theorem II to $F(x)$, it follows from (4.5) that $H(e^{ix}) = f(x)$ almost everywhere.

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THE GENERALIZED ABEL-TAUBER THEOREM

By V. RAMASWAMI.

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This paper is the outcome of an effort to meet comments on a proposed proof of Schmidt's theorem, simpler than that given by Vijayaraghavan*. My thanks are due to Mr. Ingham, who drew my attention to the work of Karamata referred to below, whose method for proving more general results seemed much akin to mine. I found that I was able, by my method, to obtain the further refinements in Theorems I.1 and I.2 below. I am also indebted to Mr. Ingham for valuable criticism regarding technique of exposition.

In what follows, a_n is real, $\limsup |a_n|^{1/n} = 1$, s_t denotes a function of t, of bounded variation in every finite interval in $(0, \infty)$, and with $s_0 = 0$; and, for any λ greater than unity, and any set of points E extending to infinity,

$$\psi(\lambda) = \liminf_{u,\, t\to\infty;\ t<u\leqslant\lambda t} (s_u - s_t),$$

$$\psi(E, \lambda) = \liminf_{u,\, t\to\infty;\ u,\, t\subset E;\ t<u\leqslant\lambda t} (s_u - s_t).$$

It is easily seen that the definition of $\psi(\lambda)$ implies one of the following mutually exclusive propositions: either (i) $\psi(\lambda) = 0$ for every λ, or (ii) $\psi(\lambda) = -\infty$ for every λ, or (iii) $0 > \psi(\lambda) > -\infty$ for every λ, or (iv) $\psi(\lambda) > \delta > 0$ for every λ and some fixed δ†.

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* *Journal London Math. Soc.* 1 (1926), 113–119.

† For given λ and a given set E, $\psi(E, \lambda)$ may not exist, on account of the existence of arbitrarily large t such that the component of E in $(t < u \leqslant \lambda t)$ is null. But under the conditions imposed on E in our context (Theorem II below), $\psi(E, \lambda)$ exists for every λ, and, as with $\psi(\lambda)$, one only of the possibilities (i) to (iv) above can be correct.

The above *a priori* analysis depends on the following obvious properties: (*a*) $\psi(\lambda)$ is a non-increasing function of λ; (*b*) if $\psi(\lambda) > \delta > 0$ for some λ, then $\psi(\lambda) > \delta$ for every λ; (*c*) if $\psi(\lambda_1) > -\delta < 0$, then $\psi(\lambda) \geqslant -2r\delta$ for $1 < \lambda \leqslant \lambda_1^r$ and r integral and positive.

Actually, however, we have, for every λ in our context, $\psi(\lambda) \leqslant 0$ and $\psi(E, \lambda) \leqslant 0$, but the proof of this depends on the theorem that the discontinuities of a monotonic function form an enumerable set. If this view is taken, the sign "$\geqslant$" in the statement of Theorem II, and in the condition (*f*) of Theorem I.2, may be replaced by "$=$", and in that case the first line in the proof of Theorem II should be replaced by "(i) If $\psi(\lambda_1) = 0$, then $\psi(\lambda_2) > 0$, and the theorem follows", but the rest of the argument remains unaffected.

The original condition $na_n = o(1)$ of Tauber* for the "o-inversion" of the Abel-limit equation

$$\lim_{x\to 1-0} \Sigma a_n x^n = 0, \tag{1}$$

was generalized by Littlewood† to $|na_n| < \kappa$, later by Hardy and Littlewood‡ to $na_n > -\kappa$, meanwhile by Landau§, in a different direction, to the pair of conditions

$$s_n = a_1 + a_2 + \ldots + a_n = O(1),$$

$$\lim_{\lambda\to 1+0} \max_{|m-n| \leqq n(\lambda-1)} |s_m - s_n| = 0,$$

and finally by R. Schmidt|| to the condition

$$\lim_{\lambda\to 1+0} \liminf_{m,n\to\infty;\ n<m\leqq n\lambda} (s_m - s_n) \geqq 0.$$

Theorem I.2 below is a theorem on oscillation and implies the validity of the "o-inversion" of the completely generalized integral equivalent of (1), viz.,

$$\lim_{\sigma\to 0} \int_0^\infty \phi(\sigma t)\, d(s_t) = 0, \tag{2}$$

under the conditions (i) $\phi(t)$ is continuous and monotonic in $(0, \infty)$, and $\phi(0) \neq 0$, (ii) $\int_1^\infty \phi(t)\, d(\log t)$ exists, (iii) a positive number h exists such that 0 is a limit-point of $\psi(\lambda)/\log(\lambda)$ in $1 < \lambda < e^h$. The last seems to be the best condition which does not involve any further restriction on $\phi(t)$, especially in view of the fact that the generalizations of Hardy and Littlewood or Schmidt above depend on special properties of e^{-t}. The incidental Theorem I.1 is a considerable improvement on an "O-inversion" theorem due to Karamata¶: of the two conditions (1) of his Satz A (*loc. cit.*) the second, equivalent to the existence of $\int_0^1 [\phi(0) - \phi(t)]\, d(\log t)$, is dispensed with, the first is replaced by its integral equivalent (b) below, and his condition of the existence and boundedness as $\sigma \to 0$ of $\int_0^\infty \phi(\sigma t)\, d(s_t)$ is replaced by the apparently broader condition (e) in Theorem I.1 below; the conclusion being the same as his.

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* *Monatshefte für Math. und Phys.*, 8 (1897), 273.
† *Proc. London Math. Soc.* (2), 9 (1910), 434–448.
‡ *Proc. London Math. Soc.* (2), 13 (1914), 174–191.
§ *Rend. di Palermo*, 35 (1913), 265–276.
|| *Math. Zeitschrift*, 22 (1925), 89–152.
¶ *Math. Zeitschrift*, 37 (1933), 582–588.

Theorem II below implies the validity of the "o-inversion" *over a set* E of the generalized integral equivalent of (1), viz.,

$$\lim_{\sigma\to 0}\int_0^\infty e^{-\sigma t}d(s_t)=0, \tag{3}$$

under the conditions: (i) $\psi(\lambda)>-\infty$ for some λ, (ii) $\lim_{\lambda\to 1+0}\psi(E,\lambda)\geqslant 0$, (iii) the complement of E in (o, t) is of measure $o(t)$ as $t\to\infty$. At the same time it is proved that the effect of (3) on the above-mentioned *a priori* distributions of the values of $\psi(\lambda)$ is to replace the alternatives (iii) and (iv) by $-\infty<2\lim_{\lambda\to 1+0}\psi(\lambda)\leqslant\lim_{\lambda\to\infty}\psi(\lambda)<0$, while, by Theorem I.2, the alternative (i) implies $s_t=o(1)$.

THEOREM I.1. *If*

(*a*) $\phi(t)$ *is continuous and monotonic in* $(0,\infty)$ *and* $\phi(0)\neq 0$,

(*b*) $\int_1^\infty \phi(t)\,d(\log t)$ *exists*,

(*c*) s_t *is of bounded variation in every finite interval in* $(0,\infty)$ *and* $s_0=0$; *and a number* λ *greater than unity, positive numbers* ω, H, δ, *and a function* $H(\sigma)$ *exist such that*

(*d*) $s_u-s_t>-\omega \quad (t\leqslant u\leqslant\lambda t)$,

and

(*e*) *to every* σ *in* $(0<\sigma\leqslant\delta)$ *corresponds an ascending divergent sequence* $P_n=P_{n,\sigma}$ *with the property*

$$H>H(\sigma)=\lim_{n\to\infty}\int_0^{P_{n,\sigma}}\phi(\sigma t)\,d(s_t)>-H,$$

then

$$s_T=O(1) \quad as \quad T\to\infty. \tag{4}$$

LEMMA 1. *For fixed positive* σ,

$$\lim_{t\to\infty}\phi(\sigma t)\log t=0.$$

For

$$\phi(\sigma t)\log t=\phi(\sigma t)\,[\log(t\sigma)-\log(\sigma)]\sim\phi(\sigma t)\log(\sigma t)\to 0,$$

by (*b*) and the integral analogue of Abel's theorem on monotonic convergent series.

Hence we may suppose that $\phi(0)>0$, and that $\phi(t)$ is non-negative and non-increasing, and so $\sum_{\nu=0}^{\infty}\phi(\alpha\lambda^{\nu})$ is convergent for $\alpha>0$, by (b) and Cauchy's integral test. Also a number κ exists such that

$$\phi(\kappa\lambda^{-1})>\phi(\kappa)\geqq 0;$$

for otherwise $\phi(0)$ would be zero, contrary to (a).

Notation and conventions:

$$\gamma=\phi(\kappa\lambda^{-1})-\phi(\kappa)\ \ (>0),\quad 0<\alpha<\kappa\lambda^{-1},\quad B_{\alpha}=\phi(0)-\phi(\alpha),$$

$$F(\alpha)=\sum_{\nu=0}^{\infty}\phi(\alpha\lambda^{\nu}),\quad M_1=\frac{1}{\phi(\kappa\lambda^{-1})}+\frac{1}{H}\max_{0\leqq u\delta\leqq\kappa}|s_u|,$$

$$M_2=M_1[H+\omega F(\alpha)],\quad M_3=\phi(0)-\phi(\kappa\lambda^{-1}),\quad m_t=-\min_{0\leqq u\leqq t}s_u.$$

LEMMA 2. (i) *m_t is non-negative and non-decreasing.*

(ii) $$m_{\lambda t}-m_t\leqq\omega.$$

(iii) $$m_t=O(\log t)\quad (t>\lambda).$$

(i) is obvious by the definition of m_t. To prove (ii) we observe that, if $m_{\lambda t}>m_t$, then $m_{\lambda t}=-\min_{t<u\leqq\lambda t}s_u$, also $-m_t\leqq s_t$, and

$$m_{\lambda t}-m_t=-\min_{t<u\leqq\lambda t}s_u-m_t\leqq-\min_{t<u\leqq\lambda t}s_u+s_t=-\min_{t<u\leqq\lambda t}(s_u-s_t)\leqq\omega,$$

by (d); and (iii) follows immediately.

LEMMA 3. $$-\int_T^P m_t\,d\phi(\sigma t)\leqq m_T\phi(\kappa)+\omega F(\kappa)\quad(\kappa\sigma^{-1}=T<P).$$

For

$$-\int_T^P m_t\,d\{\phi(t\kappa T^{-1})\}=m_T\phi(\kappa)-m_P\phi(P\kappa T^{-1})+\int_T^P\phi(t\kappa T^{-1})\,d(m_t)$$

$$\leqq m_T\phi(\kappa)+\omega\sum_{\nu=0}^{\infty}\phi(\kappa\lambda^{\nu}),$$

by Lemma 2 (i); and, by Lemma 2 (ii), and the first mean value theorem applied to the ranges $(T, T\lambda)$, $(T\lambda, T\lambda^2)$, ..., the last expression is equal to

$$m_T\phi(\kappa)+\omega F(\kappa).$$

LEMMA 4. $$s_t<s_T+\omega\qquad(T\lambda^{-1}\leqq t\leqq T).$$

For, by (d), $$s_T-s_t>-\omega\quad(t\leqq T\leqq t\lambda)$$

i.e., $$s_t<s_T+\omega\ \text{ in }\ (T\lambda^{-1}\leqq t\leqq T).$$

We now proceed to the proof of the theorem.

For $\alpha\delta^{-1} \leqslant \alpha\sigma^{-1} = T < P_n$, we have

$$\int_0^{P_n} \phi(\alpha t)\, d(s_t) = \int_0^T \phi(0)\, d(s_t) - \int_0^T [\phi(0) - \phi(\sigma t)]\, d(s_t) + \int_T^P \phi(\sigma t)\, d(s_t).$$

Make $P_n \to \infty$, using (e); apply the second mean value theorem to the second integral, and, by means of (d), to the third integral in the ranges $(T, T\lambda)$, $(T\lambda, T\lambda^2)$, ...; we then have

$$H > \phi(0)\, s_T - B_\alpha(s_T + m_T) - \omega F(\alpha) \quad (T \geqslant \alpha\delta^{-1}),$$

i.e.

$$s_T < \frac{1}{\phi(\alpha)} [B_\alpha m_T + \omega F(\alpha) + H] \quad (T \geqslant \alpha\delta^{-1}).$$

Hence

$$s_T < M_1 B_\alpha m_T + M_2 \quad (T \geqslant 0), \tag{5}$$

on "majorising" in $(0, \alpha\delta^{-1})$ and noting that $0 < \alpha < \kappa\lambda^{-1} < \kappa$, and consequently $\phi(\alpha) \geqslant \phi(\kappa\lambda^{-1})$. Also

$$s_t < M_1 B_\alpha m_T + M_2 \quad (t \leqslant T), \tag{6}$$

since m_T is non-negative and non-decreasing; and

$$s_T = O(\log T) \quad (T > \lambda) \tag{7}$$

by (5), the definition of m_t, and Lemma 2 (iii).

Now, for $\kappa\delta^{-1} \leqslant \kappa\sigma^{-1} = T < P = P_n$, we have

$$\int_0^P \phi(\sigma t)\, d(s_t) = \phi(\sigma P)\, s_P - \left[\int_0^{T/\lambda} + \int_{T/\lambda}^T + \int_T^P\right] s_t\, d\{\phi(\sigma t)\}.$$

We now "substitute" by means of (6) in the first integral, by means of Lemma 4 in the second integral, and by means of (5) in the last integral, and estimate the resulting integrals with the aid of Lemma 3; then, making $P\,(= P_{n,\sigma}) \to \infty$ and using (7) and Lemma (1), we have, by (e),

$$-H < (M_1 B_\alpha m_T + M_2) M_3 + (S_T + \omega)\gamma + M_1 B_\alpha [m_T \phi(\kappa) + \omega F(\kappa)] + M_2 \phi(\kappa)$$
$$(T \geqslant \kappa\delta^{-1}).$$

Hence

$$s_T > -M_1 B_\alpha m_T \gamma^{-1} [M_3 + \phi(\kappa)]$$
$$-\gamma^{-1}[M_2 M_3 + M_1 B_\alpha \omega F(\kappa) + M_2 \phi(\kappa) + H] - \omega \quad (T \geqslant \kappa\delta^{-1}). \tag{8}$$

If now $m_T \to \infty$ as $T \to \infty$, then, on the one hand, obviously,

$$\liminf_{T\to\infty} \frac{s_T}{m_T} = -1, \tag{9}$$

by the definition of m_T; and on the other,

$$\liminf_{T\to\infty} \frac{s_T}{m_T} \geqslant -M_1\gamma^{-1}[M_3+\phi(\kappa)]\,B_\alpha, \tag{10}$$

by (8), and hence,

$$-1 = \liminf_{T\to\infty} \frac{s_T}{m_T} \geqslant -M_1\gamma^{-1}[M_3+\phi(\kappa)]\,B_\alpha \to 0 \quad \text{as} \quad \alpha\to 0.$$

This is a contradiction, and proves that

$$m_T = O(1). \tag{11}$$

By (5), (8), and (11) the theorem now follows.

LEMMA 5. *If*

$$1<\lambda_p<e^h, \quad \min(\delta, \delta h^{-1})>2\epsilon_p>0, \quad \mu_p=\exp[\tfrac{1}{2}\{\delta/2\epsilon_p-h\}],$$

*and** $\psi(\lambda_p)/\log\lambda_p > -\epsilon_p$; *then* $T(\epsilon_p)$ *exists such that*

$$s_u > s_T-\delta \quad [T(\epsilon_p)<T\leqslant u\leqslant T\mu^2{}_p], \tag{i}$$

$$s_u < s_T+\delta \quad [T\geqslant u\geqslant T\mu_p^{-2}>T(\epsilon_p)]. \tag{ii}$$

By the hypothesis,

$$\Delta = [\delta/(2\epsilon_p\log\lambda_p)]\geqslant 1.$$

Let $u_r = T\lambda_p{}^r$, for r integral and non-negative. Then, for

$$T=u_0\leqslant u_{r-1}\leqslant u\leqslant u_r\leqslant u_\Delta,$$

we have

$$s_u-s_T = s_u-s_{u_{r-1}}+[(s_{u_{r-1}}-s_{u_{r-2}})+(s_{u_{r-2}}-s_{u_{r-3}})+\ldots, (r-1 \text{ terms})]. \tag{12}$$

By the hypothesis, $T(\epsilon_p)$ exists such that

$$s_u-s_T > -2\epsilon_p\log\lambda_p \quad [T\lambda_p\geqslant u\geqslant T>T(\epsilon_p)]. \tag{13}$$

Hence

$$s_u-s_T > -2\epsilon_p\log\lambda_p\,\Delta \quad [T\lambda_p{}^\Delta\geqslant u\geqslant T>T(\epsilon_p)]$$

[by (12) and (13)]

$$\geqslant -2\epsilon_p\log\lambda_p\,\frac{\delta}{2\epsilon_p\log\lambda_p} \quad [T\lambda_p^{-1+\delta/2\epsilon_p\log\lambda_p}\geqslant u\geqslant T>T(\epsilon_p)]$$

and so

$$s_u-s_T > -\delta \qquad [T\mu^2{}_p\geqslant u\geqslant T>T(\epsilon_p)], \tag{14}$$

* *Cf.* the introduction.

since $1<\lambda_p<e^h$; this proves (i), and (ii) follows by interchange of T and u in (14).

THEOREM I.2. *If conditions (a), (b), (c), and (e) of Theorem* I.1 *are satisfied, and a positive number h and a sequence λ_p exist such that*

$$(f)\quad e^h>\lambda_p>1;\quad \lim_{p\to\infty}\frac{\psi(\lambda_p)}{\log\lambda_p}\geqslant 0,$$

then

$$\operatorname*{osc}_{\sigma\to 0} H(\sigma)=|\phi(0)|\operatorname*{osc}_{t\to\infty} s_t. \tag{15}$$

Proof. Obviously, (f) implies (d) of Theorem I.1, and hence

$$|s_t|<\kappa, \tag{16}$$

say. We may also suppose $\phi(t)$ non-negative.

Let $$\limsup_{t\to\infty} s_t=M \quad\text{and}\quad \liminf_{t\to\infty} s_t=m.$$

Then two ascending divergent sequences T_q and T_r exist such that

$$\lim_{q\to\infty} s_{T_q}=M \quad\text{and}\quad \lim_{r\to\infty} s_{T_r}=m.$$

Let δ be any positive number. Then $T_1(\delta)$, $T_2(\delta)$ exist such that

$$s_T<M+\delta,\quad s_{T_q}>M-\delta\quad [T,\ T_q>T_1(\delta)], \tag{17a}$$

$$s_T>m-\delta,\quad s_{T_r}<m+\delta\quad [T,\ T_r>T_2(\delta)]. \tag{17b}$$

By (f), a positive vanishing sequence ϵ_p can be found such that

$$\frac{\psi(\lambda_p)}{\log(\lambda_p)}>-\epsilon_p.$$

Now, for sufficiently large p, the conditions of Lemma 5 are satisfied, and we have

$$s_u>s_{T_q}-\delta>M-2\delta\quad [T_q\mu_p{}^2\geqslant u\geqslant T_q>T_1(\delta),\ T(\epsilon_p)] \tag{18a}$$

by Lemma 5 (i) and (17a); and

$$s_u<s_{T_r}+\delta<m+2\delta\quad [T_r\geqslant u\geqslant T_r\mu_p^{-2}>T_2(\delta),\ T(\epsilon_p)] \tag{18b}$$

by Lemma 5 (ii) and (17b).

Now, for $\delta^{-1}\leqslant\sigma^{-1}=T<T\mu_p<P=P_{n,\sigma}$, we have

$$\int_0^P \phi(\sigma t)\,d(s_t)=\phi(\sigma P)\,s_P-\left[\int_0^{T/\mu_p}+\int_{T/\mu_p}^{T\mu_p}+\int_{T\mu_p}^{P}\right] s_t\,d\{\phi(\sigma t)\};$$

making $P\to\infty$ in this, using (7), Lemma I, and (16), we have

$$H(T^{-1})<\kappa[\phi(0)-\phi(\mu_p^{-1})]+\phi(\mu_p^{-1})\max_{t\geqslant T\mu_p^{-1}} s_t, \tag{19a}$$

$$H(T^{-1})>-\kappa[\phi(0)-\phi(\mu_p^{-1})]+[\phi(\mu_p^{-1})-\phi(\mu_p)]\min_{T\mu_p^{-1}\leqslant t\leqslant T\mu_p} s_t-\kappa\phi(\mu_p), \tag{20a}$$

$$H(T^{-1})>-\kappa[\phi(0)-\phi(\mu_p^{-1})]+\phi(\mu_p^{-1})\min_{t\geqslant T\mu_p^{-1}} s_t \tag{19b}$$

$$H(T^{-1})<\kappa[\phi(0)-\phi(\mu_p^{-1})]+[\phi(\mu_p^{-1})-\phi(\mu_p)]\max_{T\mu_p^{-1}\leqslant t\leqslant T\mu_p} s_t+\kappa\phi(\mu_p). \tag{20b}$$

Now make $T\to\infty$ in (19 a) and (19 b), use (17 a) and (17 b), put $T=T_q\mu_p$ in (20 a) and $T=T_r\mu_p^{-1}$ in (20 b); then, making q and $r\to\infty$, and using (18 a) and (18 b), we have, writing $L=\limsup_{\sigma\to 0} H(\sigma)$, and $l=\liminf_{\sigma\to 0} H(\sigma)$,

$$L\leqslant\kappa[\phi(0)-\phi(\mu_p^{-1})]+\phi(\mu_p^{-1})(M+\delta)\to\phi(0)(M+\delta) \quad \text{as} \quad p\to\infty,$$

$$L\geqslant-\kappa[\phi(0)-\phi(\mu_p^{-1})]+[\phi(\mu_p^{-1})-\phi(\mu_p)](M-2\delta)-\kappa\phi(\mu_p)$$

$$\to\phi(0)(M-2\delta) \quad \text{as} \quad p\to\infty.$$

$$l\geqslant-\kappa[\phi(0)-\phi(\mu_p^{-1})]+\phi(\mu_p^{-1})(m-\delta)\to\phi(0)(m-\delta) \quad \text{as} \quad p\to\infty,$$

$$l\leqslant\kappa[\phi(0)-\phi(\mu_p^{-1})]+[\phi(\mu_p^{-1})-\phi(\mu_p)](m+2\delta)+\kappa\phi(\mu_p)$$

$$\to\phi(0)(m+2\delta) \quad \text{as} \quad p\to\infty;$$

since $\mu_p=\exp[\frac{1}{2}\{\delta/2\epsilon_p-h\}]\to\infty$ as $p\to\infty$. Hence

$$\phi(0)(M+\delta)\geqslant L\geqslant\phi(0)(M-2\delta), \tag{21a}$$

$$\phi(0)(m+2\delta)\geqslant l\geqslant\phi(0)(m-\delta), \tag{21b}$$

and (15) follows since δ is arbitrary.

LEMMA A. *If $\sigma T=r$, a positive integer, then*

$$\max_{0\leqslant t<\infty}\frac{\sigma^{r+1}}{r!}t^r e^{-\sigma t}=O(r^{\frac{1}{2}}T^{-1}).$$

This follows from the known result that $\max_{t\geqslant 0}(t^r e^{-t})=r^r e^{-r}$, and Stirling's asymptotic formula for $r!$.

The following two lemmas are due to Littlewood (*loc. cit.*).

LEMMA B. *For any fixed p greater than unity, and $\sigma > 0$,*

$$\frac{\sigma^{r+1}}{r!}\int_{r/p\sigma}^{rp/\sigma} t^r e^{-\sigma t}\,dt = 1 - f(r)$$

and

$$\frac{\sigma^{r+1}}{r!}\int_0^{r/p\sigma} t^r e^{-\sigma t}\,dt + \frac{\sigma^{r+1}}{r!}\int_{rp/\sigma}^{\infty} t^r e^{-\sigma t}\,dt = f(r),$$

where $f(r)\to 0$ as $r\to\infty$.

LEMMA C. *If $\sigma G(\sigma)\to 0$ as $\sigma\to 0$, and for every r, $\sigma^{r+1}G^r(\sigma)$ is bounded as $\sigma\to 0$, then, for every r, $\sigma^{r+1}G^r(\sigma)\to 0$ as $\sigma\to 0$.*

THEOREM II. *Suppose that s_t is of bounded variation in every finite interval in $(0,\infty)$, $s_0 = 0$, and that either* (i) *$\psi(\lambda_1) < 2\psi(\lambda_2)$ for some λ_1, λ_2, or* (ii) *$\psi(\lambda) > -\infty$ for some λ, and $\lim_{\lambda\to 1+0}\psi(E,\lambda) \geqslant 0$, E being a set of points whose complement $\bar{E}$ is of measure $o(t)$ in (o,t); and $s_t \neq o(1)$ as $t\to\infty$ in E.*

Then $H(\sigma) = \int_0^\infty e^{-\sigma t}\,d(s_t)$ cannot tend to zero as $\sigma\to 0$.

Proof. (i) If $\psi(\lambda_1) \geqslant 0$, the $\psi(\lambda_2) > 0$, $s_t\to\infty$, and $H(\sigma)\to\infty$, and the theorem follows.

If $\psi(\lambda_1) = -2\delta < 0$, then $\psi(\lambda_2) > -\delta + 3\epsilon$, by a proper choice of ϵ, where $0 < \epsilon < \delta$. Then a divergent sequence t_n exists such that $|s_t| > \delta - \epsilon > 0$ for $t = t_n$. Corresponding to t_n there exists a divergent sequence μ_n such that, in any one interval $\mu_n \leqslant t \leqslant \lambda_2\mu_n$, s_t keeps the same sign and $|s_t| > \epsilon$.

(ii) With a proper choice of ϵ and λ_2, it is easily seen that in this case also the same conclusion holds, *provided that $t \subset E$.*

If now $H(\sigma)\to 0$, then, by Theorem I.1, s_t is bounded, say $|s_t| < M$, and

$$H(\sigma) = \sigma\int_0^\infty s_t e^{-\sigma t}\,dt = \sigma G(\sigma)\to 0, \tag{22}$$

where

$$|\sigma^{r+1}G^r(\sigma)| = \left|\sigma^{r+1}\int_0^\infty s_t t^r e^{-\sigma t}\,dt\right| < M\,.\,r!\,; \tag{23}$$

Let now $T_n = \mu_n\lambda_2^{\frac{1}{2}}$, $p = \lambda_2^{\frac{1}{2}}$. Then, for $\sigma_n = rT_n^{-1}$, and r a fixed positive integer,

$$\left|\frac{\sigma_n^{r+1}}{r!}G^r(\sigma_n)\right| = \left|\frac{\sigma_n^{r+1}}{r!}\int_0^\infty s_t t^r e^{-\sigma_n t}\,dt\right|$$

$$= \left|\frac{\sigma_n^{r+1}}{r!}\left[\int_0^{r/p\sigma_n} + \int_{E}^{rp/\sigma_n} + \int_{\bar{E}}^{rp/\sigma_n} + \int_{rp/\sigma_n}^{\infty}\right] s_t t^r e^{-\sigma_n t}\,dt\right|$$

$$\geqslant \epsilon[1 - f(r) - O(r^{\frac{1}{2}}T_n^{-1})\,.\,o(T_n)] - M\,.\,O(r^{\frac{1}{2}}T_n^{-1})\,.\,o(T_n) - Mf(r)$$

by Lemmas A and B, and the measure condition satisfied by E. Hence, making $n \to \infty$, we have

$$(24) \quad \limsup_{\sigma \to 0} \left| \frac{\sigma^{r+1}}{r!} G^r(\sigma) \right| \geqslant \liminf_{n \to \infty} \left| \frac{\sigma_n^{r+1}}{r!} G^r(\sigma_n) \right| \geqslant \epsilon - (M+\epsilon) f(r) > \tfrac{1}{2}\epsilon$$

for sufficiently large r, by Lemma B.

(22), (23), and (24) contradict Lemma C, and the theorem is proved.

COROLLARY. *If* $\lim_{\sigma \to 0} \int_0^\infty e^{-\sigma t} d(s_t) = 0$, s_t *is of bounded variation in every finite interval, and* $s_0 = 0$, *then*

either (i) $\psi(\lambda) = 0$ *for every* λ [*and* $s_t = o(1)$ *by Theorem* I. 2],

or (ii) $\psi(\lambda) = -\infty$ *for every* λ,

or (iii) $-\infty < 2 \lim_{\lambda \to 1+0} \psi(\lambda) \leqslant \lim_{\lambda \to \infty} \psi(\lambda) < 0$.

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KINEMATICAL RELATIVITY (I) : RELATIVELY STATIONARY OBSERVERS

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1. *Introduction and summary.*

The object of Kinematical Relativity is the description of motion in terms of the observer's immediate awareness of a temporal experience. The aim of these and of previous notes is to set this subject as far as possible on an *a priori* basis. In particular, we have already considered the cases in which it is possible for two observers to graduate their clocks so as to describe one another as in uniform relative motion*. We now consider the cases in which it is possible for observers to choose clocks so as to describe each other as at fixed distances apart on the basis of time observations alone; it will be found that the results have a bearing on the meaning of uniform time and its relation to unique time-keeping.

In the case of two observers who can graduate their clocks so as to describe each other as equivalent and relatively stationary, it will be seen that it is not possible to determine a unique time-scale in this manner. Moreover, it is not possible to determine uniquely the running of either clock in terms of the other. There appears in the analysis a periodic function, which possesses the property that the rise from minimum to the succeeding associated maximum is steeper than the fall from maximum to the succeeding associated minimum. An interpretation of this unexpected asymmetry will be given in terms of the Doppler effect.

In the case of three observers in line it will be shown that, if one of them can choose his time-scale so as to describe the other two as relatively stationary, then it is always possible for the other two to choose clocks so that all three are equivalent in pairs. Moreover, this choice of time-scale is unique.

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* Whitrow, *Quart. J. of Math.* (Oxford), 6 (1935), 249–61; also *Zeitschrift für Astrophysik*, 12 (1936), 47–55.

In this way it is possible to say what is meant by uniform time without appealing to indefinable concepts.

2. *Two observers.*

As Milne has already remarked, if we dispense with the concept of the rigid measuring rod it is not at once evident what is meant by saying that two observers are "relatively stationary"*. We therefore begin from first principles.

A is defined to be an observer who can correlate the totality of possible events in his temporal experience with the continuum of real numbers. A clock is defined to be any mechanism for effecting such a correlation. All clocks run in the same sense. Consequently, if the analytical relation between any two clocks t, τ is

$$t = \psi(\tau),$$

then $\psi(\tau)$ may be any monotonic increasing function, subject to the following conditions. It must exist for all values of τ and must be capable of taking all values, *i.e.* it must diverge with τ. It must also be single valued and continuous. Moreover, a finite interval in t must correspond to a finite interval in τ and conversely.

Let B be another observer. We assume that A, B can communicate with each other by light signalling, the light-paths being unique†. If all other particles are excluded from consideration, no further physical restrictions need be imposed‡. If not, we then stipulate that each observer finds for all time that the other maintains a fixed orientation. Alternatively, we may regard them as occupants of a one-dimensional universe.

A emits in the direction of B at time t_1 a signal which is instantaneously reflected on arrival and returns to A at t_2. This process can be repeated indefinitely and A can tabulate for each signal the corresponding time-

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* Milne, *Relativity, gravitation, and world-structure* (Oxford, 1935), § 14. In future references this will be referred to as WS. I should like to express my thanks to Prof. Milne for allowing me access to some of his unpublished investigations. These are acknowledged *seriatim*.

† Robertson, *Astrophysical Journal*, 82 (1935), 288. This condition may be relaxed by stipulating that if at each instant signals are sent from A along every possible light-path to B, then B need consider "first arrivals" only. This can easily be arranged if A emits signals of frequency $\nu(t)$, where $\nu(t)$ is a monotonic function of A's local time. I owe this point to Dr. A. G. Walker. See also Robb, *Geometry of time and space* (Cambridge, 1936), 21.

‡ Milne, in an unpublished investigation, has found that two equivalent observers at a fixed distance apart cannot discover whether they are rotating about each other, merely from observations on each other. Moreover, since the observers considered are located at particles, it is not possible for them to determine directions by observations on themselves.

difference t_2-t_1. Each entry will be either positive or zero. If, when A, B are not coincident, each entry is zero, the system is said to be Newtonian. Excluding this case, it follows that, if A, B never coincide, then t_2-t_1 is strictly positive. We now suppose that over a long period of observation the fluctuations, if any, in the value of this time-difference have definite upper and lower bounds. In other words, A can determine a maximum m and a minimum M such that

$$m \leqslant t_2-t_1 \leqslant M. \tag{1}$$

If A regraduates his time-scale from t to τ by the formula

$$t = \psi(\tau), \tag{2}$$

let the epochs t_1, t_2 recorded by the first clock correspond to epochs τ_1, τ_2 recorded by the second. From (1) it follows that

$$\psi(\tau_2)-\psi(\tau_1) \geqslant m. \tag{3}$$

Consequently $\tau_2-\tau_1$ must have a positive lower bound*. If not, then for an arbitrarily small ϵ we can find a value of τ_1 so that the corresponding time-difference $\tau_2-\tau_1 < \epsilon$. Hence an arbitrarily small interval on the τ clock corresponds to a definite finite interval on the t clock, and this is incompatible with our physical postulates.

Similarly we can prove that $\tau_2-\tau_1$ must have an upper bound. If not, then for an arbitrarily large η there is a value of τ_1 so that the corresponding time-difference $\tau_2-\tau_1 > \eta$. Since, for this value of τ_1,

$$\psi(\tau_2)-\psi(\tau_1) \leqslant M, \tag{4}$$

an arbitrarily large interval on the τ clock corresponds to a finite interval on the t clock; this is excluded. We thus find that, if t_2-t_1 is bounded above and below, this feature of B's motion will persist throughout all possible regraduations of A's clock.

We now investigate the conditions under which B will automatically observe the same feature in A's motion. It will be found that the following three postulates are sufficient:

(i) that signals are received by either observer in the order in which they were emitted by the other;

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* Dingle, *Philosophy*, 11 (1936), 53, remarks that "however the distance of a body is 'changing' we can always choose a clock so that $\frac{1}{2}c(t_2-t_1)$ remains constant in time." It is clear from the considerations above that this statement must be qualified, for the choice will not be open if the original $\frac{1}{2}c(t_2-t_1)$ had a zero.

(ii) that a signal, which is emitted at a finite epoch by one observer, cannot be received by the other at an epoch which is negatively infinite;

(iii) that a signal, which is received at a finite epoch by one observer, cannot have been emitted by the other at an epoch which is positively infinite.

It follows from (i) that if the function expressing t', the time of reception by B, according to B's clock, of a signal leaving A at t, is

$$t' = \theta_{12}(t), \tag{5}$$

then θ_{12} must be monotonic increasing. Moreover, since B considers "first arrivals" only, θ_{12} will be single-valued. It is also continuous, but may diverge for a finite value of t.

If the signal which leaves A at t_1 is instantaneously reflected by B at t_3' and returns to A at t_2, it follows from postulates (ii) and (iii) that if t_1, t_2 are finite then t_3' must be finite. Similarly, if the signal is instantaneously reflected by A and arrives again at B at t_4', then t_4' must be finite. We thus find that in this case θ_{12} must be a function such that a finite interval in t' corresponds to a finite interval in t, and conversely. Hence θ_{12} obeys the same analytical conditions as ψ, and so it follows that, if t_2-t_1 is bounded above and below, then $t_4'-t_3'$ is similarly bounded and this is true in any time-scale chosen by B.

3. *Regraduation of the time-scale.*

Two observers A, B are said to be equivalent if the totality of observations which A can make on B can be described by A in the same way as the totality of observations which B can make on A can be described by B*. It is assumed that A, B are provided only with clocks and with apparatus for emitting and reflecting light signals. From the definition of equivalence it follows that the functions relating the times of emission and reception of signals passing from A to B and from B to A respectively must be identical.†

Following Milne, we call $\frac{1}{2}c(t_2-t_1)$, where c is an arbitrarily chosen constant, the distance of B according to A at the conventional epoch $\frac{1}{2}(t_2+t_1)$. A particular case of bounded t_2-t_1 therefore occurs when A regards B as relatively stationary. We now prove that, if in any particular

* Milne, WS, § 11.

† Whitrow, *Quart. J. of Math.* (Oxford), 6 (1935), 250.

time-scale the interval t_2-t_1 is bounded, then it is always possible for A, B to choose time-scales so as to regard each other as equivalent and relatively stationary.

We begin by drawing attention to the following lemma. Let A, B choose new clocks τ, τ' respectively. Consider a signal emitted by A at τ_1, instantaneously reflected by B at τ_3', and returning to A at τ_2. Then, if

$$\tau_3' = \Omega(\tau_1), \quad \tau_2 = \Omega(\tau_3'),$$

where
$$\Omega(\tau) \equiv \tau + a, \tag{6}$$

a being a constant, A and B will describe each other as equivalent and relatively stationary. Moreover, their clocks will read the same time. The proof of this lemma is immediate.

Returning to the original situation, in which A, B possess clocks t, t' respectively, we consider, as before, a signal emitted by A at t_1, instantaneously reflected by B at t_3', and returning to A at t_2. Let the functional relations between these times be

$$t_3' = \theta_{12}(t_1), \quad t_2 = \theta_{21}(t_3') \tag{7}$$

respectively. If A, B regraduate their time-scales by the formulae

$$t = \psi_1(\tau), \quad t' = \psi_2(\tau') \tag{8}$$

respectively, so that τ_1, τ_2, τ_3' correspond to t_1, t_2, t_3' respectively, it follows that

$$\tau_3' = \psi_2^{-1}\theta_{12}\psi_1(\tau_1), \quad \tau_2 = \psi_1^{-1}\theta_{21}\psi_2(\tau_3').$$

We therefore look for functions ψ_1, ψ_2 such that

$$\psi_2^{-1}\theta_{12}\psi_1(\tau) \equiv \tau + a \equiv \psi_1^{-1}\theta_{21}\psi_2(\tau).$$

Since ψ_1, ψ_2 and their inverses must be single-valued, it follows that

$$\theta_{12}\psi_1(\tau) \equiv \psi_2(\tau + a), \quad \theta_{21}\psi_2(\tau) \equiv \psi_1(\tau + a). \tag{9}$$

From these relations it follows immediately that*

$$\left.\begin{aligned} \theta_{21}\theta_{12}\psi_1(\tau) &\equiv \psi_1(\tau + 2a), \\ \theta_{12}\theta_{21}\psi_2(\tau) &\equiv \psi_2(\tau + 2a). \end{aligned}\right\} \tag{10}$$

* These relations are independent if A, B choose clocks merely to describe each other as at the same fixed distance apart. The additional relation between ψ_1, ψ_2 is to ensure equivalence.

It is clearly sufficient to consider the former equation only, for, if this equation is solved for $\psi_1(\tau)$, then $\psi_2(\tau)$ is given by

$$\psi_2(\tau) \equiv \theta_{21}^{-1}\psi_1(\tau+\alpha). \tag{11}$$

It is easily verified that if the equation

$$\theta_{21}\theta_{12}\psi_1(\tau) \equiv \psi_1(\tau+2\alpha) \tag{12}$$

determines a possible clock regraduation $t=\psi_1(\tau)$, then $\{\theta_{21}\theta_{12}(\tau)-\tau\}$ must be automatically a positive bounded function. If we define new functional operators Φ, Ψ by the relations

$$\Phi \equiv \theta_{21}\theta_{12}, \quad \Psi \equiv \psi_1^{-1}, \tag{13}$$

then, since

$$\tau = \Psi(t), \tag{14}$$

equation (12) may be written in the form

$$\Psi\Phi(t) \equiv \Psi(t)+2\alpha. \tag{15}$$

Assuming that α may be any arbitrarily chosen positive number, our object is to construct the function $\Psi(t)$ knowing the form of $\Phi(t)$ for past values of t only.

We are given that the signal emitted by A at any arbitrarily chosen instant t_1, by the original clock, is instantaneously reflected by B and returns to A at

$$t_2 = \Phi(t_1). \tag{16}$$

For the range of values of t given by

$$t_1 \leqslant t \leqslant \Phi(t_1),$$

let

$$\tau = \Psi(t)$$

be any continuous curve inclined everywhere at an acute angle to the axes and obeying the boundary conditions*

$$\Psi\Phi(t_1) = \Psi(t_1)+2\alpha, \quad \left[\frac{d\{\Psi\Phi(t)\}}{dt}\right]_{t=t_1} = \left[\frac{d\Psi(t)}{dt}\right]_{t=t_1}, \quad \ldots. \tag{17}$$

Since

$$\Phi(t) \geqslant t+m,$$

it follows that

$$\Phi\Phi(t_1) \geqslant \Phi(t_1)+m \geqslant t_1+2m$$

and, generally,

$$\Phi^n(t_1) \geqslant t_1+nm.$$

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* These conditions are imposed merely to secure continuity, differentiability, and so forth, at all instants.

Consequently we can always find a positive integer n so that, for any

$$t > \Phi(t_1),$$

$$\Phi^n(t_1) \leqslant t \leqslant \Phi^{n+1}(t_1).$$

The value of τ corresponding to this t is then defined by the relation

$$\Psi(t) = \Psi\{\Phi^{-n}(t)\} + 2n\alpha. \tag{18}$$

Hence, given Φ, the value of $\Psi(t)$ at any instant may be deduced from its behaviour for past values of t only.

The value to be assigned to τ corresponding to any given t may now be determined by the following rule. Let the instant at which the signal returning to A at $t = T_1$ was emitted by A be $t = T_2$. Let the instant at which the signal returning to A at $t = T_2$ was emitted by A be $t = T_3$, and so on. If, after repeating this process $n-1$ times, we arrive at an instant T_n in the range

$$t_1 \leqslant T_n \leqslant \Phi(t_1), \tag{19}$$

then the value to be assigned to τ at $t = T_1$ is

$$\tau = \Psi(T_n) + 2n\alpha. \tag{20}$$

As stated this rule applies only if $t \geqslant t_1$, but the process of extrapolation for $t < t_1$ is immediate and requires no further comment.

We therefore see that if, adopting any arbitrary time-scale A finds that $t_2 - t_1$ is positive and bounded, then it is always possible for A, B to choose time-scales so as to describe each other as equivalent and relatively stationary. This choice of clocks is not unique. To prove this, we first consider the case already treated in which the clocks are chosen so as to read the same time. If $\psi_1(\tau)$ be the general solution of the equation

$$\psi_1(\tau + 2\alpha) \equiv \theta_{21}\theta_{12}\psi_1(\tau), \tag{21}$$

let $\omega(\tau)$ be defined so that

$$\psi_1(\tau) \equiv \psi\{\tau + \omega(\tau)\}, \tag{22}$$

where $\psi(\tau)$ is any particular solution of the equation*. Substituting this form for $\psi_1(\tau)$, we find that

$$\psi\{\tau + 2\alpha + \omega(\tau + 2\alpha)\} \equiv \theta_{21}\theta_{12}\psi\{\tau + \omega(\tau)\}. \tag{23}$$

* A similar substitution is used by Cherry, *Proc. Camb. Phil. Soc.*, 21 (1923), 715.

By replacing τ by $\tau+\omega(\tau+2a)$ in (21), we see that ψ must satisfy the relation

$$\psi\{\tau+2a+\omega(\tau+2a)\}\equiv\theta_{21}\theta_{12}\psi\{\tau+\omega(\tau+2a)\}. \tag{24}$$

Hence, since θ_{21}, θ_{12}, ψ are single valued increasing functions, it follows by comparing (23) and (24) that

$$\omega(\tau+2a)\equiv\omega(\tau). \tag{25}$$

Consequently the general solution of (21) is

$$\psi_1(\tau)\equiv\psi\{\tau+\omega(\tau)\},$$

where $\psi(\tau)$ is a particular solution and $\omega(\tau)$ is any function of period $2a$ such that $\tau+\omega(\tau)$ is monotonic increasing. This proves that A's choice of clock is not unique, and similarly for B.

4. *Synchronization of clocks by two observers.*

If A, B describe each other as equivalent and relatively stationary, it is not obvious *a priori* that their clocks must read the same time. In point of fact, it will be shown that in general this is not the case.

Let the distance apart of A, B be a constant multiple a of the signal velocity c. Then the function Ω relating the times of emission and reception of any signal passing from A to B or from B to A satisfies the equation

$$\Omega\Omega(t)\equiv t+2a. \tag{26}$$

This equation may be solved graphically*. Alternatively, the characteristic features of the general solution may be established thus.

Writing $\Omega(t)$ for t, we find that

$$\Omega(t+2a)\equiv\Omega(t)+2a. \tag{27}$$

Hence
$$\Omega(t)\equiv t+\omega(t), \tag{28}$$

where $\omega(t)$ is a function of period $2a$†. It is not, however, an arbitrary periodic function, but satisfies the equation

$$\omega(t)+\omega\{t+\omega(t)\}\equiv 2a. \tag{29}$$

Replacing t by $t+\omega(t)$, we can immediately verify that any solution of this equation is of period $2a$. Consequently $\omega(t)$ is any solution of this equation such that $t+\omega(t)$ is monotonic increasing.

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* Hardy, *Orders of infinity* (Cambridge, 1924), 31.

† This $\omega(t)$ is not the same as the $\omega(\tau)$ of § 3.

An equation equivalent to (29) was first considered in this connection by Milne. We therefore state the leading properties of its solution briefly. Since, as may easily be verified, $\omega(t)$ cannot take the values 0, $2a$, and since*

$$\Omega(t) > t, \tag{30}$$

it follows that $\omega(t)$ is restricted to some range of values $(a-b, a+b)$, where $0 \leqslant b < a$. If t_0 is a minimum, then $t_0+\omega(t_0)$ is the succeeding associated maximum and t_0+2a is the succeeding associated minimum. The surprising feature is that the interval from minimum to succeeding associated maximum is less than half the period. It was this result which originally led Milne to apply this equation to the theory of Cepheid variation†.

If $f(t)$ is the function expressing the time t' of an event at either observer in terms of the hypothetical epoch t assigned to it by the other observer, then, since

$$t = \tfrac{1}{2}(t_2+t_1),$$

where $$t' = \Omega(t_1), \quad t_2 = \Omega(t'),$$

it follows that $$f^{-1}(t) \equiv \tfrac{1}{2}\{\Omega(t)+\Omega^{-1}(t)\}. \tag{31}$$

Hence, if $$\Omega\Omega(t) \equiv t+2a,$$

it is readily found that

$$f^{-1}(t) \equiv \Omega(t)-a \equiv t+\{\omega(t)-a\}. \tag{32}$$

Since $\omega(t)$ is not necessarily identically a, it follows that the clocks kept by two equivalent observers who describe each other as relatively stationary do not necessarily agree‡. Since f is not uniquely specified, we conclude that the fact that two observers describe each other as equivalent and at a given fixed distance apart is not sufficient to determine uniquely the running of the clock kept by either in the time-experience of the other.

5. *The Doppler effect.*

It follows from the analysis of the previous section that the Doppler shift ratio of an atom at one observer keeping the proper time of that observer is seen by the other to be

$$\frac{\lambda_0+\delta\lambda}{\lambda_0} = \frac{\text{apparent wave-length}}{\text{proper wave-length}} = \Omega'(t) = 1+\omega'(t). \tag{33}$$

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* This follows from the fact that $\Omega(t)$ is monotonic increasing and $\Omega\Omega(t) > t$.

† Milne, *Quart. J. of Math.* (Oxford), 4 (1933), 258–77.

‡ This result was first obtained by Milne. Since we have a constant a of the dimensions of time at our disposal, we cannot restrict the form of $\omega(t)$ by any dimensional argument.

Hence $$\omega'(t) = \frac{\delta\lambda}{\lambda_0}. \tag{34}$$

Since $$\omega(t)+\omega\{t+\omega(t)\} \equiv 2a, \tag{35}$$

it follows that $$\omega'\{t+\omega(t)\} \equiv -\left\{\frac{\omega'(t)}{1+\omega'(t)}\right\}, \tag{36}$$

and so, if $$\omega'(t_1) > 0,$$

then $$\omega'\{t_1+\omega(t_1)\} < 0.$$

Consequently, if $\omega'(t_1)$ is the greatest value which $\omega'(t)$ attains, then $\omega'\{t_1+\omega(t_1)\}$ is the least value. Moreover, if

$$\omega'(t_1) = k_1, \quad \omega'\{t_1+\omega(t_1)\} = -k_2,$$

then, from (36), $$(1+k_1)(1-k_2) = 1.$$

Hence, writing $$1+k_1 = \frac{\lambda_{\max}}{\lambda_0}, \quad 1-k_2 = \frac{\lambda_{\min}}{\lambda_0},$$

we see that $$\lambda_{\max}\lambda_{\min} = \lambda_0{}^2. \tag{37}$$

We thus find that any particular line in the spectrum of an atom keeping the proper time of one observer is seen by the other to oscillate in such a way that the proper wave-length is the geometric mean of the greatest and least apparent wave-lengths. Since the arithmetic mean of two unequal numbers is greater than their geometric mean, it follows that

$$\lambda_{\max}-\lambda_0 > \lambda_0-\lambda_{\min}, \tag{38}$$

and hence the rise of $\omega(t)$ from minimum to succeeding associated maximum is steeper than the fall from maximum to succeeding associated minimum. *Thus, although the observers regard their relative distance to be stationary, as judged by light signalling experiments, they may each see a fluctuating Doppler effect, assuming that their atomic clocks keep the same time as their macroscopic clocks.*

6. *Three observers.*

Let A, B, C be three collinear observers. Then, with an obvious extension of our previous notation, if A can graduate his clock so as to regard both B and C as relatively stationary, there must exist a function ψ_1 such that

$$\theta_{21}\theta_{12}\psi_1(\tau) \equiv \psi_1(\tau+2a_{12}), \quad \theta_{31}\theta_{13}\psi_1(\tau) \equiv \psi_1(\tau+2a_{13}), \tag{39}$$

where α_{12}, α_{13} are suitably chosen constants. Hence, if there exist graduations of A's clock for which B is relatively stationary and graduations for which C is relatively stationary, it will not, in general, be the case that any of these scales coincide. For if there is a common time-scale, then from (39) we see that the operators $\theta_{21}\theta_{12}$, $\theta_{31}\theta_{13}$ respectively must be expressible in the form

$$\theta_{21}\theta_{12}(t) \equiv \psi_1\{\psi_1^{-1}(t)+2\alpha_{12}\}, \quad \theta_{31}\theta_{13}(t) \equiv \psi_1\{\psi_1^{-1}(t)+2\alpha_{13}\}.$$

It is easily verified that a necessary, but not a sufficient, condition for this to be possible is that $\theta_{21}\theta_{12}$, $\theta_{31}\theta_{13}$ are commutative; in general, of course, this will not be so.

We now confine attention to the case in which it is assumed *ab initio* that A can graduate his clock so as to regard both B and C as relatively stationary. *It will be found that it is then possible for B and C to choose clocks so that all three observers are equivalent in pairs.* We have already seen that A's choice of clock is determined by equations (39). Let B and C regraduate their clocks by the formulae

$$t' = \psi_2(\tau'), \quad t'' = \psi_3(\tau''), \tag{40}$$

respectively, where ψ_2, ψ_3 are determined in terms of ψ_1 and the signal functions θ_{21}, θ_{31} by the relations

$$\theta_{21}\psi_2(\tau) \equiv \psi_1(\tau+\alpha_{12}), \quad \theta_{31}\psi_3(\tau) \equiv \psi_1(\tau+\alpha_{13}). \tag{41}$$

From the first equation of (39) and the first equation of (41) we readily find that

$$\theta_{21}\theta_{12}\psi_1(\tau) \equiv \theta_{21}\psi_2(\tau+\alpha_{12}).$$

Similarly we can prove that

$$\theta_{31}\theta_{13}\psi_1(\tau) \equiv \theta_{31}\psi_3(\tau+\alpha_{13}).$$

Hence
$$\theta_{12}\psi_1(\tau) \equiv \psi_2(\tau+\alpha_{12}), \quad \theta_{13}\psi_1(\tau) \equiv \psi_3(\tau+\alpha_{13}). \tag{42}$$

From equations (41) and (42) it is immediately seen that if, in terms of the new clocks, a signal is emitted by A at τ_1, passes B at τ_2', and is instantaneously reflected by C at τ_3'' so that it passes B again at τ_4' and returns to A at τ_5, then

$$\left.\begin{aligned} \tau_2' &= \tau_1+\alpha_{12}, \quad \tau_3'' = \tau_1+\alpha_{13}, \\ \tau_5 &= \tau_4'+\alpha_{12}, \quad \tau_5 = \tau_3''+\alpha_{13}. \end{aligned}\right\} \tag{43}$$

From these equations it follows immediately that

$$\tau_3'' = \tau_2' + a_{23}, \quad \tau_4' = \tau_3'' + a_{23},$$

where a_{23} is such that

$$a_{12} + a_{23} = a_{13}.$$

It is now evident that with the new choice of clocks A, B, and C regard each other as relatively stationary and equivalent in pairs. Moreover, their clocks keep the same time and the distances which they assign to each other are additive.

Incidentally we observe that if, with the new time-scale, the coordinates assigned by A, B to any collinear event E are (σ, τ) and (σ', τ') respectively, then

$$\tau' - \sigma'/c = \tau - \sigma/c + a, \quad \tau' + \sigma'/c = \tau + \sigma/c - a,$$

where a is written for a_{12}. Consequently $\tau = \tau'$ and

$$\sigma = \sigma' + ca.$$

Hence A, B assign the same epoch to any collinear event and the distances between A, B, E are additive.

7. *Synchronization of clocks by three observers.*

We now inquire if the fact that three collinear observers describe each other as equivalent and relatively stationary implies that their clocks must read the same time. Let the signal function* connecting A, B be Ω_{12} and that connecting B, C be Ω_{23}. If A, B are at a distance ca apart and B, C at a distance cb apart, then

$$\Omega_{12}\,\Omega_{12}(t) \equiv t + 2a, \quad \Omega_{23}\,\Omega_{23}(t) \equiv t + 2b. \tag{44}$$

If Ω_{13} is the signal function connecting A, C, it follows that

$$\Omega_{13} \equiv \Omega_{23}\,\Omega_{12} \equiv \Omega_{12}\,\Omega_{23}. \tag{45}$$

Hence $$\Omega_{13}\,\Omega_{13}(t) \equiv \Omega_{12}\,\Omega_{12}\,\Omega_{23}\,\Omega_{23}(t) \equiv t + 2(a+b),$$

and so the distances apart of A, B, C must be additive†.

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* As before, this means that if a signal is emitted by A at t_1 and received by B at t_2', then $t_2' = \Omega_{12}(t_1)$, and so forth.

† Since A, B, C are equivalent, Ω_{12}, Ω_{23} must be commutative.

Operating on both sides of the first equation of (44) with Ω_{13}, we get

$$\Omega_{23}(t+2a) \equiv \Omega_{23}\,\Omega_{12}\,\Omega_{12}(t),$$

and hence, since Ω_{12}, Ω_{23} are commutative,

$$\Omega_{23}(t+2a) \equiv \Omega_{12}\,\Omega_{12}\,\Omega_{23}(t) \equiv \Omega_{23}(t)+2a.$$

It follows that

$$\Omega_{23}(t) \equiv t+\omega(t), \tag{46}$$

where $\omega(t)$ is of period $2a$; but, from the second equation of (44), it follows that $\omega(t)$ must be of period $2b$. Since a, b are in general incommensurable it follows from a well-known theorem that, if $\omega(t)$ is continuous, it must be identically constant, and hence must be equal to b. Hence

$$\Omega_{23}(t) \equiv t+b, \tag{47}$$

and similarly

$$\Omega_{12}(t) \equiv t+a. \tag{48}$$

Consequently the clocks kept by A, B, C must read the same time*.

8. *Four observers.*

If A, B, C are three collinear observers who can choose clocks so as to describe each other as equivalent and if A, B regard each other as relatively stationary, it does not necessarily follow that B, C must be relatively stationary. For, if

$$\Omega_{12}\,\Omega_{12}(t) \equiv t+2a,$$

then $\Omega_{23}(t)$ must satisfy the equation

$$\Omega_{23}(t+2a) \equiv \Omega_{23}(t)+2a.$$

Hence

$$\Omega_{23}(t) \equiv t+\omega(t),$$

where $\omega(t)$ is any function of period $2a$ such that $t+\omega(t)$ is monotonic increasing. Consequently

$$\Omega_{23}\,\Omega_{23}(t) \equiv t+\omega(t)+\omega\{t+\omega(t)\}, \tag{49}$$

and since, in general,

$$\omega\{t+\omega(t)\}+\omega(t) \not\equiv \text{constant}, \tag{50}$$

* This result was first obtained by Milne, working in terms of the function f. In actual practice it is only possible to approximate to measurements by rational numbers, and so a function of extremely small period will persist in the synchronization formula.

it follows that B, C do not, in general, describe each other as relatively stationary*. If, however, A, B, C are all given to be relatively stationary, and if their mutual distances are incommensurable, then any fourth equivalent observer D must also be relatively stationary.

9. *Uniform time.*

Finally, if A, B, C are three observers who describe each other as equivalent and relatively stationary, then their respective time-scales are effectively unique. In time-scale t let A, B and B, C be at distances ca, cb apart respectively. Let each observer regraduate his clock by the formula

$$t = \psi(\tau). \tag{51}$$

If, with the new time-scale, A, B and B, C describe each other as at ca, $c\beta$ apart respectively, then

$$\left.\begin{aligned} \psi(\tau+2a)-\psi(\tau) &\equiv 2a, \\ \psi(\tau+2\beta)-\psi(\tau) &\equiv 2b. \end{aligned}\right\} \tag{52}$$

The general solution of this pair of equations is

$$\psi(\tau) \equiv \kappa\tau + \varpi(\tau), \tag{53}$$

where

$$a\kappa = a, \quad \beta\kappa = b, \tag{54}$$

and $\varpi(\tau)$ admits periods $2a$, 2β. Hence a, β must be in the same ratio as a, b and $\varpi(\tau)$ must be identically constant. Since a transformation of time-scale of the form $t = \psi(\tau)$, where

$$\psi(\tau) \equiv \kappa\tau + \kappa', \tag{55}$$

merely signifies a change in time zero and in time unit, it follows that the clocks t, τ are effectively the same. Consequently, if an observer describes two other collinear observers as relatively stationary, then, in general, the time-numbering of his clock, and the ratio of the distances which he assigns to the other observers, are unique. Moreover, as we have already seen, the two other observers can each choose clocks so as to be equivalent to the first observer and to each other. The time-numberings of these clocks will also be unique and all three clocks will keep the same time.

* Contrast the case of collinear observers in uniform relative motion. Any third observer equivalent to two equivalent observers in uniform relative motion must also be in uniform relative motion. This is readily proved.

This unique time-numbering they will all call uniform time. Hence, as Milne has remarked, "uniform time is a convention based on the simplest set of phenomena occurring in nature, as is obvious when we examine its astronomical definition in terms of the earth's rotation, or in terms of dynamical clocks assumed to remain invariable"*. Uniform time, in this sense, is essentially of a macroscopic or gravitational nature. Whether it coincides with the time kept by atomic clocks is, in the present state of our knowledge, an unverifiable postulate, since we have no experimental means of comparing an atomic frequency with any macroscopic standard of time†. Consequently, Dingle's method of defining relatively stationary observers as those which mutually exhibit no Doppler shift in their emission spectra is open to criticism, in so far as it is possible *a priori* for two such observers to have any relative motion whatsoever, if no assumptions are made about the behaviour of atomic clocks‡. It follows that the definition adopted in this paper§, although in some cases more difficult to put into practice than Dingle's, is ideally simpler.

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* Milne, *Philosophy*, 9 (1934), 29.

† K. T. Birge, *Nature*, 134 (1934), 771.

‡ Dingle, *loc. cit.* He also appears to confuse the congruent division of a rigid body with the fixity of its ends. In point of fact it is not possible to use his method to determine the mid-point of a rod. Moreover, if the far end of a rod is relatively fixed, it does not follow that we can transport it and use it as a length measure.

§ [*Added, July* 16, 1936.] The arguments on bounded functions, in this paper, are valid for finite epochs only.

SUPERPOSITIONS OF FUNCTIONS (II): TRANSFINITE SUPERPOSITIONS OF ABSOLUTELY CONTINUOUS FUNCTIONS

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1. *Introduction.* In this note we continue the study of transfinite superpositions of absolutely continuous functions which was commenced in **4**. We indicate, in § 3, a class† of functions included in $\Sigma(\omega+1;\ I)$ and, in § 5, a particular member of this class which belongs to $S(\omega+1;\ I)$. This function was obtained by a modification of an example suggested by Mlle Nina Bary‡. In § 6 we shall see that this function is, in certain respects, a best possible example since any "less singular" function of its type belongs to $\Sigma(\omega;\ I)$, Naturally, the results and methods of **4** play an important part in the present paper.

2. *An elementary lemma.* In this paragraph we shall prove the following

LEMMA. *Every continuous function with but a finite set of maxima and minima is a polygonal function of a monotone continuous function. Further, this polygonal function may be assumed to have its sides (alternately) parallel to the bisectors of the angles between the axes.*

We use the terms "maxima", "minima" in the sense of Hobson (**3**, 348). The lemma can be extended to the case when there is an infinity (necessarily countable) of maxima and minima provided that they form a reducible (**3**, 125) set and that the function is of bounded variation in

† The notations introduced in **4** will be used except that, instead of $S(\alpha;\ I)$ and $\Sigma(\alpha;\ I)$, we shall write $S(\alpha)$ and $\Sigma(\alpha)$.

‡ I take this opportunity of thanking Mlle Bary for her personal interest in my researches in this field. My indebtedness to her published work will be apparent to those familiar with it.

every closed interval in the interior of its interval of definition. These conditions are required so that we can piece together the monotone inner functions corresponding to certain sub-intervals. The conclusion regarding the sides of the polygonal function is replaced by the result that the outer function has its derived numbers bounded by ± 1. These generalizations are of no immediate importance in the present connection and we prove the lemma in the finite case only.

We assume $f(x)$ to be defined and continuous for $a \leqslant x \leqslant b$ and to have one maximum and no minimum in this interval.

Suppose, in fact, that $f(x)$ is strictly increasing in (a, c), constant in (c, d), and strictly decreasing in (d, b), where $a < c \leqslant d < b$.

Consider the functions f_1, P_1 defined by the equations

$$f_1(x) = f(x) \quad (a \leqslant x \leqslant d), \qquad f_1(x) = 2f(d) - f(x) \quad (d \leqslant x \leqslant b),$$

$$P_1(z) = z \quad [f(a) \leqslant z \leqslant f(d)], \qquad P_1(z) = 2f(d) - z \quad [f(d) \leqslant z \leqslant 2f(d) - f(b)].$$

It is clear that we have (i) $f_1(x)$ is monotone in (a, b), (ii) $y = P_1(x)$ is a polygonal function of the proper type, and (iii) $f(x) = P_1\{f_1(x)\}$. So our lemma is proved when the function has one maximum and no minimum. The case when the function has one minimum and no maximum is reduced to the present case by considering its negative. We complete the proof from this by finite induction.

Assume, then, that our lemma is proved for functions with not more than n maxima or minima and consider a function $f(x)$ with $n+1$ maxima or minima in the interval (a, b). We shall further suppose that the last (*i.e.* that furthest to the right) of these is an interval (c, d) (possibly reducing to a point) which is a maximum; the other case is reduced to the present by the consideration of $-f(x)$.

The function $f(x)$ considered only in (a, d) has n maxima or minima there and so can be represented in the form $f(x) = P\{g(x)\}$ $(a \leqslant x \leqslant d)$, where $g(x)$ is monotone and continuous and $y = P(x)$ is a polygonal function of the proper type. We may assume that g is increasing, and then the last side of $y = P(x)$ is necessarily parallel to $y = x$.

We now consider the functions g^*, P^* defined by the equations

$$g^*(x) = g(x) \qquad (a \leqslant x \leqslant d),$$

$$g^*(x) = g(d) + f(d) - f(x) \quad (d \leqslant x \leqslant b),$$

$$P^*(z) = P(z) \qquad [g(a) \leqslant z \leqslant g(d)],$$

$$P^*(z) = g(d) + f(d) - z \quad [g(d) \leqslant z \leqslant g(d) + f(d) - f(b)].$$

It is clear that we have

$$f = P^*\{g^*(x)\}, \quad (a \leqslant x \leqslant b),$$

where g^*, P^* satisfy the conditions of the lemma, which we can now assume established in all cases.

3. *A sub-class of* $\Sigma(\omega+1)$. Since a polygonal function is absolutely continuous, we have, from the preceding lemma and the result of **4**, the following

THEOREM 1. *Every continuous function with but a finite set of maxima and minima belongs to* $\Sigma(\omega+1)$.

Each successive generalization of the lemma gives us a larger class of functions of $\Sigma(\omega+1)$, but these are not important in the present connection. A refinement of this theorem is available.

THEOREM 2. *Every continuous function with but a finite set of maxima and minima belongs to* $S(1)$ *or* $S(\omega)$ *or* $S(\omega+1)$.

This includes a refinement of the result that *every monotone continuous function belongs to* $\Sigma(\omega)$, which was not explicitly mentioned in **4**. It was shown there that a necessary and sufficient condition for a monotone continuous function to belong to $S(\omega)$ was that it should not satisfy the condition N. It is, however, well-known† that a continuous function which satisfies the condition N and is of bounded variation is absolutely continuous. So *every monotone continuous function belongs either to* $S(1)$ *or to* $S(\omega)$. We use this result in the proof of Theorem 2. Any f satisfying the conditions can be expressed in the form $f = P\{g\}$, where P is polygonal and either (i) $g \in S(1)$ or (ii) $g \in S(\omega)$. In the first case, P having bounded derived numbers, we can apply a familiar result‡ to deduce that $f \in I = S(1)$. So, to establish our theorem, it remains to prove that, if $g \in S(\omega)$, then $g \supset S(2)$. Now, since $g \in S(\omega)$, we have§ $g \supset N$, and so‖ $f = P\{g\} \supset N$ (except in the trivial case when P is constant).

We now notice that we must have $f \supset N$ on some one at least of the sub-intervals on which it is monotone, and therefore f cannot belong to

† Compare (**5**, 157) or (**6**, 583).

‡ If $f, g \in I$ and f has bounded derived numbers (*i.e.* satisfies a Lipschitz condition of order 1), then $f\{g\} \in I$ (see **6**, 555).

§ We denote by N the class of functions which satisfy the condition N of Lusin, *i.e.* transform nul sets into nul sets.

‖ For sets of positive outer measure are transformed by not constant polygonal functions into sets with positive outer measure (see **6**, 341–6).

$\Sigma(2)$ (indeed cannot belong to I) in this interval and so, *a fortiori*, f cannot belong to $\Sigma(2)$ in the whole interval.

4. *Preliminary remarks.* We begin with a few remarks† on the representation of functions $f \epsilon \Sigma(\omega)$. For such a function there is a certain sequence of absolutely continuous functions $\{\phi_n\}$, such that

$$f = \lim f_n,$$

where $f_n = \phi_n\{f_{n-1}\}$ $(n = 2, 3, \ldots)$ and $f_1 = \phi_1$. We notice that, for each y in Δ_n, the interval of values of $f_n(x)$, the limit‡

$$\lim_r \phi_{n+r}\{\phi_{n+r-1} \cdots \phi_{n+2}[\phi_{n+1}(y)] \ldots\}$$

exists. There is, therefore, a function $\rho_n(y)$, defined and continuous§ in this interval, such that

$$f(x) = \rho_n\{f_n(x)\}; \tag{A}$$

this for all $n = 1, 2, \ldots$, and all significant x.

In the next two paragraphs we shall be concerned with functions f, defined and continuous in $(0, 1)$ and such that $f(0) = f(1) = 0$, $f(\frac{1}{2}) = \frac{1}{2}$, which are strictly monotone in $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$. We also suppose that $f \epsilon \Sigma(\omega)$.

Consider now the representation of $f(x)$ in the form (A) for x in the interval $(0, \frac{1}{2})$. Our first point is that f_n is strictly monotone there. For, if not, by continuity there would be a pair of distinct points x_1, x_2 in this interval such that $f_n(x_1) = f_n(x_2)$, and this would imply that $f(x_1) = \rho_n\{f_n(x_1)\} = \rho_n\{f_n(x_2)\} = f(x_2)$, in contradiction with the strict monotony of f. It follows that $\rho_n(y)$ is strictly monotone in $\Delta_n^{(1)}$, the interval of values of $f_n(x)$ for x in $(0, \frac{1}{2})$. Similarly, $f_n(x)$ is strictly monotone in $(\frac{1}{2}, 1)$ and $\rho_n(y)$ in $\Delta_n^{(2)}$, the interval of values of $f_n(x)$ for x in $(\frac{1}{2}, 1)$.

From the fact that the intervals of values of $f(x)$ for x in $(0, \frac{1}{2})$ and for x in $(\frac{1}{2}, 1)$ are identical [each being $(0, \frac{1}{2})$], it follows that the

† Compare 2, 329.

‡ It should be noticed that ϕ_{n+1} is the innermost function and that the second sequence of dots indicates a sequence of brackets.

§ For, if ρ_n is not continuous, there is a sequence $\{y_\nu\}$ of points of Δ_n such that $\lim_\nu y_\nu = y$, but $\lim_\nu \rho_n(y_\nu) \neq \rho_n(y)$. From this sequence we could pick out a sub-sequence $\{y_i'\}$ such that $y_i' = f_n(x_i')$ and $\lim x_i' = x$. We should therefore have

$$\lim_i f(x_i') = \lim_i \rho_n\{f_n(x_i')\} \neq \rho_n\{f_n(x)\} = f(x),$$

contradicting the continuity of f.

intervals $\Delta_n^{(1)}$ and $\Delta_n^{(2)}$ either (i) coincide, or (ii) have exactly one point $[f_n(\frac{1}{2})]$ in common. For if this were not so, an end point of one of these would lie in the interior of the other, and this we show to be impossible. As typical, take the case when $f_n(1)$ is in the interior of $\Delta_n^{(1)}$, so that $f_n(1) = f_n(x^*)$ for some x^* with $0 < x^* < \frac{1}{2}$. This means that the minimum $f(x^*) = f(1) = 0$, which is impossible. Since $f = \lim f_n$, it is clear that the case (i) must obtain for all sufficiently large n. This is our second point†.

5. *A member of* $S(\omega+1)$. Let $F(x)$ be a strictly decreasing continuous function which transforms the interval $(\frac{1}{2}, 1)$ into the interval $(0, \frac{1}{2})$ in such a way that a nul set in $(\frac{1}{2}, 1)$ transforms into a set of full measure in $(0, \frac{1}{2})$ (and reciprocally). Such functions exist‡.

Let $f(x)$ be a function coinciding with x in $(0, \frac{1}{2})$ and with $F(x)$ in $(\frac{1}{2}, 1)$. We shall show§ that $f \in S(\omega+1)$. Since $f \supset I$, it is clear, from Theorem 2, that we have only to show that $f \supset S(\omega)$. We do this by assuming that $f \in S(\omega)$ and obtaining a contradiction by means of the results noticed in § 4.

Now, in $(\frac{1}{2}, 1)$, using the representation in the form (A) of $f(x)$, we have

$$f_n(x) = \rho_n^{-1}\{F(x)\},$$

where the inverse function ρ_n^{-1} is well defined, since ρ_n has been proved strictly monotone. Since F transforms a certain nul set Z into a set $F(Z)$ of full measure, and f_n, being absolutely continuous [it is increasing and belongs to $\Sigma(n)$], transforms Z into a nul set $f_n(Z)$, it appears that ρ_n^{-1} must transform $F(Z)$ into $f_n(Z)$. So ρ_n^{-1} must transform the nul set $CF(Z)$ into the set $Cf_n(Z)$ of full measure. Thus we see that $\rho_n^{-1} \supset N$ on $(0, \frac{1}{2})$, *i.e.*, $\rho_n^{-1} \supset I$ on $(0, \frac{1}{2})$. From this we obtain the contradiction desired. For we have seen that, for all sufficiently large n, we must have $\Delta_n^{(1)} = \Delta_n^{(2)}$; this gives, for x in $(0, \frac{1}{2})$, $\rho_n^{-1}(x) \equiv f_n(x)$, an absolutely continuous function, since for these values of x we have

$$f_n(x) = \rho_n^{-1}\{f(x)\} = \rho_n^{-1}\{x\}.$$

† The results of this and the preceding paragraph are special cases of more general ones about functions which are "finalement monotone" established by Mlle Bary (2, 334). A function $f(x)$ continuous in (a, b) is said to be "finalement monotone" there if its values for x in the interior of (a, b) lie strictly between the values $f(a)$, $f(b)$.

‡ They can be obtained, for example, by linear transformations on functions constructed by Carathéodory (6, 357) or Denjoy, *Comptes rendus* (Paris), 194 (1932), 44–46.

§ From this it will follow that the result "*if* $f \in Lip\ 1$ *and* $g \in \Sigma(\alpha)$, *then* $f\{g\} \in \Sigma(\alpha)$", true for ordinals α with an immediate predecessor $\alpha-1$, is false when α is a limiting number.

6. The condition imposed on the behaviour in $(\frac{1}{2}, 1)$ of the function $f(x)$ of § 5 is in a certain sense necessary in order to get a function (of this type) belonging to $S(\omega+1)$. In particular, it is not enough that f shall not satisfy the condition N in $(\frac{1}{2}, 1)$.

We shall show how to exhibit any such function $f(x)$ whose inverse [in $(\frac{1}{2}, 1)$] is absolutely continuous [in $(0, \frac{1}{2})$] as a formal superposition of order ω of absolutely continuous functions.

In this paragraph† we confine our attention to values of x in $(\frac{1}{2}, 1)$. There are sequences of functions $P_n(x)$, $p_n(y)$, $\rho_n(y)$ such that each P_n is polygonal and $\lim P_n = f$, while each p_n and ρ_n is defined in the interval of values of $P_{n-1}(x)$ and, moreover, each p_n is polygonal there. Also, we have, for each $n = 1, 2, \ldots,$

$$P_{n+1} = p_{n+1}\{P_n\}, \quad \rho_n(y) = \lim_r p_{n+r}\{p_{n+r-1} \cdots p_{n+2}[p_{n+1}(y)] \cdots\},$$

$$f = \rho_n\{P_n\}.$$

We also know that each P_n and ρ_n (with f) is strictly monotone and continuous, and we can write

$$P_n(x) = \rho_n^{-1}\{f(x)\}, \quad \rho_n^{-1}\{y\} = P_n\{f^{-1}(y)\}.$$

We now define a sequence of functions $f_n(x)$ for x in $(0, 1)$ by writing

$$f_n(x) = \rho_n^{-1}(x) \quad \text{for } x \text{ in } (0, \tfrac{1}{2}),$$

$$f_n(x) = P_n(x) \quad \text{for } x \text{ in } (\tfrac{1}{2}, 1).$$

That $\lim f_n = f$ is clear. We show that f is a formal superposition of order ω of absolutely continuous functions. In fact, we have

$$f_{n+1} = P_{n+1} = p_{n+1}\{P_n\} = p_{n+1}\{f_n\},$$

and

$$f_{n+1} = \rho_{n+1}^{-1} = P_{n+1}\{f^{-1}\} = p_{n+1}[P_n\{f^{-1}\}] = p_{n+1}\{\rho_n^{-1}\} = p_{n+1}\{f_n\},$$

when the (suppressed) argument x is in $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$ respectively. Recalling the definitions given in **4**, we see that it is now only necessary to show that each $f_n \in \Sigma(m)$ for some $m = m(n) < \omega$. This m is necessarily 1 from Theorem 2, and we can complete our proof that $f \in \Sigma(\omega)$ by showing that $\rho_n^{-1} \in N$.

If $\rho_n^{-1} \ni N$, there is a nul set E on $(0, \frac{1}{2})$ such that $\rho_n^{-1}(E)$ is not nul. Consider the well defined sub-set $\mathfrak{E} = f^{-1}(E)$ of the interval $(\frac{1}{2}, 1)$ which is

† The results used here follow from **4** and §§ 3, 4 above.

nul in virtue of the assumed absolute continuity of f^{-1}. We have, then, that

$$\rho_n^{-1}(E) = P_n\{f^{-1}(E)\} = P_n(\mathfrak{E}),$$

which is impossible, as the transform of a nul set by a polygonal function is itself nul.

To justify the statements made at the beginning of this section, we notice that it is easy to construct functions which do not satisfy the condition N, but which, nevertheless, have their inverse functions absolutely continuous. In particular, there are (6, 356) such functions, continuous and strictly decreasing from $\frac{1}{2}$ to 0 for x in $(\frac{1}{2}, 1)$, which transform a nul set into a set whose measure is as near $\frac{1}{2}$ as we please.

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THE BEHAVIOUR OF CERTAIN SERIES ASSOCIATED WITH LIMITING CASES OF ELLIPTIC THETA-FUNCTIONS: (III)

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1. In previous papers with the same title* I have considered the analytic character of the functions defined by series of the types

$$(1.1) \qquad \Sigma(\pm 1)^n e^{\pm \pi i n^2 \theta} \phi(n, s), \quad \Sigma e^{\pm \pi i (n+\frac{1}{2})^2 \theta} \phi(n+\tfrac{1}{2}, s),$$

where θ is real and $\phi(n, s)$ is either n^{-s} or e^{-ns}. In this paper I consider the series (1.1) when the convergence factors $\phi(n, s)$ approach zero (i) more rapidly and (ii) more slowly than in either of the above cases. Other cases will be considered in later papers.

The results of the earlier papers were that

(i) if θ is rational and $\phi(n, s) = e^{-ns}$, the functions are meromorphic with simple poles on $\Re(s) = 0$;

(ii) if θ is irrational and $\phi(n, s) = e^{-ns}$, the functions have $\Re(s) = 0$ as a singular line;

(iii) if θ is rational and $\phi(n, s) = n^{-s}$, the only possible singularities are simple poles at $s = 1, -1, -3, -5, \ldots$;

(iv) if θ is a quadratic irrational and $\phi(n, s) = n^{-s}$, the possible singularities are simple poles at the points†

$$s = \tfrac{1}{2} - 2q + \frac{(8\mu - \gamma)\pi i}{4\rho \log \Theta} \quad (q = 0, 1, 2, \ldots;\ \mu = 0, \pm 1, \pm 2, \ldots),$$

in which γ is one of the numbers 0, 1, 2, ..., 7;

(v) for certain other types of irrational θ, with $\phi(n, s) = n^{-s}$, the function is regular to the right of a vertical singular line which lies in the region $\frac{1}{2} < \Re(s) < 1$.

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* 2 and 3.

† The notation is that of 2; see Theorem 6, 418. γ and ρ are not always the same for the six functions of the set (1.1).

In this paper I consider the cases in which $\phi(n, s)$ is of the forms

(i) $\exp(-sn^a)$ with $a > 1$;

(ii) $n^{-\frac{1}{2}} \exp\{-s(\log\log n)^b\}$ with $b \geqslant 1$ and θ a quadratic irrational.

In the first case the functions all have a singular line $\Re(s) = 0$.

In the second case, when $b = 1$, each series (1.1) represents an integral function except possibly when the corresponding γ is zero. We consider first more complicated series of which they are special cases.

2. *The case* $\phi(n, s) = \exp(-sn^a)$ $(a > 1)$.

The axis of convergence of (1.1) is $\Re(s) = 0$. Since $(n+1)^a - n^a \to \infty$ as $n \to \infty$, the axis of convergence is a singular line (for any real θ), by Wennberg's theorem*. The same is obviously true for any more rapidly converging factors $\phi(n, s)$.

3. *The case* $\phi(n, s) = n^{-\frac{1}{2}} \exp\{-s(\log\log n)^b\}$ $(b \geqslant 1)$.

When θ is a quadratic irrational, the axis of convergence of each of the series (1.1) lies in the region $0 \leqslant \Re(s) \leqslant 1$†. In the special case of $b = 1$, we can apply general theorems due to Hardy and Fekete‡ to the results stated above for the case $\phi(n, s) = n^{-s}$. These show that, if the sum of a series (1.1) with $\phi(n, s) = n^{-s}$ is regular at $s = \frac{1}{2}$, which is certainly the case when $\gamma \neq 0$, the corresponding series (1.1) with $\phi(n, s) = n^{-\frac{1}{2}}(\log n)^{-s}$ will be an integral function, and otherwise will have a simple pole at $s = 1$ and no other singularity. We give a direct proof of a more general result, deducing it from certain transformation formulae which we now establish.

4. *The transformation formulae.*

We now establish a transformation formula for the series

$$F(s;\ \theta;\ a, b, c, d) = \sum_{3}^{\infty} e^{\pi i n^2 \theta} n^{-a} e^{-s(l_2 n)^b} (ln)^{-c} (l_2 n)^{-d}, \tag{4.1}$$

in which $0 < \theta < 1$, a, b, c, d are real, $b \geqslant 1$, and a is such that the series

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* 5, 3–7.

† Using Abel's transformation and the Theorems 2.141 and 2.30 of 4. On page 418 of 2 there is a misprint in a corresponding argument; $O(1)$ in lines 14 and 15 should be replaced by $O(n^{\frac{1}{2}-\sigma})$.

‡ See, *e.g.*, Theorems V and VI in 1, 179.

converges for some*, but not all values of s; $ln, l_2 n$ denote $\log n$ and $\log\log n$ respectively. We use the formula when θ is a quadratic irrational, in which case $a=\frac{1}{2}$ and, when $b=1$, the series is convergent if $\Re(s)>1-c$ and is not convergent if $\Re(s)\leqslant -c$.

The sum of the first $n-2$ terms of the series (4.1) is†

$$(4.2)\qquad \int_{L(n+\frac{1}{2})} - \int_{L(\frac{1}{2})} \frac{z^{-a}\exp\{\pi i z^2\theta - s(l_2 z)^b - cl_2 z - dl_3 z\}}{e^{2\pi i z}-1}\,dz.$$

In this, those values of the logarithms are to be taken which are continuous in the half-plane to the right of the line $L(e)$ and which are real on the real axis in this half-plane.

The second integral in (4.2) is an integral function of s. In the first integral we substitute

$$\frac{1}{e^{2\pi i z}-1} = \sum_{\nu=1}^{M} e^{-2\nu\pi i z} + \frac{e^{-2M\pi i z}}{e^{2\pi i z}-1},$$

where $M=[(n+\frac{1}{2})\theta]$. We write the result as

$$\int_{L(n+\frac{1}{2})} = \sum_{\nu=1}^{M} f_\nu(s) + R_M(s).$$

In any bounded region of s, $|R_M(s)|$ does not exceed a multiple of $n^{-\frac{1}{2}a}$. Therefore we have

$$(4.3)\qquad F(s;\ \theta;\ a, b, c, d) = I(s) + \sum_{\nu=1}^{\infty} f_\nu(s),$$

where $I(s)$ denotes an integral function;

$$(4.4)\quad f_\nu(s) = \int_{L(n+\frac{1}{2})} z^{-a}\exp\{\pi i z^2\theta - 2\nu\pi i z - s(l_2 z)^b - cl_2 z - dl_3 z\}\,dz$$

$$= \int_{L(\nu/\theta)}$$

$$= e^{-\pi i\nu^2/\theta}\int_{L(0)}\left(z+\frac{\nu}{\theta}\right)^{-a}\exp\left[\pi i z^2\theta - s\left\{l_2\left(z+\frac{\nu}{\theta}\right)\right\}^b - cl_2\left(z+\frac{\nu}{\theta}\right) - dl_3\left(z+\frac{\nu}{\theta}\right)\right]dz$$

$$= i^{\frac{1}{2}}\,e^{-\pi i\nu^2/\theta}\int_{-\infty}^{\infty} e^{-\pi\theta x^2} g_\nu(x)\,dx,$$

---

* This is the only case of interest; but the formulae hold if the series converges for all values of s. The series cannot converge for any value of s if $a<\frac{1}{2}$.

† We use the notation of the first paper of the series (2); $L(x)$ is a straight line through the point $z=x$, making an angle $\frac{1}{4}\pi$ with the positive real axis; the positive direction along $L(x)$ is that in which $\Re(z)$ increases; $l_3 z$ denotes $\log l_2 z$. We omit some of the details when the argument is similar to that in paper 2.

in which $i^{\frac{1}{2}}$ means $(1+i)/\sqrt{2}$; and

$$(4.4.1)\quad g_\nu(x)=\left(\frac{\nu}{\theta}+i^{\frac{1}{2}}x\right)^{-a}$$
$$\times\exp\left[-s\left\{l_2\left(\frac{\nu}{\theta}+i^{\frac{1}{2}}x\right)\right\}^b-cl_2\left(\frac{\nu}{\theta}+i^{\frac{1}{2}}x\right)-dl_3\left(\frac{\nu}{\theta}+i^{\frac{1}{2}}x\right)\right].$$

We first examine this when $|x|\leqslant\frac{1}{2}\nu/\theta$. Then we have

$$\left(1+\frac{i^{\frac{1}{2}}x\theta}{\nu}\right)^{-a}=1-\frac{i^{\frac{1}{2}}ax\theta}{\nu}+x^2\theta^2O\left(\frac{1}{\nu^2}\right).$$

Also

$$l_2\left(\frac{\nu}{\theta}+i^{\frac{1}{2}}x\right)=l_2\left(\frac{\nu}{\theta}\right)+\log\left\{1+\frac{\log\{1+(i^{\frac{1}{2}}x\theta)/\nu\}}{\log(\nu/\theta)}\right\}$$
$$=l_2\left(\frac{\nu}{\theta}\right)+\frac{i^{\frac{1}{2}}x\theta}{\nu\log(\nu/\theta)}+x^2\theta^2O\left(\frac{1}{\nu^2\log(\nu/\theta)}\right),$$

and so

$$\exp\left[-s\left\{l_2\left(\frac{\nu}{\theta}+i^{\frac{1}{2}}x\right)\right\}^b+s\left(l_2\frac{\nu}{\theta}\right)^b\right]$$
$$=1-\frac{i^{\frac{1}{2}}bsx\theta}{\nu\log(\nu/\theta)}\left(l_2\frac{\nu}{\theta}\right)^{b-1}+x^2\theta^2O\left[\frac{A(s)\{l_2(\nu/\theta)\}^{2b-2}}{\nu^2\log(\nu/\theta)}\right],$$

where $A(s)=|s|+|s|^2$.

We have a similar expression for the term involving c. We also have

$$\exp\left[-dl_3\left(\frac{\nu}{\theta}+i^{\frac{1}{2}}x\right)+dl_3\frac{\nu}{\theta}\right]=1-\frac{i^{\frac{1}{2}}dx\theta}{\nu l(\nu/\theta)\,l_2(\nu/\theta)}+x^2\theta^2O\left(\frac{1}{\nu^2\,l(\nu/\theta)\,l_2(\nu/\theta)}\right).$$

Combining these results and substituting in (4.4.1), we find, when $|x|\leqslant\frac{1}{2}\nu/\theta$,

$$(4.4.2)\quad g_\nu(x)=\frac{\theta^a}{\nu^a}\exp\left[-s\left\{l_2\left(\frac{\nu}{\theta}\right)\right\}^b-cl_2\left(\frac{\nu}{\theta}\right)-dl_3\left(\frac{\nu}{\theta}\right)\right]$$
$$\times[1-x\theta\phi(\nu)+x^2\theta^2O(1/\nu^2)],$$

where the O is uniform for s in any bounded region, and $\phi(\nu)$ involves a, b, c, d, s.

Substituting this in (4.4), we find

$$(4.5)\quad f_\nu(s)=\frac{i^{\frac{1}{2}}\theta^a}{\nu^a}\exp\left[-\frac{\pi i\nu^2}{\theta}-s\left\{l_2\left(\frac{\nu}{\theta}\right)\right\}^b\right.$$
$$\left.-cl_2\left(\frac{\nu}{\theta}\right)-dl_3\left(\frac{\nu}{\theta}\right)\right]\int_{-\infty}^{\infty}e^{-\pi\theta x^2}E\,dx\quad\text{(say)}$$

and

$$\int_{-\infty}^{\infty} e^{-\pi\theta x^2} E dx = \int_{-\frac{1}{2}\nu/\theta}^{\frac{1}{2}\nu/\theta} + \int_{|x|\geqslant\frac{1}{2}\nu/\theta}$$

$$= \int_{-\frac{1}{2}\nu/\theta}^{\frac{1}{2}\nu/\theta} e^{-\pi\theta x^2}\left\{1+x^2 O\left(\frac{1}{\nu^2}\right)\right\} dx + \int_{|x|\geqslant\frac{1}{2}\nu/\theta} e^{-\pi\theta x^2} E\, dx.$$

Now $\int_{-\frac{1}{2}\nu/\theta}^{\frac{1}{2}\nu/\theta} e^{-\pi\theta x^2} dx$ differs from $\theta^{-\frac{1}{2}}$ by

$$(4.5.1) \qquad 2\int_{\frac{1}{2}\nu/\theta}^{\infty} e^{-\pi\theta x^2} dx \leqslant 2e^{-\frac{1}{8}\pi\nu^2/\theta}\int_0^{\infty} e^{-\frac{1}{2}\pi\theta x^2} dx = \left(\frac{2}{\theta}\right)^{\frac{1}{2}} e^{-\frac{1}{8}\pi\nu^2/\theta}.$$

Similarly,

$$(4.5.2) \qquad \left|\int_{-\frac{1}{2}\nu/\theta}^{\frac{1}{2}\nu/\theta} x^2 e^{-\pi\theta x^2} dx - \frac{1}{2\pi\theta^{\frac{3}{2}}}\right| \leqslant \frac{1}{2\pi}\left(\frac{2}{\theta}\right)^{\frac{3}{2}} e^{-\frac{1}{8}\pi\nu^2/\theta}.$$

Also

$$\left|\int_{|x|\geqslant\frac{1}{2}\nu/\theta} e^{-\pi\theta x^2} E\, dx\right| \leqslant e^{-\frac{1}{8}\pi\nu^2/\theta}\int_{-\infty}^{\infty} e^{-\frac{1}{4}\pi\theta x^2}\left|e^{-\frac{1}{4}\pi\theta x^2} E\right| dx$$

$$\leqslant e^{-\frac{1}{8}\pi\nu^2/\theta} A\int_{-\infty}^{\infty} e^{-\frac{1}{4}\pi\theta x^2} dx$$

$$(4.5.3) \qquad = 2A\theta^{-\frac{1}{2}} e^{-\frac{1}{8}\pi\nu^2/\theta},$$

where we have assumed that s is restricted to lie in some bounded region D, and A is a constant depending on this region.

Substituting these results in (4.5), we have, in D,

$$(4.6) \quad f_\nu(s) = \frac{i^{\frac{1}{2}}\theta^{a-\frac{1}{2}}}{\nu^a}\exp\left[-\frac{\pi i\nu^2}{\theta} - s\left\{l_2\left(\frac{\nu}{\theta}\right)\right\}^b - cl_2\left(\frac{\nu}{\theta}\right) - dl_3\left(\frac{\nu}{\theta}\right)\right]\left\{1+\theta O\left(\frac{1}{\nu^2}\right)\right\}$$

$$= f_\nu^{(1)}(s) + f^{(2)}(s) \quad \text{(say)}.$$

The series $\Sigma f_\nu^{(2)}(s)$ is absolutely and uniformly convergent in D and so represents an integral function*.

We have finally to examine the series $\Sigma f_\nu^{(1)}(s)$. We transform it into a sum of series each of which is of the same form as the original series (4.1). Writing $N = (\log\theta^{-1})/\log\nu$, we have

$$e^{-cl_2(\nu/\theta)} = (\log\nu)^{-c}(1+N)^{-c}$$

$$= (\log\nu)^{-c}\left\{1 - cN + \binom{c+1}{2}N^2 - \ldots \pm \binom{c+n-1}{n}N^n + O(N^{n+1})\right\},$$

* We can note that this integral function is, for any positive number ϵ, $O(\theta^{a+\frac{1}{2}-\epsilon})$ in D, uniformly for any bounded ranges of b, c, d.

and

$$e^{-dl_3(\nu/\theta)} = (l_2\nu)^{-d}\left\{1+\frac{N}{l_2\nu}-\frac{N^2}{2l_2\nu}+\ldots\pm\frac{N^n}{nl_2\nu}+O\left(\frac{N^{n+1}}{l_2\nu}\right)\right\}^{-d}$$

$$= (l_2\nu)^{-d}\left\{1-\frac{dN}{l_2\nu}+\frac{N^2P_1(l_2\nu)}{(l_2\nu)^2}+\ldots+\frac{N^nP_{n-1}(l_2\nu)}{(l_2\nu)^n}+O\left(\frac{N^{n+1}}{l_2\nu}\right)\right\},$$

in which $P_r(x)$ denotes a polynomial in x of degree r at most. We expand the factor $e^{-s\{l_2(\nu/\theta)\}^b}$ similarly, and, combining the results, we find that

$$e^{-s\{l_2(\nu/\theta)\}^b-cl_2(\nu/\theta)-dl_3(\nu/\theta)}$$

$$= e^{-s(l_2\nu)^b-cl_2\nu-dl_3\nu}\left[1+Q_1(s,\, l_2\nu)N+Q_2(s,\, l_2\nu)N^2+\ldots+Q_n(s,\, l_2\nu)N^n\right.$$

$$\left.+O\{|Q_{n+1}(s,\, l_2\nu)|N^{n+1}\}\right],$$

in which $Q_r(s,\, l_2\nu)$ denotes a polynomial in s, of degree r at most, in which each coefficient is the sum of a finite number of multiples of positive and negative powers of $l_2\nu$. We substitute this in $f_\nu^{(1)}(s)$ when $\nu \geqslant 3$ and then, combining (4.6) and (4.2) and remembering that $\Sigma f_\nu^{(2)}(s)$ represents an integral function, we obtain the following theorem.

THEOREM 1. *The function*

$$F(s;\ \theta;\ a,\ b,\ c,\ d) = \sum_3^\infty e^{\pi in^2\theta}n^{-a}e^{-s(l_2n)^b}(\log n)^{-c}(l_2n)^{-d}$$

satisfies the equation

$$F(s;\ \theta;\ a,\ b,\ c,\ d)$$

$$= i^{\frac{1}{2}}\theta^{a-\frac{1}{2}}F(s;\ -\theta^{-1};\ a,\ b,\ c,\ d)$$

$$+\theta^{a-\frac{1}{2}}\sum_{m=1}^{n}(\log\theta^{-1})^m\sum_d P_m(s,\, d)\,F(s;\ -\theta^{-1};\ a,\ b,\ c+m,\ d)$$

$$+R_{n+1}(s;\ c+n),$$

where $P_m(s,\, d)$ denotes a polynomial in s of degree m at most, each of the inner sums is taken over a finite number of different values of d, and $R_{n+1}(s,\, c+n)$ is regular in at least the same region as $F(s;\ -\theta^{-1};\ a,\ b,\ c+n+1,\ d)$.

We can similarly establish corresponding formulae for the other five functions of the set (1.1) with the same convergence factor $\phi(n,\, s)$. We distinguish the six functions (1.1) by suffixes 1, 2, ..., 6 according to the scheme

$$F_1 = \Sigma e^{\pi in^2\theta}\phi(n,\, s); \qquad F_2 = \Sigma(-1)^n e^{\pi in^2\theta}\phi(n,\, s);$$

$$F_3 = \Sigma e^{-\pi in^2\theta}\phi(n,\, s); \qquad F_4 = \Sigma(-1)^n e^{-\pi in^2\theta}\phi(n,\, s);$$

$$F_5 = \Sigma e^{\pi i(n+\frac{1}{2})^2\theta}\phi(n+\tfrac{1}{2},\, s); \qquad F_6 = \Sigma e^{-\pi i(n+\frac{1}{2})^2\theta}\phi(n+\tfrac{1}{2},\, s).$$

The general transformation formula for the set of functions may be stated as follows.

THEOREM 2. *If $F_\alpha(s;\ \theta;\ a, b, c, d)$ is any one of the six functions defined above and $0 < \theta < 1$, $\theta_1 = \theta^{-1} - [\theta^{-1}]$, we have the formula*

$$F_\alpha(s, \theta;\ a, b, c, d)$$
$$= e^{\frac{1}{2}\pi j}\theta^{a-\frac{1}{2}} F_\beta(s;\ \theta_1;\ a, b, c, d)$$
$$+\theta^{a-\frac{1}{2}} \sum_{m=1}^{n} (\log \theta^{-1})^m \sum_d P_m(s, d)\, F_\beta(s;\ \theta_1;\ a, b, c+m, d)$$
$$+R_{n+1}(s;\ c+n),$$

in which $P_m(s, d)$, $R_{n+1}(s;\ c+n)$ have the same meaning as in Theorem 1, the inner sums over d are for a finite number of different values of d, and j, β are given by the following table in terms of α, $[\theta^{-1}]$:

	$\mu = [\theta^{-1}]$ *even*						$\mu = [\theta^{-1}]$ *odd*					
α	1	2	3	4	5	6	1	2	3	4	5	6
β	3	6	1	5	4	2	4	6	2	5	3	1
j	1	$1-\mu$	7	$7+\mu$	1	7	1	$1-\mu$	7	$7+\mu$	1	7

These values of β, j are the same as the corresponding results in Theorems 1, 2, 3, 4 of the first paper of the series*.

5. We now apply these transformation formulae to the case in which θ is a quadratic irrational and show that, except in some special cases, the functions $F_\alpha(s;\ \theta;\ \frac{1}{2}, 1, c, d)$ are integral functions. The result can be stated as follows:

THEOREM 3. *If θ is a quadratic irrational, the function $F_\alpha(s;\ \theta;\ \frac{1}{2}, 1, c, d)$ is an integral function, provided that none of the points which are possible poles of the corresponding function $\Sigma(\pm 1)^n e^{\pm \pi i n^2 \theta} n^{-s}$ or $\Sigma e^{\pm \pi i (n+\frac{1}{2})^2 \theta}(n+\frac{1}{2})^{-s}$ lies on the real axis.*

If we take $c = d = 0$, we have the result stated at the end of § 1.

* 2, 415–417.

By Theorem 2 we have, first for $\Re(s) > 1-c$, and then wherever the functions concerned are all regular,

$$(5.1)\quad F_\alpha(s;\ \theta;\ \tfrac{1}{2},\ 1,\ c,\ d)$$
$$= e^{\frac{1}{2}\pi j} F_\beta(s;\ \theta_1;\ \tfrac{1}{2},\ 1,\ c,\ d)$$
$$+ \sum_{m=1}^{n}{}' (\log \theta^{-1})^m \sum_d P_m(s;\ d)\, F_\beta(s;\ \theta_1;\ \tfrac{1}{2},\ 1,\ c+m,\ d)$$
$$+ R_{n+1}(s;\ c+n),$$

and so, in particular,

$$(5.2)\qquad F_\alpha(s;\ \theta;\ \tfrac{1}{2},\ 1,\ c,\ d) = e^{\frac{1}{2}\pi j} F_\beta(s;\ \theta_1;\ \tfrac{1}{2},\ 1,\ c,\ d) + R(s,\ c),$$

the last term denoting a function regular for $\Re(s) > -c$.

Let the expression for θ as a simple continued fraction have its periodic part of length m, and let $\theta_1 = \theta^{-1} - [\theta^{-1}]$, $\theta_{\nu+1} = \theta_\nu^{-1} - [\theta_\nu^{-1}]$, so that $\theta_{k+m} = \theta_k$ when k is sufficiently large, say when $k \geqslant h$. Applying repeatedly the formula (5.2) we find, δ being some integer,

$$(5.3)\qquad F_\alpha(s;\ \theta;\ \tfrac{1}{2},\ 1,\ c,\ d) = e^{\frac{1}{2}\pi\delta} F_{\alpha_k}(s;\ \theta_k;\ \tfrac{1}{2},\ 1,\ c,\ d) + R(s,\ c).$$

We can find integers $g \geqslant h$ and $\rho = 1, 2, 3, 4, 5$, or 6 such that, taking (5.3) as defining α_k for all integers k, we have*

$$\alpha_{g+\rho m} = \alpha_g,$$

and so, of course, for any integer r,

$$\alpha_{g+r\rho m} = \alpha_g.$$

Now, writing $A = \alpha_g$, and applying (5.2) again, we have

$$(5.4)\qquad (1 - e^{\frac{1}{2}\gamma\pi i})\, F_A(s;\ \theta_g;\ \tfrac{1}{2},\ 1,\ c,\ d) = R(s,\ c),$$

γ being an integer. If γ is a multiple of 8, the equation (5.4) gives us no information, since it reduces to the form $0 = 0$. This is the exceptional case given in the statement of the theorem†. An example of this is given by the function F_1 for the irrational

$$\theta = \frac{1}{2+}\ \frac{1}{2+}\ \frac{1}{2+}\ \cdots.$$

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* If m is odd, ρ must be even.

† See page 419 of the first paper of the series (2), the notation corresponds, and γ is the same in both papers.

An example of γ not being a multiple of 8 is given by the function F_1 for the irrational

$$\theta = \frac{1}{2+}\ \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{3+}\ \cdots,$$

which has no non-recurring part and for which m is 4; here we have $\gamma = 7$.

If we take a case for which γ is not a multiple of 8, (5.4) shows that F_A is regular for $\Re(s) > -c$ and then (5.3) shows that the same is true for F_α and for each $F_{\alpha_k}(s;\ \theta_k;\ \ldots)$.

In deducing (5.4) we used only the first term on the right-hand side of (5.1) together with a remainder term. If we use more terms we see that (5.4) can be refined to take the form

$$(5.5)\quad (1-e^{\frac{1}{2}\gamma\pi i})\, F_A(s;\ \theta_g;\ \tfrac{1}{2},\ 1,\ c,\ d)$$

$$= \sum_{m=1}^{n} \sum_{d,\ \lambda,\ \mu} P_m(s, d, \lambda, \mu)\, F_\lambda(s;\ \theta_\mu;\ \tfrac{1}{2},\ 1,\ c+m,\ d) + R_{n+1}(s, c+n),$$

in which the inner sum is taken over a finite number of values of d, λ, μ and $P_m(s, d, \lambda, \mu)$ is a polynomial in s; each of the functions F on the right-hand side of (5.5) is one of the functions F_{α_k} on the right-hand side of (5.3). We have just proved that each $F_{\alpha_k}(s;\ \theta_k;\ \frac{1}{2},\ 1,\ c,\ d)$ is regular for $\Re(s) > -c$. Therefore the right-hand side of (5.5) is regular for $\Re(s) > -1-c$, and so the same is true of $F_A(s;\ \theta_g;\ \frac{1}{2},\ 1,\ c,\ d)$ and thence of each of the other functions in (5.3).

Repetitions of the argument prove regularity step by step for

$$\Re(s) > -2-c,\quad -3-c,\quad -4-c,\quad \ldots,$$

and so we deduce that the functions are integral functions.

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ON AN EXPONENTIAL SUM

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Introduction.

Let q and k be positive integers with $q > 1$, $k > 2$. For $(a, q) = 1$, we write

$$S_{a,q} = \sum_{x=1}^{q} e_q(ax^k) \quad [e_q(t) = e^{2\pi it/q}].$$

Our object in this note is to prove the inequality

$$\sum_{x=1}^{m} e_q(ax^k) = \frac{m}{q} S_{a,q} + O(q^{\frac{3}{4}+\epsilon}),$$

for every $\epsilon > 0$*. This inequality has a certain interest in connection with the "major arcs" in Waring's problem, and in a recent paper† we have made use of it in the case $k = 4$.

Lemmas.

For any integer b, we write

$$S(a, b, q) = \sum_{x=1}^{q} e_q(ax^k + bx).$$

LEMMA 1. *If* $p \nmid a$, *then*

$$S(a, b, p) = O(p^{\frac{3}{4}}).$$

* The constant implied in the symbol O depends only on k and ϵ.

† "On Waring's problem for fourth powers", *Proc. London Math. Soc.* (2), 41 (1936), 143–150.

Proof (Mordell*). We have the identity:

$$(1)\quad \sum_{u=0}^{p-1}\sum_{v=0}^{p-1}|S(u,v,p)|^4$$

$$=\sum_{u=0}^{p-1}\sum_{v=0}^{p-1}\sum_{x_1=0}^{p-1}\dots\sum_{x_4=0}^{p-1}e_p\Big(u(x_1{}^k+x_2{}^k-x_3{}^k-x_4{}^k)+v(x_1+x_2-x_3-x_4)\Big)$$

$$=\sum_{\substack{x_1=0\\ x_1+x_2\equiv x_3+x_4(p)\\ x_1{}^k+x_2{}^k\equiv x_3{}^k+x_4{}^k(p)}}^{p-1}\dots\sum_{x_4=0}^{p-1}p^2=p^2\sum_{w=0}^{p-1}\sum_{t=0}^{p-1}R^2(w,t),$$

where $R(w,t)$ denotes the number of solutions of the congruences

$$x+y\equiv w \pmod p,$$

$$x^k+y^k\equiv t \pmod p.$$

This pair of congruences has at most k solutions, unless $w\equiv t\equiv 0$, in which case it has at most p solutions†. Hence the right-hand side of (1) is $O(p^4)$. Now for any $\lambda\not\equiv 0 \pmod p$ we have $S(a,b,p)=S(a\lambda^k, b\lambda, p)$. Hence

$$|S(a,b,p)|^4=\frac{1}{p-1}\sum_{\lambda=1}^{p-1}|S(a\lambda^k,b\lambda,p)|^4\leqslant\frac{k}{p-1}\sum_{u=0}^{p-1}\sum_{v=0}^{p-1}|S(u,v,p)|^4=O(p^3),$$

which establishes the lemma.

LEMMA 2. *If* $\nu>1$, $p+a$, $p^\nu+b^4$, *then*

$$S(a,b,p^\nu)=O(p^{\frac{3}{4}\nu}).$$

Proof. Let $b=p^\beta b'$, where $p+b'$, and write $\tau=[\frac{1}{2}(\nu+1)]$; then we have $\beta<\frac{1}{4}\nu$ and

$$\beta\leqslant\tfrac{1}{4}(\nu-1)<\tfrac{1}{2}(\nu-1)\leqslant\nu-\tau.$$

In the definition of $S(a,b,p^\nu)$ replace x by $p^\tau y+z$. Since $2\tau\geqslant\nu$, we obtain

$$S(a,b,p^\nu)=\sum_{z=1}^{p^\tau}e_{p^\nu}(az^k+bz)\sum_{y=0}^{p^{\nu-\tau}-1}e_{p^{\nu-\tau}}\Big((kaz^{k-1}+b)y\Big)$$

$$=p^{\nu-\tau}\sum_{\substack{z=1\\ kaz^{k-1}+b\equiv 0(p^{\nu-\tau})}}^{p^\tau}e_{p^\nu}(az^k+bz).$$

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* *Quart. J. of Math.*, 3 (1932), 161–167.

† The congruence $x^k+(w-x)^k\equiv t \pmod p$ can only be an identity in x if $w\equiv t\equiv 0 \pmod p$, or if $k\equiv 0 \pmod p$, and in the latter case the number of solutions does not exceed $p\leqslant k$.

Hence $$|S(a, b, p^\nu)| \leqslant p^{\nu-\tau} \frac{p^\tau}{p^{\nu-\tau}} N,$$

where N is the number of solutions of the congruence

$$kaz^{k-1}+b \equiv 0 \pmod{p^{\nu-\tau}}. \tag{2}$$

Now let $k = p^\kappa k'$, where $p \nmid k'$. Clearly $0 \leqslant \kappa < k-1$. Since $p \nmid a$, and $\beta < \nu-\tau$, the congruence (2) can have solutions only if $\beta = \kappa+(k-1)\rho$, where $\rho \geqslant 0$ is an integer, and in this case (2) is the same as

$$k' a(z/p^\rho)^{k-1}+b' \equiv 0 \pmod{p^{\nu-\tau-\beta}}.$$

This congruence* has not more than $2(k-1)$ solutions (mod $p^{\nu-\tau-\beta}$); hence

$$N = O\left(\frac{p^{\nu-\tau}}{p^{\nu-\tau-\beta+\rho}}\right) = O(p^{\beta-\rho}).$$

Hence $$S(a, b, p^\nu) = O(p^{\tau+\beta-\rho}).$$

If $\kappa = 0$, we have

$$\tau+\beta-\rho = \tau \leqslant \tfrac{1}{2}(\nu+1) \leqslant \tfrac{3}{4}\nu, \quad \text{if} \quad \rho = 0,$$

$$\tau+\beta-\rho \leqslant \tau+\beta-1 \leqslant \tfrac{1}{2}(\nu+1)+\tfrac{1}{4}(\nu-1)-1 < \tfrac{3}{4}\nu \quad \text{if} \quad \rho > 0.$$

If $\kappa > 0$, so that $p \mid k$, we have

$$S(a, b, p^\nu) = O(p^{\tau+\beta}) = O(p^{\frac{1}{2}(\nu+1)+\frac{1}{4}(\nu-1)}) = O(p^{\frac{3}{4}\nu}).$$

LEMMA 3. *If* $(a, q) = 1$ *and* $b \neq 0$, *then* $S(a, b, q) = O\left(q^{\frac{3}{4}+\epsilon}(q, b)\right)$.

Proof. If $(q_1, q_2) = 1$, then

$$S(a, b, q_1 q_2) = \sum_{x=1}^{q_1} \sum_{y=1}^{q_2} e_{q_1 q_2}\left(a(xq_2+yq_1)^k+b(xq_2+yq_1)\right)$$

$$= S(aq_2^{k-1}, b, q_1)\, S(aq_1^{k-1}, b, q_2).$$

By Lemmas 1 and 2, if $p^\beta \| b$ and $\beta < \frac{1}{4}\nu$,

$$S(a, b, p^\nu) = O(p^{\frac{3}{4}\nu}) = O\left(p^{\frac{3}{4}\nu}(p^\nu, b)\right).$$

Also, if $\beta \geqslant \frac{1}{4}\nu$,

$$S(a, b, p^\nu) = O(p^\nu) = O\left(p^{\frac{3}{4}\nu}(p^\nu, b)\right).$$

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* The congruence $x^l \equiv c \pmod{p^s}$, where $p \nmid c$, has at most l solutions, unless $p = 2$, in which case it has at most $2l$ solutions.

The result follows, on noting that

$$\prod_{p \mid q} O(1) = O(q^{\epsilon}),$$

and that $(q_1 q_2, b) = (q_1, b)(q_2, b)$ if $(q_1, q_2) = 1$.

LEMMA 4. *Let $f(x)$ be any polynomial with integral coefficients, and let*

$$\sum_{x=1}^{q} e_q\big(f(x)+nx\big) = F(n).$$

Then

$$\sum_{x=1}^{m} e_q\big(f(x)\big) = \frac{m}{q} F(0) + O\Big(\sum_{n=1}^{q-1} \frac{1}{n} \big(|F(n)|+|F(-n)|\big)\Big) + O(\log q).$$

Proof. Without loss of generality, we can suppose that $1 \leqslant m \leqslant q$. It is well known that, if t is not an integer,

$$\{t\} = t-[t]-\tfrac{1}{2} = -\frac{1}{2\pi i} \sum_{n=-(q-1)}^{q-1}{}' \frac{1}{n} e^{2\pi i n t} + O\Big(\frac{1}{q\|t\|}\Big), \tag{3}$$

where the summation excludes $n = 0$, and $\|t\|$ denotes the distance of t from the nearest integer. Now the expression

$$\Big[\frac{x-\frac{1}{2}}{q}\Big] - \Big[\frac{x-m-\frac{1}{2}}{q}\Big] = \frac{m}{q} - \Big\{\frac{x-\frac{1}{2}}{q}\Big\} + \Big\{\frac{x-m-\frac{1}{2}}{q}\Big\}$$

has the value 1 for $1 \leqslant x \leqslant m$ and 0 for $m+1 \leqslant x \leqslant q$. Hence

$$\sum_{x=1}^{n} e_q\big(f(x)\big) = \frac{m}{q} F(0) - \sum_{x=1}^{q} \Big\{\frac{x-\frac{1}{2}}{q}\Big\} e_q\big(f(x)\big) + \sum_{x=1}^{q} \Big\{\frac{x-m-\frac{1}{2}}{q}\Big\} e_q\big(f(x)\big).$$

Now if h denotes either of the numbers $\frac{1}{2}$, $m+\frac{1}{2}$, we have, by (3),

$$\begin{aligned}\sum_{x=1}^{q} \Big\{\frac{x-h}{q}\Big\} e_q\big(f(x)\big) &= -\frac{1}{2\pi i} \sum_{n=-(q-1)}^{q-1}{}' \frac{1}{n} e_q(-nh) F(n) + O\Big(\sum_{x=1}^{q} \frac{1}{q\Big\|\frac{x-h}{q}\Big\|}\Big) \\ &= O\Big(\sum_{n=1}^{q-1} \frac{1}{n} \big(|F(n)|+|F(-n)|\big)\Big) + O(\log q).\end{aligned}$$

This establishes the lemma.

Proof of the inequality.

THEOREM. *With the notation of the introduction,*

$$\sum_{x=1}^{m} e_q(ax^k) = \frac{m}{q} S_{a,q} + O(q^{\frac{3}{4}+\epsilon}).$$

Proof. By Lemma 4 with $f(x) = ax^k$, it suffices to prove that

$$\sum_{n=1}^{q-1} \frac{1}{n} \left(|S(a, n, q)| + |S(a, -n, q)| \right) = O(q^{\frac{1}{2}+\epsilon}).$$

By Lemma 3, the sum on the left is

$$O\left(\sum_{n=1}^{q-1} \frac{1}{n} q^{\frac{1}{2}+\epsilon}(q, n) \right) = O\left(q^{\frac{1}{2}+\epsilon} \sum_{d|q} d \sum_{m=1}^{(q-1)/d} \frac{1}{md} \right)$$

$$= O\left(q^{\frac{1}{2}+\epsilon} \log q \sum_{d|q} 1 \right) = O(q^{\frac{1}{2}+2\epsilon}).$$

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THREE TRITANGENT PLANES OF THE QUADRICUBIC CURVE

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The nine points of contact of three tritangent planes of a quadricubic curve (the canonical curve of genus four) may or may not lie on a quadriquadric curve, *i.e.* on a quadric other than that containing the quadricubic curve. If they do, the quadriquadric curve meets the quadricubic curve in three further points which are the points of contact of a fourth tritangent plane, and any two of the four tritangent planes together form a degenerate contact quadric of the same system as that formed by the other two.

In this paper* I develop a direct algebraic treatment which brings out clearly the distinction between the two cases, and leads on to further results. The algebra is extremely heavy, and I have only outlined the method and noted the results.

It will be convenient, throughout this paper, to denote reciprocals by dashed letters; thus p' stands for $1/p$.

1. We are given three tritangent planes of a quadricubic curve; we take them as the planes $x = 0$, $y = 0$, $z = 0$, respectively, their point of intersection being the vertex T of the tetrahedron of reference. The curve lies on a single quadric S; if we take for $t = 0$ the polar plane of T with respect to S, we may write the equation of S in the form

$$S \equiv t^2 - v_2 = 0, \tag{1}$$

* The algebraic treatment was suggested some years ago by H. W. Richmond, who obtained the two types of solution given at the end of § 1, but did not proceed to prove that they correspond respectively to the two distinct cases mentioned in the text. Mr. Richmond has recently [*Proc. Edinburgh Math. Soc.* (2), 4 (1935), 159–169] given a simplified algebraic treatment of the first case, in which four tritangent planes arise, but his new method does not apply to the second case, and thus does not give an algebraic criterion to distinguish the two cases.

where, without loss of generality, we may take

$$v_2 \equiv A^2x^2+B^2y^2+C^2z^2+(P+P')BCyz$$
$$+(Q+Q')CAzx+(R+R')ABxy. \quad (2)$$

The cubic surfaces through the curve form a linear system of freedom four; there is thus a single cubic surface through the curve with a node at T, and its equation is

$$tu_2-u_3=0, \quad (3)$$

where

$$u_2 \equiv x^2+y^2+z^2+(p+p')yz+(q+q')zx+(r+r')xy, \quad (4)$$

$$u_3 \equiv ax^3+by^3+cz^3+3a_2x^2y+3a_3x^2z+3b_3y^2z$$
$$+3b_1y^2x+3c_1z^2x+3c_2z^2y+6kxyz. \quad (5)$$

The projection of the quadricubic curve from T on to the plane $t=0$ is the sextic curve

$$u_2^2v_2-u_3^2=0, \quad (6)$$

which has nodes at the six intersections of the conic $u_2=0$ and the cubic curve $u_3=0$.

This plane sextic must have $x=0$, $y=0$, $z=0$ as tritangents. If the three points of contact with $x=0$ are given by

$$a_0y^3+3a_1y^2z+3a_2yz^2+a_3z^3=0, \quad (7)$$

and the arbitrary factor is suitably chosen, it is easily seen that we must have

$$(By+PCz)(By+P'Cz)(y+pz)^2(y+p'z)^2$$
$$\equiv (by^3+3b_3y^2z+3c_2yz^2+cz^3)^2-(a_0y^3+3a_1y^2z+3a_2yz^2+a_3z^3)^2. \quad (8)$$

The two cubic forms on the right-hand side cannot have a common factor, if $x=0$ is to be a true tritangent, because the vanishing of such a factor would give one of the nodes; hence we must have

$$2(by^3+3b_3y^2z+3c_2yz^2+cz^3)$$
$$\equiv m(By+PCz)(y+pz)^2+m'(By+P'Cz)(y+p'z)^2, \quad (9)$$

$$2(a_0y^3+3a_1y^2z+3a_2yz^2+a_3z^3)$$
$$\equiv m(By+PCz)(y+pz)^2-m'(By+P'Cz)(y+p'z)^2, \quad (10)$$

where m is a numerical factor.

Similarly, expressing that $y=0$, $z=0$ are tritangents, we get the identities

$$2(cz^3+3c_1z^2x+3a_3zx^2+ax^3)$$
$$\equiv n(Cz+QAx)(z+qx)^2+n'(Cz+Q'Ax)(z+q'x)^2, \quad (11)$$

$$2(ax^3+3a_2x^2y+3b_1xy^2+by^3)$$
$$\equiv l(Ax+RBy)(x+ry)^2+l'(Ax+R'By)(x+r'y)^2. \quad (12)$$

The coefficients of z^3 in (9) and (11) are the same; hence we have

$$mp^2P+m'p'^2P'=n+n',$$

whence $P=m'n'p'^2$ or $m'np'^2$.

Similarly we get

$$Q=n'l'q'^2 \quad \text{or} \quad n'lq'^2, \qquad R=l'm'r'^2 \quad \text{or} \quad l'mr'^2.$$

It is easily seen, on detailed examination, that, of the eight possible combinations, only two are essentially distinct; we may take these to be:

Solution I:

$$P=m'n'p'^2, \quad Q=n'l'q'^2, \quad R=l'm'r'^2. \quad (13)$$

Solution II:

$$P=m'np'^2, \quad Q=n'lq'^2, \quad R=l'mr'^2. \quad (14)$$

2. *Solution I.* Using the values for P, Q, R given by (13), we find for $2u_3$ the expression

$$\begin{aligned}2u_3\equiv{}&(l+l')Ax^3+(m+m')By^3+(n+n')Cz^3+6Kxyz\\
&+\{2(lr+l'r')\ A+(mr^2+m'r'^2)\ B\}x^2y\\
&+\{2(lq+l'q')\ A+(nq^2+n'q'^2)\ C\}x^2z\\
&+\{2(mp+m'p')B+(np^2+n'p'^2)\ C\}y^2z\\
&+\{2(mr+m'r')\ B+(lr^2+l'r'^2)\ A\}y^2x\\
&+\{2(nq+n'q')\ C+(lq^2+l'q'^2)\ A\}z^2x\\
&+\{2(np+n'p')\ C+(mp^2+m'p'^2)B\}z^2y. \quad (15)\end{aligned}$$

The points of contact of the plane x with the quadricubic curve are given by

equation (7), together with $tu_2-u_3=0$. If we put $x=0$ in the latter equation, and use (9), with the values of P, Q, R from (13), we get

$$2t(y+pz)(y+p'z)$$
$$=m(By+m'n'p'^2Cz)(y+pz)^2+m'(By+mnp^2Cz)(y+p'z)^2.$$

Also, from (10), equation (7) becomes

$$m(By+m'n'p'^2Cz)(y+pz)^2=m'(By+mnp^2Cz)(y+p'z)^2.$$

Thus the three points of contact are given by

$$\frac{y+pz}{y+p'z}=\frac{m'By+np^2Cz}{t}=\frac{t}{mBy+n'p'^2Cz}. \tag{16}$$

Similarly for the points of contact of the planes y and z, we have

$$\frac{z+qx}{z+q'x}=\frac{n'Cz+lq^2Ax}{t}=\frac{t}{nCz+l'q'^2Ax} \tag{17}$$

and

$$\frac{x+ry}{x+r'y}=\frac{l'Ax+mr^2By}{t}=\frac{t}{lAx+m'r'^2By}. \tag{18}$$

It is now easily verified that the twisted cubic T_{yz} through the six points of contact of the two planes y, z with the quadricubic curve is

$$\frac{x+r'y+q'z}{x+ry+qz}=\frac{lAx+m'r'^2By+n'q'^2Cz}{t}=\frac{t}{l'Ax+mr^2By+nq^2Cz}, \tag{19}$$

since these equations reduce to (17) and (18) respectively if we put $y=0$ and $z=0$.

It may also be verified that the quadric

$$S'\equiv tU_1-U_2=0, \tag{20}$$

where

$$U_1\equiv(qr+p)x+(rp+q)y+(pq+r)z, \tag{21}$$

$$\begin{aligned}U_2\equiv{}&(lqr+l'p)Ax^2+(mrp+m'q)By^2+(npq+n'r)Cz^2\\&+\{(mrp^2+m'p'q)B+(np^2q+n'rp')\ C\}yz\\&+\{(npq^2+n'q'r)\ C+(lq^2r+l'\ pq')\ A\}zx\\&+\{(lqr^2+l'\ r'\ p)\ A+(mr^2p+m'qr')B\}xy,\end{aligned} \tag{22}$$

contains all the nine points of contact; for example, if we put $x = 0$, we get

$$q\{t(y+pz)-(y+p'z)(m'By+np^2Cz)\}$$
$$+rp\{t(y+p'z)-(y+pz)(mBy+n'p'^2Cz)\}=0,$$

which is satisfied by (16).

We have thus proved that

Solution I corresponds to the case in which the nine points of contact of the three tritangent planes lie on a pencil of quadrics.

2.1. It is known* that the cubic surface which contains the quadricubic curve and the twisted cubic through the points of contact of two tritangent planes is four-nodal. For the twisted cubic T_{yz}, it may be verified that the equation of the cubic surface is

$$\Gamma_1 \equiv \Gamma - \Phi_1 S = 0, \tag{23}$$

where

$$(m'nq^2r'^2+mn'q'^2r^2-mnp^2-m'n'p'^2)\,BC\Phi_1$$
$$\equiv 2\{(lqr+l'q'r')A+(mrp+m'r'p')B+(npq+n'p'q')C-3K\}x$$
$$+\{2(mq'r+m'qr'-mp-m'p')B+(nq^2r'^2+n'q'^2r^2-np^2-n'p'^2)C\}y$$
$$+\{2(nqr'+n'q'r-np-n'p')C+(mq'^2r^2+m'q^2r'^2-mp^2-m'p'^2)B\}z$$
$$+2(p+p'-qr'-q'r)t. \tag{24}$$

This may also be written in the form of a symmetrical determinant

$$\Gamma_1 \equiv \begin{vmatrix} Al'x+Bmr^2y+Cnq^2z & t & x+ry+qz \\ t & Alx+Bm'r'^2y+Cn'q'^2z & x+r'y+q'z \\ x+ry+qz & x+r'y+q'z & \Phi_1 \end{vmatrix} = 0,$$

which verifies that it is four-nodal and puts in evidence the fact that it contains the curve T_{yz}.

Similarly we may obtain the equations of the quadrinodal cubic surfaces Γ_2, Γ_3 containing the quadricubic curve and T_{zx}, T_{xy} respectively.

2.2. It is known† that two twisted cubic curves T, T' with three common points have a single common biscecant. It may also be proved that the planes which meet the cubics in points P, Q, R and P', Q', R' which

* W. P. Milne, *Proc. London Math. Soc.* (2), 22 (1921), 373–380.

† Cremona, *Journal für Math.*, 60 (1862), 188–192; *Opere matematiche*, 2 (1915), 45.

lie on a conic pass through a fixed point U on this bisecant. For P, Q, R must be the intersections, other than the three common points, of T with a quadric through T'; they thus form a set of a linear series of freedom 2 on T, so that the planes PQR pass through a fixed point U. Clearly U lies on the common bisecant, since every plane through the bisecant satisfies the condition.

The twisted cubics T_{zx}, T_{xy} have in common the three points in the plane x; hence there is a point U_{yz} on their common bisecant l_{yz}, such that all planes which meet T_{zx}, T_{xy} in six points of a conic pass through U_{yz}. The coordinates of this point are found to be

$$x = l(p - qr), \quad y = 0, \quad z = 0, \quad t = (p - l^2 qr)A. \tag{24}$$

We may prove that U_{yz} lies on the plane $z = 0$ as follows. The nine points of contact lie on a pencil of quadrics; there is one quadric of this pencil containing T_{xy} entirely; hence the plane z meets T_{xy} in three points lying on a conic with the three points of contact in z, they being the three points in which z meets T_{zx}. Hence the plane z passes through U_{yz}. Similarly for the plane y.

It is easily verified that U_{yz} lies upon the quadrinodal cubic surfaces Γ_2, Γ_3, but not upon the quadric S. It therefore lies on the residual intersection of Γ_2, Γ_3 other than the quadricubic curve, *i.e.* on a plane cubic C_1.

The enveloping cone from a point of a quadrinodal cubic surface breaks up into two quadric cones, which project two twisted cubics on the surface. Thus any such twisted cubic on the surface meets a plane section in the points of contact of a tritangent conic of the plane cubic; and since any two such twisted cubics form together the complete section of the cubic surface by a quadric all such tritangent conics belong to the same family*.

We thus get on C_1 a family of tritangent conics arising from Γ_2 and another arising from Γ_3. But since U_{yz} lies in the plane of C_1 the two triads of points in which this plane meets T_{zx} and T_{xy} lie on a conic; hence the two families of tritangent conics of C_1 are the same. Hence

the enveloping quadric cones of Γ_2 *and* Γ_3 *meet* C_1 *in tritangent conics belonging to the same family.*

It may also be verified that the line $y = z = 0$ meets Γ_2 in two further points, in addition to U_{yz}, which belong to the involution of pairs of points determined on the line by the quadrics of the pencil S, S'. Similarly for Γ_3.

* See, for instance, Baker, *Principles of geometry*, 3 (Cambridge, 1923), 214.

2.3. The quadriquadric curve $S = S' = 0$ meets the quadricubic curve again in the three points of contact of a fourth tritangent plane*, which we may call w. The two pairs of planes y, z and x, w form contact quadrics of the same system; hence the quadrinodal cubic surface Γ_1 contains the twisted cubic T_{xw} as well as the twisted cubic T_{yz}. The quadric cone S_{yz} which touches Γ_1 along T_{yz} belongs to the pencil of quadrics determined by S and the plane-pair y, z; hence its vertex lies on the polar line of $y = z = 0$ with respect to S.

The cones S_{yz}, S_{xw} touch at the point, other than the four nodes of Γ_1, in which T_{yz}, T_{xw} meet.

The equation of S_{yz} is found to be

$$S_{yz} \equiv (l' Ax + mr^2 By + nq^2 Cz)(lAx + m' r'^2 By + n' q'^2 Cz) - t^2 = 0. \quad (25)$$

We may thus verify that the cones S_{zx}, S_{xy} touch at two points lying on a line m_{yz} which passes through the point $x = y = z = 0$ and lies in the plane x. The cones meet in two conics, one lying in the plane x and the other in a plane π_{yz} which contains m_{yz} and $y = z = 0$. The three planes π_{yz}, π_{zx}, π_{xy} have a line in common.

The following result is also of interest. Let y, z and x, w be two of the twenty-eight tritangent plane-pairs arising from a quadrinodal cubic surface Γ_1; let z, x and y, w arise similarly from Γ_2, and x, y and z, w from Γ_3. Then, starting with Γ_1, Γ_2, Γ_3 so isolated, we know that we can pick out four tritangent planes so related to Γ_1, Γ_2, Γ_3 in six ways, x, y, z, w *being one of the set*†. *It thus follows that the twelve points* U_{yz}, U_{xw} *belonging to the six tetrahedra lie on the cubic curve* C_1 *common to* Γ_2 *and* Γ_3 *and hence are coplanar.*

3. *Solution II.* If we now use the values of P, Q, R given by the formulae (14), we find that the nine points of contact of the three tritangent planes do not lie on a quadric, other than S. We may also verify that in this case the point U_{yz} arising from the twisted cubics T_{zx} and T_{xy} no longer lies in the plane of the cubic C_1. Hence

Solution II corresponds to the case in which the nine points of contact of the three tritangent planes do not lie on a pencil of quadrics. In this case the enveloping cones of the quadrinodal cubic surfaces Γ_2, Γ_3 *meet the plane of the residual intersection* C_1 *of* Γ_2 *and* Γ_3 *in tritangent conics belonging to different families.*

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* W. P. Milne, *loc. cit.*

† A. Krazer, *Lehrbuch der Thetafunktionen* (1903), 303, § 36.

In this case, also, the coefficients of y^3 and z^3 in the cubic form (10), whose vanishing gives the points of contact of $x = 0$ with the plane sextic (6), are immediately seen to be

$$2a_0 = (m-m')\,B, \quad 2a_3 = (n-n')\,C;$$

with similar results for the tritangents y and z. It thus follows that the nine points of contact of the three tritangent lines with the plane sextic lie on a cubic curve (other than that formed by the lines themselves). We thus have the following result, due to Richmond:

If the nine points of contact of three tritangent planes of the quadricubic curve do not lie on a pencil of quadrics, the nine lines joining the points of contact to the point of intersection of the three planes lie on a pencil of cubic cones.

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THE SOLUTION OF SETS OF SIMULTANEOUS INTEGRAL EQUATIONS RELATED TO THE EQUATION OF VOLTERRA

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It is the purpose of the present paper to extend the solution given by Fock (**1**) for an integral equation of the Volterra type containing one unknown function to the more general case of n integral equations containing n unknown functions. From the solution so obtained conditions are found sufficient to ensure that these functions all tend to zero as the independent variable tends to infinity. These conditions are of importance in connection with the stability of the physical or biological systems described by a set of integral equations of the type under discussion. From this result may be derived a generalization of the theorem recently established by Paley and Wiener (**2** and **3**) for the case of the simple Volterra equation*. It is shown that this generalization may be deduced independently from Paley and Wiener's result for the single equation. The relation between the non-homogeneous Volterra equation and the corresponding homogeneous equation of the Fredholm type is then considered, and it is shown that the most important condition for stability (that is, to ensure that the unknown functions tend to zero as the independent variable tends to infinity) is the same in the two cases. The general conditions under which such a correspondence would continue to hold are then stated, and the stability of systems of integral equations still more general than those so far treated is discussed, although in dealing with these complete vigour is not attempted. However, the close mathematical and physical analogy to the less general cases previously discussed renders it probable that the conclusions arrived at are essentially correct. Finally, an example is given which illustrates this more general theory.

* We have to thank Prof. H. Bateman for drawing our attention to this paper.

I. *Solution of a set of two simultaneous "Volterra" equations.*

Let us consider the system of simultaneous non-homogeneous integral equations in the unknown functions $\phi_1(x)$ and $\phi_2(x)$, namely

$$\phi_1(x)+\int_0^x \phi_1(x-\xi)\,k_{11}(\xi)\,d\xi+\int_0^x \phi_2(x-\xi)\,k_{12}(\xi)\,d\xi=f_1(x), \tag{1}$$

$$\phi_2(x)+\int_0^x \phi_1(x-\xi)\,k_{21}(\xi)\,d\xi+\int_0^x \phi_2(x-\xi)\,k_{22}(\xi)\,d\xi=f_2(x), \tag{2}$$

where f_1 and f_2, k_{11}, k_{12}, k_{21}, and k_{22} are known functions of x, which, together with their first derivatives, are bounded and absolutely integrable in the interval $(0, \infty)$.

If we write K to denote the operation which transforms $\phi(x)$ into $\int_0^x k(\xi)\,\phi(x-\xi)\,d\xi$, or, what is the same thing, $\int_0^x k(x-\xi)\,\phi(\xi)\,d\xi$, the above equations may be written in the form

$$(1+K_{11})\,\phi_1(x)+K_{12}\,\phi_2(x)=f_1(x), \tag{3}$$

$$K_{21}\,\phi_1(x)+(1+K_{22})\,\phi_2(x)=f_2(x). \tag{4}$$

We shall call operators defined as above, operators of type A.

Certain properties of these operators are readily established.

(1) The commutative law holds, namely if K and L are any two operators of type A, then $KL=LK$, since

$$\begin{aligned} KL\phi(x) &= \int_0^x \int_0^{x-\xi} k(\xi)\,l(\theta)\,\phi(x-\xi-\theta)\,d\theta\,d\xi \\ &= \int_0^x \int_0^{x-\theta} l(\theta)\,k(\xi)\,\phi(x-\theta-\xi)\,d\xi\,d\theta \\ &= LK\phi(x). \end{aligned} \tag{5}$$

(2) The operator $M=KL=LK$ is itself an operator of type A; in fact,

$$M\phi(x)=\int_0^x m(\xi)\,\phi(x-\xi)\,d\xi,$$

where $$m(\xi)=\int_0^\xi k(\lambda)\,l(\xi-\lambda)\,d\lambda=\int_0^\xi l(\lambda)\,k(\xi-\lambda)\,d\lambda.$$

This is readily proved by suitably transforming the expression for $KL\phi(x)$ already given. Further, it may be proved that, if $k(x)$ and $l(x)$ are

bounded and integrable in the range $(0, \infty)$, then $m(x)$ and its first derivative are also bounded and integrable in the range $(0, \infty)$, *cf.* Titchmarsh (**4**), p. 391, 12·61, and Bochner (**5**), p. 46.

$$(3) \qquad \int_0^\infty e^{-\alpha x} Kf(x)\,dx = k_{(\alpha)} f_{(\alpha)}, \quad \text{where} \quad k_{(\alpha)} = \int_0^\infty k(\xi)\, e^{-\alpha\xi}\, d\xi, \tag{6}$$

and $f_{(\alpha)} = \int_0^\infty f(\xi)\, e^{-\alpha\xi}\, d\xi$, and where f satisfies the conditions applicable to f_1 and f_2 above, and $R(\alpha) \geqslant 0$. This follows from the well-known Dirichlet formula.

(4) If $M = KL$, and therefore $m(\xi) = \int_0^\xi k(\lambda)\, l(\xi - \lambda)\, d\lambda$, then

$$m_{(\alpha)} = k_{(\alpha)} l_{(\alpha)}. \tag{7}$$

(5) It follows from these results that any identity in the operators K, L, M, etc., and a function $\phi(x)$, implies a corresponding identity in the functions $k_{(\alpha)}$, $l_{(\alpha)}$, $m_{(\alpha)}$, etc., and $\phi_{(\alpha)}$.

Operating on equation (3) by $1+K_{22}$, and on equation (4) by K_{12} and subtracting, we have

$$\begin{vmatrix} 1+K_{11}, & K_{12} \\ K_{21}, & 1+K_{22} \end{vmatrix} \phi_1(x) = (1+K_{22}) f_1(x) - K_{12} f_2(x), \tag{8}$$

the determinantal notation being justified by the property (1) above.

This may be written in the form

$$(1+F)\,\phi_1(x) = g(x), \tag{9}$$

where F, that is $K_{11}+K_{22}+K_{11}K_{22}-K_{21}K_{12}$, in virtue of property (2) above, is an operator of type A, and $g(x) = (1+K_{22}) f_1(x) - K_{12} f_2(x)$ is a known function of x.

Thus equation (9) is an equation of the simple Volterra type treated by Fock (**1**), so that

$$\phi_1(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\alpha x} h_{1(\alpha)}\, d\alpha, \tag{10}$$

where

$$h_{1(\alpha)} = \frac{g_{(\alpha)}}{1+f_{(\alpha)}}, \tag{11}$$

and the integration is taken along the line $\alpha = c$, where c is a positive

number sufficiently large, that all the singularities of the function $h_{1(\alpha)}$ lie to the left of it. But, by (9) above, since

$$g(x) = (1+K_{22})f_1(x) - K_{12}f_2(x), \quad \text{and} \quad 1+F = \begin{vmatrix} 1+K_{11}, & K_{12} \\ K_{21}, & 1+K_{22} \end{vmatrix},$$

it follows that

$$\frac{g_{(\alpha)}}{1+f_{(\alpha)}} = \frac{\begin{vmatrix} f_{1(\alpha)}, & k_{12(\alpha)} \\ f_{2(\alpha)}, & 1+k_{22(\alpha)} \end{vmatrix}}{\begin{vmatrix} 1+k_{11(\alpha)}, & k_{12(\alpha)} \\ k_{21(\alpha)}, & 1+k_{22(\alpha)} \end{vmatrix}} = \frac{\Delta_{1(\alpha)}}{\Delta_{(\alpha)}}, \tag{12}$$

where $\Delta_{1(\alpha)}$ and $\Delta_{(\alpha)}$ are the respective determinants.

Similarly $$\phi_2(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\alpha x} k_{2(\alpha)}\, d\alpha,$$

$h_{1(\alpha)}$ and $h_{2(\alpha)}$ being the solution of the two simultaneous linear equations

$$\left.\begin{aligned} (1+k_{11(\alpha)})h_{1(\alpha)} + k_{12(\alpha)}h_{2(\alpha)} &= f_{1(\alpha)}, \\ k_{21(\alpha)}h_{1(\alpha)} + (1+k_{22(\alpha)})h_{2(\alpha)} &= f_{2(\alpha)}. \end{aligned}\right\} \tag{13}$$

Further, it may be shown that $\phi_1(x)$ and $\phi_2(x)$ so obtained do actually satisfy equations (1) and (2), for by Fock's theorem they satisfy the equations

$$(1+F)\phi_1(x) = (1+K_{22})f_1(x) - K_{12}f_2(x)$$

and $$(1+F)\phi_2(x) = -K_{21}f_1(x) + (1+K_{11})f_2 x.$$

Hence, premultiplying the first by $1+K_{11}$ and the second by K_{12}, adding, and noticing that the operations commute, we find that

$$(1+F)\{(1+K_{11})\phi_1(x) + K_{12}\phi_2(x)\} = (1+F)f_1(x).$$

Thus $$(1+F)j(x) = 0,$$

where $$j(x) = (1+K_{11})\phi_1(x) + K_{12}\phi_2(x) - f_1(x).$$

This is a Fock equation for $j(x)$, and, by using his solution, or otherwise, it follows that $j(x) \equiv 0$, whence equations (1) and (2) are satisfied.

II. *Solution of a set of n simultaneous "Volterra" equations.*

It appears from the above that the equations for $h_{1(\alpha)}$ and $h_{2(\alpha)}$ correspond in form to equations (3) and (4) for $\phi_1(x)$ and $\phi_2(x)$. This, it will be seen, is a consequence (i) of the properties of the operators of type A stated above, from which it follows that the process of reduction of the two equations (3)

and (4) to the single equation (9) is formally equivalent to the corresponding process in the case of simultaneous linear algebraical equations, and (ii) of the fact that the integrand $h_{1(\alpha)}$ in Fock's solution corresponding to the integral equation $(1+F)\phi_{(x)} = g(x)$ is the solution of the corresponding equation $(1+f_{(\alpha)})h_{1(\alpha)} = g_{(\alpha)}$. This observation makes it evident that the theorem holds for the general case of n equations involving n unknown functions. Thus, if the equations are

$$\phi_s(x) + \sum_{r=1}^{n} \int_0^x \phi_r(x-\xi)\, k_{sr}(\xi)\, d\xi = f_s(x) \quad (s = 1, 2, \ldots, n), \tag{14}$$

where k_{sr} and f_s satisfy conditions analogous to those mentioned above, so that

$$\phi_s(x) + \sum_{1}^{n} K_{sr}\, \phi_r(x) = f_s(x), \tag{15}$$

then

$$\phi_s(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\alpha x} h_{s(\alpha)}\, d\alpha, \tag{16}$$

where the functions $h_{s(\alpha)}$ are given by the n simultaneous linear equations

$$h_{s(\alpha)} + \sum_{1}^{n} k_{sr(\alpha)} h_{r(\alpha)} = f_{s(\alpha)} \tag{17}$$

so that

$$h_{s(\alpha)} = \frac{\Delta_{s(\alpha)}}{\Delta_\alpha}, \tag{18}$$

where

$$\Delta_\alpha = \begin{vmatrix} 1+k_{11(\alpha)}, & k_{12(\alpha)}, & k_{13(\alpha)}, & \cdots \\ k_{21(\alpha)}, & 1+k_{22(\alpha)}, & k_{23(\alpha)}, & \cdots \\ k_{31(\alpha)}, & k_{32(\alpha)}, & 1+k_{33(\alpha)}, & \cdots \\ \vdots & & & \end{vmatrix} \tag{19}$$

and $\Delta_{s(\alpha)}$ is a determinant involving $f_{s(\alpha)}$ and $k_{sr(\alpha)}$.

This extension of Fock's result appears to have been rigorously established by the above method. Because of its bearing on subsequent work (see Section VI below), it is useful to point out that, if we multiply the equations (15) by $e^{-\alpha x}$ and integrate from 0 to ∞ according to x, we obtain, by making use of the relations already established,

$$\phi_{s(\alpha)} + \sum_{r=1}^{n} k_{sr(\alpha)}\, \phi_{r(\alpha)} = f_{s(\alpha)}, \tag{20}$$

so that

$$\int_0^\infty \phi_s(\xi)\, e^{-\alpha\xi}\, d\xi \equiv \phi_{s(\alpha)} = h_{s(\alpha)}, \tag{21}$$

since $h_{s(\gamma)}$ is defined as the solution of the set of equations now shown to

be satisfied by $\phi_{s(\alpha)}$. Therefore, by using the properties of the Laplace transforms, we have

$$\phi_s(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\alpha x} h_{s(\alpha)} d\alpha. \tag{22}$$

A rigorous proof would require, however, that it should be shown that $\phi_s(x)$ so defined does actually satisfy the integral equations. The labour of doing this is avoided by the method of the previous section, which reduces the problem to the simple case already treated rigorously by Fock.

Reference may here be made to the general discussion of integral equations of the "Faltungstypus" by Doetsch (**6**), (**7**), and (**8**), which is of wide application, and which, when applied to our equations, would yield a solution in the above form; but we have not been able to extract therefrom an explicit statement, in a form applicable to our problem, of the conditions under which the solution exists and satisfies the equations.

III. *Conditions for stability.*

(*a*) We shall first prove that $\phi_s(x)$ belongs to L_2, that is the Lebesgue integral $\int_0^\infty |\phi(x)|^2 dx$ exists [*cf.* Titchmarsh (**4**), 384] (apart possibly from a null set of points), provided that an additional condition is imposed, namely that all the roots of the equation $\Delta_{(\alpha)} = 0$ have negative real parts, that is $\Delta_{(\alpha)} = 0$ implies $R(\alpha) < 0$.

Let the functions $k_{sr}(x)$ and $f_s(x)$ be represented by $J(x)$. Since the function $J(x)$ is bounded and absolutely integrable, it belongs both to L_1 and L_2.

Let
$$J_{(\alpha)} = \int_0^\infty e^{-\alpha x} J(x)\, dx,$$

so that, if $\alpha = \beta + iy$,

$$J_{(\alpha)} = \int_0^\infty e^{-\beta x} J(x)\, e^{-iyx} dx.$$

Clearly, $e^{-\beta x} J(x)$ belongs both to L_1 and L_2 for $\beta \geqslant 0$. Since $e^{-\beta x} J(x)$ belongs to L_1,

$$\int_0^\infty e^{-\beta x} J(x)\, e^{-iyx} dx$$

exists everywhere for $\beta \geqslant 0$, so that $J_{(\alpha)}$ is an analytic function regular over the whole of the right half of the complex plane including the y axis.

Also, by the Riemann-Lebesgue lemma,

$$\lim_{y\to\infty} J_{(\beta+iy)} = 0,$$

and, by the theorem of Plancherel [Wiener (**9**), 69],

$$\text{l.i.m.}_{A\to\infty} \int_{-A}^{A} e^{-(\beta+iy)x} J(x)\, dx$$

belongs to L_2. Since $J_{(\beta+iy)}$ exists, it is identical almost everywhere with

$$\text{l.i.m.}_{A\to\infty} \int_{-A}^{A} e^{-(\beta+iy)x} J(x)\, dx,$$

and being continuous and bounded everywhere it must also belong to L_2.

Since $\Delta_{s(\alpha)}$ and $\Delta_{(\alpha)}-1$ are sums of products of the functions $J_{(\alpha)}$, these functions are also regular everywhere over the y axis and over the right half of the complex plane. Further, they tend to zero when $y\to\infty$ $(\beta \geqslant 0)$. Also $\Delta_{(\alpha)}$ has no zeros on or to the right of the y axis; thus the function $h_{s(\alpha)}$ is regular everywhere on and to the right of the y axis, and tends to 0 when $y\to\infty$ $(\beta \geqslant 0)$. It also follows that $\Delta_{s(iy)}$ regarded as a function of y belongs to L_2, since all the component functions $J_{(iy)}$ are bounded and belong to L_2. Consequently $h_{s(iy)}\,(=\Delta_{s(iy)}/\Delta_{(iy)})$ belongs to L_2, as $\Delta_{(iy)}\to 1$ when $y\to\infty$, and has no zeros for any real value of y.

We note that, by the theorem of the last section,

$$\phi_s(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\alpha x} h_{s(\alpha)}\, d\alpha,$$

where c is a sufficiently large positive number. It follows at once from the principles of contour integration that since $h_{s(\alpha)}$ is a regular analytic function for all values of α with $R(\alpha) \geqslant 0$, and since

$$\lim_{y\to\infty} h_{s(\beta+iy)} = 0, \text{ when } \beta \geqslant 0,$$

we have

$$\int_{c-i\infty}^{c+i\infty} e^{\alpha x} h_{s(\alpha)}\, d\alpha = \int_{-i\infty}^{i\infty} e^{\alpha x} h_{s(\alpha)}\, d\alpha.$$

Thus

$$\phi_s(x) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{\alpha x} h_{s(\alpha)}\, d\alpha = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iyx} h_{s(iy)}\, dy,$$

where, as already proved, $h_{s(iy)}$ belongs to L_2.

But, by Plancherel's theorem,

$$\text{l.i.m.}_{A\to\infty} \int_{-A}^{A} e^{iyx} h_{s(iy)}\, dy$$

exists and belongs to L_2, that is to say a function belonging to L_2 exists which is identical with $\phi_s(x)$ except possibly over a set of points of zero measure. Thus $\phi_s(x)$ is defined almost everywhere and belongs to L_2.

(*b*) We now prove that, under the above conditions,

$$\lim_{x\to\infty} \{\phi_s(x) - f_s(x)\} = 0.$$

It follows from the Liouville-Neumann theory [Whittaker and Watson (**10**)] that the solution of the integral equation

$$\phi(x) + K\phi(x) = f(x)$$

is given, for any finite value of x, by the uniformly convergent series

$$\phi(x) = f(x) + \sum_{n=1}^{\infty} (-)^n K^n f(x).$$

Consequently $\phi(x)$ is bounded for any finite value of x. Now according to Section II, in the general case of n equations the function $\phi_s(x)$ satisfies the equation

$$(1+F)\,\phi_s(x) = g_s(x),$$

where the function $g_s(x)$ is bounded and integrable, and

$$F\phi_s(x) = \int_0^x \bar{K}(x-\xi)\,\phi_s(\xi)\,d\xi,$$

where $\bar{K}(\xi)$ is derived from functions $k_{sr}(\xi)$ and is bounded and integrable. Thus the functions $\phi_s(x)$ are necessarily finite for any value of x.

Since $\int_0^\infty |k_{sr}(\xi)|^2 d\xi$ is absolutely convergent, a value x_1 can be found such that $\int_x^{x+\eta} |k_{sr}(\xi)|^2 d\xi < \epsilon^2$ for all values of $x > x_1$, and for all positive values of η (ϵ being any finite number).

Similarly, since $\phi_s(x)$ is bounded over any finite range, is defined almost everywhere, and belongs to L_2, a value x_2 can be found such that, for $x > x_2$, $\int_x^{x+\eta} |\phi_r(\xi)|^2 d\xi < \epsilon^2$.

Let x be chosen to be greater than $2x_1$ and $2x_2$.

Since

$$\phi_s(x)-f_s(x) = -\sum_{r=1}^{n}\int_0^x k_{sr}(\xi)\,\phi_r(x-\xi)\,d\xi,$$

$$|\phi_s(x)-f_s(x)| \leqslant \sum_1^n \int_0^x |k_{sr}(\xi)|\,.\,|\phi_r(x-\xi)|\,d\xi,$$

i.e.
$$\leqslant \sum_1^n \left[\int_0^{\frac{1}{2}x} |k_{sr}(\xi)|\,.\,|\phi_r(x-\xi)|\,d\xi + \int_{\frac{1}{2}x}^x |k_{sr}(\xi)|\,.\,|\phi_r(x-\xi)|\,d\xi\right]$$

i.e.
$$\leqslant \sum_1^n \left[\int_0^{\frac{1}{2}x} |k_{sr}(\xi)|\,.\,|\phi_r(x-\xi)|\,d\xi + \int_0^{\frac{1}{2}x} |k_{sr}(x-\xi)|\,.\,|\phi_r(\xi)|\,d\xi\right].$$

But, by the Schwarz inequality,

$$\left[\int_0^{\frac{1}{2}x} |k_{sr}(\xi)|\,.\,|\phi_r(x-\xi)|\,d\xi\right]^2 \leqslant \int_0^{\frac{1}{2}x} |k_{sr}(\xi)|^2\,d\xi \int_0^{\frac{1}{2}x} |\phi_r(x-\xi)|^2\,d\xi$$

$$\leqslant \bar{k}_{sr}\int_{\frac{1}{2}x}^x |\phi_r(\xi)|^2\,d\xi$$

$$\leqslant \bar{k}_{sr}\,\epsilon^2,$$

where $\bar{k}_{sr} \equiv \int_0^\infty |k_{sr}(\xi)|^2\,d\xi$ exists and is finite, so that

$$\sum_{r=1}^{n}\int_0^{\frac{1}{2}x} |k_{sr}(\xi)|\,.\,|\phi_r(x-\xi)|\,d\xi \leqslant nc_1\,\epsilon,$$

where c_1 is positive and equal to the greatest of the numbers $\surd(\bar{k}_{sr})$.

Similarly

$$\left[\int_0^{\frac{1}{2}x} |k_{sr}(x-\xi)|\,.\,|\phi_r(\xi)|\,d\xi\right]^2 \leqslant \int_{\frac{1}{2}x}^x |k_{sr}(\xi)|^2\,d\xi \int_0^{\frac{1}{2}x} |\phi_r(\xi)|^2\,d\xi.$$

Since $\phi_s(x)$ is defined almost everywhere and belongs to L_2, and is in any case bounded, we have

$$\int_0^{\frac{1}{2}x} |\phi_r(\xi)|^2\,d\xi < \bar{\phi}_r,$$

where $\bar{\phi}_r$ is a finite number; so that

$$\sum_{r=1}^{n}\int_0^{\frac{1}{2}x} |k_{sr}(x-\xi)|\,.\,|\phi_r(\xi)|\,d\xi \leqslant nc_2\,\epsilon,$$

where c_2 is positive and equal to the greatest of the numbers $\sqrt{\bar{\phi}_r}$. Consequently

$$|\phi_s(x)-f_s(x)| \leqslant n\epsilon(c_1+c_2).$$

Since ϵ can be taken arbitrarily small,

$$\lim_{x\to\infty} |\phi_s(x)-f_s(x)| = 0.$$

It follows at once from this result that, if $f_s(x) = 0$, $\phi_s(x) \to 0$ as $x \to \infty$, and, more generally, for very large values of x, $\phi_s(x)-f_s(x) \to 0$.

(*c*) The above results hold, of course, only when the functions $f_s(x)$ belong to L_1. We may generalize the theorem to the case where $f_s(x)$ tends not to zero but to a constant l_s in such a way that $f_s(x)-l_s$ belongs to L_1. It is, however, necessary to assume also that $\int_x^\infty k_{sr}(\xi)\,d\xi$ belongs to L_1.

Let $\bar{\phi}_s$ $(s = 1, 2, \ldots, n)$ be the solution of the set of equations

$$\bar{\phi}_s + \sum_{r=1}^{n} \bar{\phi}_r \int_0^\infty k_{sr}(x)\,dx = l_s, \tag{23}$$

so that, writing $\int_0^\infty k_{sr}(x)\,dx = k_{sr(0)}$, the integral being in fact equal to $k_{sr(\alpha)}$ when $\alpha = 0$, we have

$$\bar{\phi}_s + \sum_{r=1}^{n} \bar{\phi}_r k_{sr(0)} = l_s. \tag{24}$$

Let $\phi_s(x) = \bar{\phi}_s + \chi_s(x)$; then the original set of equations may be written

$$\bar{\phi}_s + \chi_s(x) + \sum_{1}^{n} \int_0^x k_{sr}(\xi)\{\bar{\phi}_r + \chi_r(x-\xi)\}\,d\xi = f_s(x),$$

i.e.

$$\chi_s(x) + \sum_{1}^{n} \int_0^x k_{sr}(\xi)\,\chi_r(x-\xi)\,d\xi = f_s(x) - \bar{\phi}_s - \sum_{1}^{n} \bar{\phi}_r k_{sr(0)} + \sum_{1}^{n} \bar{\phi}_r \int_x^\infty k_{sr}(\xi)\,d\xi$$

$$= f_s(x) - l_s + \sum_{1}^{n} \bar{\phi}_r \int_x^\infty k_{sr}(\xi)\,d\xi. \tag{25}$$

But the right-hand side of (25) tends to 0 as $x \to \infty$; thus, by the theorem already proved, $\chi_s(x) \to 0$ as $x > \infty$, provided that $\Delta_{(\alpha)} = 0$ implies that the real part of α is negative, and that $f_s(x) - l_s + \sum_{r=1}^{n} \bar{\phi}_r \int_x^\infty k_{sr}(\xi)\,d\xi$

belongs to L_1. But, by hypothesis, $\{f_s(x)-l_s\}$ and $\int_x^\infty k_{sr}(\xi)\,d\xi$ both belong to L_1.

Thus the more general theorem is that, if $\phi_s(x)$ is defined by the set of equations (14), and $f_s(x)\to l_s$, as $x\to\infty$, then, if $\bar{\phi}_s$ is a solution of the set of equations (24), $\phi_s(x)\to\bar{\phi}_s$ as $x\to\infty$, provided (i) that $\Delta_{(\alpha)}=0$ implies that the real part of α is negative, and (ii) that $f_s(x)-l_s$ and $\int_x^\infty k_{sr}(\xi)\,d\xi$ belong to L_1.

IV. *The theorem of Paley and Wiener and its generalization.*

In the special case of one equation with one unknown function

$$\phi(x)+\int_0^x k(\xi)\,\phi(x-\xi)\,d\xi=f(x),$$

where $f(x)\to l$, as $x\to\infty$, this gives

$$\phi(x)\to\frac{l}{1+k_{(0)}}.$$

This last result is essentially the same as that already obtained by Paley and Wiener (**2**) and (**3**) by a different method, so that the foregoing may be considered as a generalization of their theorem. The assumptions made by us are, however, somewhat more restrictive, since in their proof of the particular case condition (ii) is not necessary. However, if we assume the theorem as given by Paley and Wiener, it is not difficult to generalize it as follows.

For simplicity, we consider the case where $n=2$, so that the system of integral equations is

$$\phi_1(x)+K_{11}\phi_1(x)+K_{12}\phi_2(x)=f_1(x),$$

$$\phi_2(x)+K_{21}\phi_1(x)+K_{22}\phi_2(x)=f_2(x).$$

The equation for $\phi_1(x)$ is then

$$(1+F)\,\phi_1(x)=g(x), \tag{26}$$

where F and g have the meanings previously assigned.

We assume that $\phi_1(x)$ and $\phi_2(x)$ are measurable and bounded over every finite range, and also that $k_{11}(x)$, $k_{12}(x)$, etc., each have a Lebesgue integral L_1, that is $\int_0^\infty |k(\xi)|\,d\xi<\infty$. These conditions are sufficient

to ensure that the theorem of Paley and Wiener may be applied to equation (26).

We have first to find the limit of $g(x)$ as $x \to \infty$.

Now

$$K_{12}f_2(x) = \int_0^x k_{12}(\xi) f_2(x-\xi)\, d\xi$$

$$= \int_0^{x_1} k_{12}(\xi) f_2(x-\xi)\, d\xi + \int_{x_1}^x k_{12}(\xi) f_2(x-\xi)\, d\xi.$$

If we keep x_1 fixed and let $x \to \infty$,

$$K_{12}f_2(x) = l_2 \int_0^{x_1} k_{12}(\xi)\, d\xi + \lim_{x\to\infty} \int_{x_1}^x k_{12}(\xi) f_2(x-\xi)\, d\xi.$$

Let $\bar{f}_2$ be the upper bound of $|f_2(x)|$ for positive values of x; then the second integral is less in magnitude than $\lim_{x\to\infty} \bar{f}_2 \int_{x_1}^x |k_{12}(\xi)|\, d\xi$, and this must tend to 0 as $x_1 \to \infty$, because $\int_0^\infty |k_{12}(\xi)|\, d\xi$ is assumed to converge.

Also $\int_0^{x_1} k_{12}(\xi)\, d\xi \to \int_0^\infty k_{12}(\xi)\, d\xi = k_{12(0)}$ as $x_1 \to \infty$.

It follows that $\lim_{x\to\infty} g(x) = (1+k_{22(0)})\, l_1 - k_{12(0)} l_2$, whence, by the theorem of Paley and Wiener,

$$\phi_1(x) \to \frac{(1+k_{22(0)})\, l_1 - k_{12(0)} l_2}{1+f_{(0)}}.$$

But, by the properties of $f_{(\alpha)}$ previously proved,

$$1+f_{(0)} = \begin{vmatrix} 1+k_{11(0)}, & k_{12(0)} \\ k_{21(0)}, & 1+k_{22(0)} \end{vmatrix}.$$

Thus, if $\bar{\phi}_1$ and $\bar{\phi}_2$ constitute the solution of the equations

$$\bar{\phi}_1 + k_{11(0)}\bar{\phi}_1 + k_{12(0)}\bar{\phi}_2 = l_1,$$

$$\bar{\phi}_2 + k_{21(0)}\bar{\phi}_1 + k_{22(0)}\bar{\phi}_2 = l_2,$$

then $\phi_1(x)$ and $\phi_2(x)$ tend to $\bar{\phi}_1$ and $\bar{\phi}_2$ respectively as $x \to \infty$, provided that $1+f_{(\alpha)} = 0$ implies that the real part of α is negative, that is provided that $\Delta_{(\alpha)} = 0$ implies that the real part of α is negative.

It is obvious that the theorem may be generalized to the case of n unknown functions without difficulty.

V. *The relation between the Volterra and Fredholm equations.*

We consider the relation between the set of equations (14) above and the corresponding set

$$\psi_s(x)+\sum_{r=1}^{n}\int_0^\infty k_{sr}(\xi)\,\psi_r(x-\xi)\,d\xi=0. \tag{27}$$

These may be conveniently referred to as the Volterra (14) and the Fredholm (27) types of equations respectively. We may represent the operation which converts $\psi(x)$ into $\int_0^\infty k_{sr}(\xi)\,\psi(x-\xi)\,d\xi$ by $\overline{K}_{sr}$; thus equation (27) may be written in the form

$$\psi_s(x)+\sum_{r=1}^{n}\overline{K}_{sr}\,\psi_r(x)=0. \tag{28}$$

We notice that
$$\overline{K}_{sr}\,e^{\alpha x}=e^{\alpha x}k_{sr(\alpha)}, \tag{29}$$

so that
$$\psi_s(x)=A_s e^{\alpha x} \tag{30}$$

is a solution of the set of equations (27) provided that

$$A_s+\sum_{1}^{n}k_{sr(\alpha)}A_r=0,$$

whence, eliminating the A's from this set of equations, we find that α is a root of the equation $\Delta_{(\alpha)}=0$, where $\Delta_{(\alpha)}$ is the determinant previously defined. Of the A's we may choose one at random; the others are then uniquely defined by the above equations. Each root α_k of the determinant provides an independent set of solutions, and, since the equations are linear and homogeneous, the sum of these solutions is also a solution. It is at once clear that, if values of α exist with real positive parts, then solutions of the equations exist which become infinite as $x\to\infty$. Thus a condition which is necessary to ensure that all solutions of the equations should tend to 0 as $x\to\infty$ is that the real parts of all the α's should be negative. It will be seen that this is the same condition as was found to be necessary to ensure the same result in the case of the set of non-homogeneous or Volterra equations (§ III).

It is of interest to consider the relations between the two sets of equations somewhat more closely. For this purpose we take the simplest case, namely that in which there is only one equation in each system and one unknown function. The equations are

$$(1+K)\,\phi(x)=f(x) \tag{31}$$

and
$$(1+\overline{K})\,\phi(x)=0. \tag{32}$$

Equation (31) may be written in the form

$$\phi(x) = f(x) - \int_0^x k(\xi)\,\phi(x-\xi)\,d\xi, \tag{33}$$

so that it may be looked upon as determining some observable ϕ at a time x in terms of the values observable at previous times, back to the time $x = 0$. Equation (32) gives the value of $\phi(x)$ in terms of its value at all previous times. However, it may be written in the form

$$\begin{aligned}\phi(x) &= -\int_0^\infty k(\xi)\,\phi(x-\xi)\,d\xi \\ &= -\left[\int_0^x k(\xi)\,\phi(x-\xi)\,d\xi + \int_0^\infty k(x+\xi)\,\phi(-\xi)\,d\xi\right].\end{aligned} \tag{34}$$

This equation becomes identical with (33), provided that

$$-\int_0^\infty k(x+\xi)\,\phi(-\xi)\,d\xi = f(x). \tag{35}$$

Now $\phi(-\xi)$ in this equation is not arbitrary, since it is a solution of (31). Thus every solution of (31) is a solution of (32) for some particular function $f(x)$, but certain functions $f(x)$ may exist such that no solution of (31) satisfies (32). As shown above, the condition that (31) should have solutions which tend to 0 as $x \to \infty$ is that $\Delta_{(\alpha)} = 0$ implies that the real part of α is negative. It is, therefore, to be expected that a condition that the solution of (32) should tend to zero as $x \to \infty$ is that $\Delta_{(\alpha)} = 0$ implies that the real part of α is negative. The proof given by Paley and Wiener establishes rigorously that this expectation is correct for the simple case of one equation, and the present paper gives the corresponding result for sets of simultaneous equations. Furthermore, if a root α_0 exists with a positive real part, it follows from (35) that $\phi(x) = Ae^{\alpha_0 x}$ is a solution of (33), if

$$f(x) = -A\int_0^\infty k(x+\xi)\,e^{-\alpha_0\xi}\,d\xi.$$

It is not difficult to prove that $f(x) \to 0$ as $x \to \infty$, provided that $k(x)$ satisfies the conditions postulated in section I. Thus a value $f(x)$ exists for which $\phi(x)$ does not tend to 0 as $x \to \infty$, although $f(x)$ itself tends to 0 as $x \to \infty$. This does not exclude the possibility that, for certain functions $f(x)$ which tend to 0 as $x \to \infty$, $\phi(x)$ may tend to 0 as $x \to \infty$, even when some roots of $\Delta_{(\alpha)} = 0$ have positive real parts; nevertheless the absence of all such roots is necessary if $\phi(x)$ is to tend to zero for all functions $f(x)$ which themselves tend to zero as $x \to \infty$.

VI. *Generalization to related types of equations.*

It will be seen from the above that the determinant $\Delta_{(\alpha)}$ is fundamental both to the homogeneous and to the non-homogeneous sets of equations, and that the fact that the same determinant appears in both types depends upon the peculiar properties of the operators K and $\bar{K}$. The characteristic properties of these operators are that

$$\int_0^\infty e^{-\alpha x} K\phi(x)\,dx = k_{(\alpha)}\phi_{(\alpha)}, \tag{36}$$

where $$\phi_{(\alpha)} = \int_0^\infty e^{-\alpha x}\phi(x)\,dx,$$

and that $$\bar{K}\,e^{\alpha x} = e^{\alpha x} k_{(\alpha)}. \tag{37}$$

In the special case $k_{(\alpha)}$ was defined as $\int_0^\infty k(\xi)\,e^{-\alpha\xi}\,d\xi$, but in the more general case now under consideration, where we start with the operators K and $\bar{K}$, $k_{(\alpha)}$ need not necessarily be defined in this way, and may be regarded simply as a function of α obtained from the operators by means of equations (36) and (37). The main point is that k_α obtained from the first relation (36) should be the same as that obtained from the second (37), and this implies that it is independent of the function $\phi(x)$. Further, if K and L are two such operators,

$$\int_0^\infty e^{-\alpha x}\,KL\phi(x)\,dx = k_{(\alpha)}(l\phi)_{(\alpha)} = k_{(\alpha)}l_{(\alpha)}\phi_{(\alpha)} = (kl)_{(\alpha)}\phi_{(\alpha)} \tag{38}$$

and $$\bar{K}\bar{L}\,e^{\alpha x} = \bar{K}e^{\alpha x}l_{(\alpha)} = e^{\alpha x}k_{(\alpha)}l_{(\alpha)}. \tag{39}$$

Thus, corresponding to the operator KL, we have a second operator $\bar{K}\bar{L}$, and the function related to both is $k_{(\alpha)}l_{(\alpha)}$. Furthermore it is easy to deduce that $KL = LK$ and $\bar{K}\bar{L} = \bar{L}\bar{K}$, so that the operators of the present type satisfy the five properties enumerated in section I above. It is clear that if we have a set of equations

$$\psi_s(x) + \sum_{r=1}^{n} \bar{K}_{sr}\psi_r(x) = 0, \tag{40}$$

where $\bar{K}_{sr}$ is any operator satisfying (37), then $\psi_s(x) = A_s e^{\alpha x}$ is a solution of (40), provided that $\Delta_{(\alpha)} = 0$, where $\Delta_{(\alpha)}$ is the determinant defined above. The difference between this set of equations and that treated earlier, to which it is formally identical, is that $\bar{K}$ is no longer limited to simple inte-

gration, as assumed previously. Similarly we have the set of non-homogeneous equations

$$\phi_s(x) + \sum_{r=1}^{n} K_{sr}\phi_r(x) = f_s(x), \tag{41}$$

where K_{sr} is any operator satisfying (36). If, as in section II, we multiply this equation by $e^{-\alpha x}$ and integrate from 0 to ∞, then, by equation (36),

$$\phi_{s(\alpha)} + \sum_{r=1}^{n} k_{sr(\alpha)}\phi_{r(\alpha)} = f_{s(\alpha)}, \tag{42}$$

whence $\phi_s(x)$ is obtained as the solution of this set of equations whether K has the limited interpretation previously assigned to it or the wider one now contemplated. In the case of the wider as well as the narrower interpretation,

$$\int_0^\infty \phi_s(x)\, e^{-\alpha x}\, dx = \frac{\Delta_{s(\alpha)}}{\Delta_{(\alpha)}}, \tag{43}$$

and therefore

$$\phi_s(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\alpha x}\frac{\Delta_{s(\alpha)}}{\Delta_{(\alpha)}}\, d\alpha, \tag{44}$$

and so, as before, if we assume that the functions $f_s(x)$ tend suitably to zero as $x \to \infty$, the condition that $\phi_s(x) \to 0$ as $x \to \infty$ is that the equation $\Delta_{(\alpha)} = 0$ implies that the real part of α is negative.

It is realized that to make the above proof absolutely rigorous it would be necessary to prove definitely that (44) is a solution of the equations in $\phi_s(x)$, as was done by Fock (1) in the case of the simple Volterra equation. This we have not attempted to do. The weakness of the above reasoning lies in the fact that we do not know *a priori* that the integrals defining $\phi_{s(\alpha)}$ exist for any particular value of α. The equations in $\phi_{s(\alpha)}$ have meaning only if the integrals $\phi_{s(\alpha)}$ exist.

VII. *Example.*

In order to illustrate the applications of the above theory, we shall give an example from the mathematical theory of epidemiology. It was in connection with this subject that we were led to consider the general problem dealt with above. The following set of five equations was encountered as representing the behaviour of a population supposed to have been in a steady state, but subject to a disturbance at the time $t = 0$. These

are

$$\bar{x} = \int_0^t \bar{u}_{t-\lambda} e^{-(\bar{K}V+\bar{\pi})\lambda} d\lambda - \bar{U} \iiint_\Sigma e^{-(\bar{K}V+\bar{\pi})\lambda} \bar{K}_\theta v_{t-\lambda+\xi-\theta} d\theta\, d\xi\, d\lambda + \int_0^t \bar{u}_{t\lambda_1} d\lambda_1, \quad (45)$$

$$x = -U \iiint_\Sigma F_\tau \Phi_\theta v_{t-\tau+\xi-\theta} \omega_\xi d\theta\, d\xi\, d\tau + \int_0^t F_\tau u_{t-\tau} d\tau + \int_0^t u_{t\tau_1} d\tau_1, \quad (46)$$

$$y = \int_0^t N_\theta v_{t-\theta} d\theta + \int_0^t v_{t\theta_1} d\theta_1, \quad (47)$$

$$u = \int_0^t L_\theta v_{t-\theta} d\theta, \quad (48)$$

$$v = \bar{x}\bar{K}V + \bar{X} \int_0^t \bar{K}_\theta v_{t-\theta} d\theta - \Phi V U \iiint_\Sigma \omega_\tau F_\tau \Phi_\theta v_{t-\tau+\xi-\theta} \omega_\xi d\theta\, d\xi\, d\tau + \Phi V \int_0^t \omega_\tau F_\tau u_{t-\tau} d\tau + UG \int_0^t \Phi_\theta v_{t-\theta} d\theta. \quad (49)$$

[For the physical meaning of the various quantities see Kermack and McKendrick (**11**).]

It will be observed that, in addition to the simple operators of the usual type, we have also operators involving multiple integration. These are

$$A = \iiint_\Sigma e^{-(\bar{K}V+\bar{\pi})\lambda} \bar{K}_\theta v_{t-\lambda+\xi-\theta} d\theta\, d\xi\, d\lambda, \quad (50)$$

$$B = \iiint_\Sigma F_\tau \Phi_\theta v_{t-\tau+\xi-\theta} \omega_\xi d\theta\, d\xi\, d\tau, \quad (51)$$

and

$$C = \iiint_\Sigma \omega_\tau F_\tau \Phi_\theta v_{t-\tau+\xi-\theta} \omega_\xi d\theta\, d\xi\, d\tau. \quad (52)$$

The field of integration Σ requires to be precisely defined, and may be represented either as the sum of two integrals, namely, in the case of A,

$$\iiint_\Sigma (\quad) d\theta\, d\xi\, d\lambda = \left[\int_0^t \int_0^\lambda \int_0^{t-\lambda+\xi} + \int_t^\infty \int_{\lambda-t}^\lambda \int_0^{t-\lambda+\xi}\right] (\quad) d\theta\, d\xi\, d\lambda,$$

or by the one triple integral

$$\int_0^\infty \int_\xi^{\xi+t} \int_0^{t-\lambda+\xi} (\quad) d\theta\, d\lambda\, d\xi, \quad (53)$$

λ being replaced by τ in B and C.

The corresponding Fredholm equations are obtained by omitting the final terms of equations (45), (46), and (47), and replacing in the integrals the limits t by ∞. In the case of the three multiple integrals, $\iiint_{\Sigma}(\ )\,d\theta\,d\xi\,d\lambda$ is replaced by $\int_0^\infty\int_0^\lambda\int_0^\infty(\ )\,d\theta\,d\xi\,d\lambda$ in A, and by $\int_0^\infty\int_0^\tau\int_0^\infty(\ )\,d\theta\,d\xi\,d\lambda$ in B and C. Let us call these multiple integrals $\bar{A}$, $\bar{B}$, and $\bar{C}$ respectively.

Let us first of all consider the operator W which converts v_t into A, corresponding to which is the operator $\overline{W}$ which converts v_t into $\bar{A}$. We shall first show that these operators do, in fact, satisfy conditions (36) and (37).

Clearly,

$$\overline{W}e^{\alpha t} = \int_0^\infty\int_0^\lambda\int_0^\infty e^{-(\bar{K}V+\bar{\pi})\lambda}\,\bar{K}_\theta\,v_{t-\lambda+\xi-\theta}\,d\theta\,d\xi\,d\lambda,$$

$$= e^{\alpha t}\,\bar{K}_{(\alpha)}\int_0^\infty\int_0^\lambda e^{-(\bar{K}V+\bar{\pi}+\alpha)\lambda}\,e^{\alpha\xi}\,d\xi\,d\lambda,$$

$$= \frac{e^{\alpha t}\,\bar{K}_{(\alpha)}}{(\bar{K}V+\bar{\pi})(\bar{K}V+\bar{\pi}+\alpha)},$$

for values of α for which the integral exists, where $\bar{K}_\alpha = \int_0^\infty e^{-\alpha\theta}\,\bar{K}_\theta\,d\theta$.

Thus
$$\overline{W}e^{\alpha t} = e^{\alpha t}\,w_{(\alpha)}, \tag{54}$$

where
$$w_{(\alpha)} = \frac{\bar{K}_{(\alpha)}}{(\bar{K}V+\bar{\pi})(\bar{K}V+\bar{\pi}+\alpha)}. \tag{55}$$

Further,

$$\int_0^\infty e^{-\alpha t}\,Wv_t\,dt$$

$$= \left[\int_0^\infty\int_0^t\int_0^\lambda\int_0^{t-\lambda+\xi} + \int_0^\infty\int_t^\infty\int_{\lambda-t}^\lambda\int_0^{t-\lambda+\xi}\right] e^{-\alpha t}\,e^{-(\bar{K}V+\bar{\pi})\lambda}\,\bar{K}_\theta\,v_{t-\lambda+\xi-\theta}\,d\theta\,d\xi\,d\lambda\,dt.$$

By suitable changes of the variable, and by reduction, the first integral may be shown to be equal to $\frac{\bar{K}_{(\alpha)}v_{(\alpha)}}{(\bar{K}V+\bar{\pi})\,\alpha}$, and the second to $\frac{-\bar{K}_{(\alpha)}v_{(\alpha)}}{(\bar{K}V+\bar{\pi}+\alpha)\,\alpha}$, whence

$$\int_0^\infty e^{-\alpha t}\,Wv_t\,dt = v_{(\alpha)}w_{(\alpha)}. \tag{56}$$

This shows that the operators W and $\overline{W}$ satisfy the fundamental conditions (36) and (37).

To prove that the other multiple integrals B and C, $\bar{B}$ and $\bar{C}$ also satisfy these conditions, we require the following lemma.

LEMMA.

Let $$H = \iiint_{\Sigma} F_1(\tau)\, F_2(\xi)\, F_3(\theta)\, F(t-\tau+\xi-\theta)\, d\theta\, d\xi\, d\tau\, dt,$$

then $$H = \int_0^\infty \int_0^\tau F_1(\tau)\, F_2(\xi)\, d\xi\, d\tau \int_0^\infty F_3(\theta)\, d\theta \int_0^\infty F(t)\, dt, \tag{57}$$

where F, F_1, F_2, and F_3 are finite functions defined for all positive values of the variable, and all the integrals exist.

The quadruple integral H may be written

$$H = \int_0^\infty \int_0^\infty \int_\xi^{\xi+t} \int_0^{t-\tau+\xi} F_1(\tau)\, F_2(\xi)\, F_3(\theta)\, F(t-\tau+\xi-\theta)\, d\theta\, d\tau\, d\xi\, dt$$

$$= \int_0^\infty \int_0^\infty \int_\xi^{\xi+t} \int_0^{t-\tau+\xi} (\quad)\, d\theta\, d\tau\, dt\, d\xi.$$

This is readily seen to be equal to

$$\int_0^\infty \int_\xi^\infty \int_{\tau-\xi}^\infty \int_0^{t-\tau+\xi} (\quad)\, d\theta\, dt\, d\tau\, d\xi,$$

since $$\int_0^\infty \int_\xi^{\xi+t} (\quad)\, d\tau\, dt = \int_\xi^\infty \int_{\tau-\xi}^\infty (\quad)\, dt\, d\tau,$$

both representing the same domain of integration.

By a similar modification the quadruple integral becomes

$$H = \int_0^\infty \int_\xi^\infty \int_0^\infty \int_{\tau-\xi+\theta}^\infty F_1(\tau)\, F_2(\xi)\, F_3(\theta)\, F(t-\tau+\xi-\theta)\, dt\, d\theta\, d\tau\, d\xi.$$

When we replace t by $t' = t-\tau+\xi-\theta$, the integral becomes

$$H = \int_0^\infty \int_\xi^\infty \int_0^\infty \int_0^\infty F_1(\tau)\, F_2(\xi)\, F_3(\theta)\, F(t')\, dt'\, d\theta\, d\tau\, d\xi,$$

$$= \int_0^\infty \int_\xi^\infty F_1(\tau)\, F_2(\xi)\, d\tau\, d\xi \int_0^\infty F_3(\theta)\, d\theta \int_0^\infty F(t')\, dt',$$

$$= \int_0^\infty \int_0^\tau F_1(\tau)\, F_2(\xi)\, d\xi\, d\tau \int_0^\infty F_3(\theta)\, d\theta \int_0^\infty F(t)\, dt.$$

Hence $$H = F_{(0)} F_{3(0)} \int_0^\infty \int_0^\tau F_1(\tau) F_2(\xi) d\xi d\tau.$$

We shall use the symbols W_1 and W_2, $\overline{W}_1$ and $\overline{W}_2$, for the operators which convert v_t into B and C, $\bar{B}$ and $\bar{C}$ respectively. It may then readily be shown, by means of the above lemma, that

$$\int_0^\infty e^{-\alpha t} W_1 v_t dt = v_\alpha \Phi_{(\alpha)} H_{(\alpha\alpha)}, \tag{58}$$

where $H_{(\alpha\alpha)}$ is written for

$$\int_0^\infty e^{-\alpha\tau} F_\tau \int_0^\tau \omega_\xi e^{\alpha\xi} d\xi d\tau,$$

while $$\overline{W}_1 e^{\alpha t} = e^{\alpha t} \Phi_{(\alpha)} H_{(\alpha\alpha)}, \tag{59}$$

that is to say W_1 and $\overline{W}_1$ satisfy conditions (36) and (37), $w_{1(\alpha)}$ having the value $\Phi_{(\alpha)} H_{(\alpha\alpha)}$.

Similarly it may be shown that

$$\int_0^\infty e^{-\alpha t} W_2 v_t dt = v_{(\alpha)} \Phi_{(\alpha)} Z_{(\alpha\alpha)}, \tag{60}$$

where $Z_{(\alpha\alpha)}$ is written for

$$\int_0^\infty \omega_\tau e^{-\alpha\tau} F_\tau \int_0^\tau \omega_\xi e^{\alpha\xi} d\xi d\tau;$$

and $$\overline{W}_2 e^{\alpha t} = e^{\alpha t} \Phi_{(\alpha)} Z_{(\alpha\alpha)}. \tag{61}$$

It follows that the equation $\Delta_{(\alpha)} = 0$ becomes, in this example,

$$\Delta_{(\alpha)} = \begin{vmatrix} -N_{(\alpha)} - \bar{K}_{(\alpha)} \bar{X}, & \bar{\mu} - (\bar{K}V + \bar{\pi} + \alpha), & \mu \\ -1 + \bar{K}_{(\alpha)} \bar{X} + L\Omega, & \bar{K}V, & 0 \\ L_{(\alpha)} F_{(\alpha)} - L \dfrac{\Phi_{(\alpha)}}{\Phi} \dfrac{H_{(\alpha\alpha)}}{G'}, & 0, & -1 \end{vmatrix} = 0, \tag{62}$$

where Ω is written for

$$\frac{\Phi_{(\alpha)}}{\Phi} \left(1 - \frac{Z_{(\alpha\alpha)}}{(G')^2}\right) + \frac{G_{(\alpha)}}{G'} \frac{L_{(\alpha)}}{L},$$

and

$$G' = \frac{1}{\Phi V}, \quad G_{(\alpha)} = \int_0^\infty e^{-\alpha\tau} \omega_\tau F_\tau d\tau, \quad F_\tau = \exp\left\{-\int_0^\tau (\Phi V \omega_\xi + \pi) d\xi\right\}.$$

This may be reduced to the more convenient form

$$\alpha\left\{1-L\left(\Omega+\pi F\,\frac{\overline{K}_{(\alpha)}}{\overline{K}}\right)-\frac{\overline{K}_{(\alpha)}}{\overline{K}}\,(D+\rho N)\right\}+\frac{m}{\overline{X}}\,(1-L\Omega)$$

$$+\frac{\mu LFV}{\overline{X}}\left\{1-L\left(\Omega+\pi F_{(\alpha)}\,\frac{L_{(\alpha)}}{L}\right)-\frac{F_{(\alpha)}}{F}\,\frac{L_{(\alpha)}}{L}\,(D+\rho N)\right\}$$

$$+\mu\overline{K}VL\,\frac{\Phi_{(\alpha)}}{\Phi}\,\frac{H_{(\alpha\alpha)}}{G'}+(\bar{\mu}-\bar{\pi})\,\overline{X}\overline{K}_{(\alpha)}$$

$$+\frac{\nu NV}{\overline{X}}\left\{1-L\left(\Omega+\pi F\,\frac{N_{(\alpha)}}{N}\right)-\frac{N_{(\alpha)}}{N}\,(D+\rho N)\right\}=0. \quad (63)$$

We have thus reduced the problem of the stability of the steady state of the population to the determination of the sign of the real parts of the roots of this equation.

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THE EXISTENCE THEOREM FOR HARMONIC INTEGRALS

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In various papers which I have published†, I have used the theory of harmonic integrals in order to deduce results concerning the Abelian and Picard integrals of the first kind associated with algebraic varieties. Herr Helmuth Kneser has recently written to me to point out that the "Methode der Parametrix" of Hilbert‡ is well suited to give a simple method of establishing the existence of the requisite number of harmonic integrals on an analytic variety. On working this method out, however, I found that the application of Hilbert's method to this problem was not quite straightforward, and the present paper is intended to show the devices to which I have had to resort in order to surmount certain difficulties which arise. Nevertheless, I desire to place on record my acknowledgments to Herr Kneser for suggesting this method of proof, which is superior in every way to the one which I originally gave (I and II).

A detailed account of every step would still be very lengthy. Much of it consists, however, of fairly obvious generalizations of the work of Hilbert, and I think that the reader will be able to obtain a better idea of the proof as a whole if I omit proofs which are either mere calculations or are straightforward generalizations of known theorems.

† W. V. D. Hodge, *Proc. London Math. Soc.* (2), 36 (1933), 257–333, referred to as I; *Proc. London Math. Soc.* (2), 38 (1934), 72–95, referred to as II; *Journal London Math. Soc.*, 8 (1933), 312–319; *Proc. Camb. Phil. Soc.*, 31 (1935), 18–25.

‡ D. Hilbert, *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen* (Leipzig, 1912).

1. The construct with which we are concerned is, in the language of topology, an absolute orientable manifold M of n dimensions, having Betti numbers R_0, R_1, ..., R_n. For the purposes of this paper, we assume that M is represented as an analytic locus in a Euclidean space E of N dimensions, so that the neighbourhood of any point of M is obtained by writing the rectangular Cartesian coordinates $(X_1, \ldots, X_N)$ as analytic functions of n parameters $(x^1, \ldots, x^n)$, which we shall refer to as local coordinates. As a sub-space in E, M can be regarded as a Riemannian space on which the metric is given by the positive definite quadratic differential form

$$g_{ij}\,dx^i\,dx^j = \sum_{i=1}^{N} dX_i^2.$$

Results borrowed from the theory of quadratic differential forms will be given in the notation of Veblen†.

Let $P_{i_1 \ldots i_p}$ $(p \leqslant n)$ be an anti-symmetric covariant absolute tensor on M, whose components are continuous functions of the local co-ordinates. From it we can form an integral

$$\frac{1}{p!} \int P_{i_1 \ldots i_p} dx^{i_1} \ldots dx^{i_p},$$

which is clearly independent of the local coordinate system used. Since this paper deals entirely with integrals, we shall find it convenient to use the notation of *forms* due to Cartan‡. We write

$$\frac{1}{p!}\, P_{i_1 \ldots i_p} dx^{i_1} \ldots dx^{i_p} = P,$$

or $P(x)$, if we wish to stress the variables used, and speak of P as a p-form. The following properties of p-forms are easily verified (assuming the existence of the necessary derivatives of the coefficients). The product of a p-form and a q-form is obtained by multiplying the two forms, say P and Q, in the given order, remembering that an odd permutation of the differentials dx^i results in a change of sign of the term concerned. Thus $PQ = R$, where

$$R_{i_1 \ldots i_{p+q}} = \frac{1}{p!\,q!}\, \delta^{j_1 \ldots j_p\, k_1 \ldots k_q}_{i_1 \ldots i_{p+q}}\, P_{j_1 \ldots j_p}\, Q_{k_1 \ldots k_q}.$$

† O. Veblen, *Invariants of quadratic differential forms*, Cambridge Tract, No. 24 (1927).

‡ E. Cartan, *Leçons sur les invariants intégraux* (Paris, 1922), ch. 7.

The derived form of a p-form P is the $(p+1)$-form $P_x = Q$, where

$$Q_{i_1 \dots i_{p+1}} = \sum_{r=1}^{p+1} (-1)^{r-1} \frac{\partial}{\partial x^{i_r}} P_{i_1 \dots i_{r-1} i_{r+1} \dots i_{p+1}}.$$

If P and Q are, respectively, a p-form and a q-form, we have the following formulae:

$$PQ = (-1)^{pq} QP,$$

$$[PQ]_x = P_x Q + (-1)^p P Q_x,$$

$$[P_x]_x = 0.$$

If we write $$P^{i_1 \dots i_p} = g^{i_1 j_1} \dots g^{i_p j_p} P_{j_1 \dots j_p},$$

and $$P^*_{j_1 \dots j_{n-p}} = \frac{1}{p!} \sqrt{g}\, \epsilon_{i_1 \dots i_p j_1 \dots j_{n-p}} P^{i_1 \dots i_p},$$

then $P^*_{j_1 \dots j_{n-p}}$ is an anti-symmetric covariant absolute tensor of rank $n-p$, and we call the form

$$P^* = \frac{1}{(n-p)!} P^*_{i_1 \dots i_{n-p}} dx^{i_1} \dots dx^{i_{n-p}}$$

the $(n-p)$-form associated with the p-form P. We notice that

$$[P^*]^* = (-1)^{p(n-p)} P.$$

In place of $[P_x]^*$ we write P^x.

The following properties of the integrals of forms have been established†. If Δ denotes a $(p+1)$-field of integration having the oriented boundary Γ (*i.e.* $\Delta \to \Gamma$, in the usual notation of topology), and P is any p-form whose coefficients have continuous derivatives, then

$$\int_\Delta P_x = \int_\Gamma P.$$

Hence, if $\Gamma \sim 0$ and $P_x = 0$,

$$\int_\Gamma P = 0.$$

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† G. de Rham, *Journal de Math.* (9), 10 (1931), 115–200.

More generally, if $\lambda\Gamma \sim 0$, *i.e.* $\Gamma \approx 0$,

$$\int_\Gamma P = 0.$$

Moreover, if there exists a $(p-1)$-form Q, such that

$$Q_x = P$$

and $\Gamma \to 0$, *i.e.* Γ is *any* p-cycle, then

$$\int_\Gamma P = 0.$$

For these reasons we shall say that P is a closed form $(P \to 0)$ if $P_x = 0$, and that P is a null-form if there exists a Q such that $P = Q_x$ $(P \sim 0)$. If P is a closed form and Γ is any p-cycle,

$$\int_\Gamma P$$

is called the period of the integral on this cycle. It is zero if $\Gamma \approx 0$, and hence it is unaltered if Γ is replaced by Γ', where $\Gamma \approx \Gamma'$.

De Rham has proved that there exists a closed p-form P, whose coefficients have continuous derivatives of the second order, such that

$$\int P$$

has arbitrarily assigned periods on R_p independent p-cycles of M.

The integral $\int P$ is said to be harmonic on M if P is defined at all points of M and each coefficient has continuous derivatives of the second order, and if, further,

$$P \to 0, \quad P^* \to 0.$$

I have proved (in II) that the periods of a harmonic integral cannot all be zero, or, what is the same thing, on account of results due to de Rham, if $\int P$ is harmonic, and

$$P \sim 0,$$

then $$P = 0.$$

The existence theorem which it is our object to prove states that there exists a harmonic p-fold integral having arbitrarily assigned periods (not all zero) on R_p independent p-cycles of M.

To prove this we first construct the de Rham integral

$$\int A$$

having the given periods. Then, if u is any $(p-1)$-form whose coefficients have continuous derivatives of the third order,

$$\int_M (A - u_x)$$

is another de Rham integral with the given periods. This is a harmonic integral if

$$u^x{}_x = A^*{}_x.$$

where

$$u^x{}_x = (u^x)_x.$$

If we can solve this equation for u, the existence theorem is established. We do this by showing that the equation

$$u^x{}_x = f, \tag{1}$$

where f is an $(n-p+1)$-form, has a solution if

$$f \sim 0.$$

For $p = 0$ a closed form is a constant, and this satisfies the condition for being harmonic. Hence we may suppose our problem solved for $p = 0$.

2. Let $K(x, y)$ denote a double form in two independent sets of variables, which is a $(p-1)$-form in x and an $(n-p+1)$-form in y, defined for all points x and y of M, except perhaps at $x = y$, where $r^{n-1} K(x, y)$ is finite, r being the distance between x and y in E. We shall make the solution of (1) depend on the solution of an integral equation for an unknown $(p-1)$-form u,

$$u(x) + \lambda \int_M K(x, y)\, u(y) = \phi(x), \tag{2}$$

where ϕ is a $(p-1)$-form defined at all points of M; and some preliminary remarks on such integral equations are necessary.

We can regard the integral equation as a set of simultaneous integral equations in the coefficients of the form u. The theory of these integral equations is exactly analogous to the theory of ordinary integral equations, and can be developed according to any of the classical methods.

For the purposes of the lemma which I require, I found that the method of Lalesco† was the most suitable. We prove that equation (2) has a unique solution, except for particular values of λ which are the characteristic values. If $\lambda = \lambda_1$ is a characteristic value, the associated homogeneous equations

$$u(x) + \lambda_1 \int_M K(x, y)\, u(y) = 0, \tag{3}$$

$$v(x) + \lambda_1 \int_M v(y)\, K(y, x) = 0, \tag{4}$$

where v is an $(n-p+1)$-form, have the same number r (>0) of linearly independent solutions. In this case (2), with $\lambda = \lambda_1$, has a solution if and only if

$$\int_M v\phi = 0$$

for each solution v of (4). Further, it is a simple deduction from Lalesco's method of solution in the case of characteristic values of the parameter, that, *if ϕ satisfies the conditions for a solution for the characteristic value $\lambda = \lambda_1$, and u denotes a solution of* (2) *for λ near λ_1, u tends to a solution of* (2) *for $\lambda = \lambda_1$, when λ tends to λ_1.*

This last result enables us to prove the following

LEMMA. *Given that*

$$u \sim 0 \quad (or \to 0)$$

implies

$$\int_M K(x, y)\, u(y) \sim 0 \quad (or \to 0),$$

then, if

$$\phi \sim 0 \quad (or \to 0),$$

equation (2) *has one solution (at least),*

$$u(y) \sim 0 \quad (or \to 0).$$

The solution of (2) for a non-characteristic value of λ is given by

$$u(x) = \phi(x) + \lambda \int_M S(x, y;\ \lambda)\, \phi(y),$$

† T. Lalesco, *Introduction à la théorie des équations intégrales* (Paris, 1912).

where
$$S(x, y; \lambda) = -\frac{D(x, y; \lambda)}{D(\lambda)},$$

$$D(x, y; \lambda) = \sum_0^\infty \frac{\lambda^n}{n!} L_n(x, y),$$

$$D(\lambda) = \sum_0^\infty \frac{\lambda^n}{n!} L_n.$$

The two series converge for all values of λ, and $D(\lambda) \neq 0$ for non-characteristic values. The following recurrence relation holds:

$$L_n(x, y) = K(x, y) L_n - n \int_M K(x, z) L_{(n-1)}(z, y),$$

where $L_0(x, y) = K(x, y)$. Hence, under our hypothesis, we find that

$$\int_M L_n(x, y) \phi(y) \sim 0 \cdot \text{ (or } \to 0),$$

and hence
$$\int_M S(x, y; \lambda) \phi(y) \sim 0 \quad \text{(or } \to 0).$$

Thus the lemma holds for non-characteristic values of the parameter.

If λ_1 is a characteristic value, and $\phi(y)$ satisfies the conditions for a solution of (2) for this value of the parameter, the solution which we have just found for a general value of λ tends, as λ tends to λ_r, to a solution of (2) with $\lambda = \lambda_1$, which satisfies the conditions of our lemma.

3. Let u, v be two $(p-1)$-forms whose coefficients have continuous derivatives of the second order in a domain Δ of M which has boundary Γ. Then, from the identity

$$(u^x v - v^x u)_x = u^x{}_x v - v^x{}_x u,$$

we have
$$\int_\Delta u^x{}_x v - \int_\Delta v^x{}_x u = \int_\Gamma u^x v - \int_\Gamma v^x u. \tag{5}$$

Let

$$\omega_{(p-1)}(x, y) = \frac{1}{[(p-1)!]^2} \frac{1}{(-2)^{p-1} r^{n-2}} \times \begin{vmatrix} \dfrac{\partial^2 S}{\partial x^{i_1} \partial y^{j_1}} & \cdots & \dfrac{\partial^2 S}{\partial x^{i_1} \partial y^{j_{p-1}}} \\ \vdots & & \\ \dfrac{\partial^2 S}{\partial x^{i_{p-1}} \partial y^{j_1}} & \cdots & \dfrac{\partial^2 S}{\partial x^{i_{p-1}} \partial y^{j_{p-1}}} \end{vmatrix} dx^{i_1} \ldots dx^{i_{p-1}} dy^{j_1} \ldots dy^{j_{p-1}},$$

where $S = r^2$, and r is the distance between x and y in E. This double form is the direct generalization of the form which I denoted by the same symbol in I, there given for a flat space†. We shall further suppose that $\omega_{(-1)}(x, y) = 0$. Then, by using normal coordinates, we prove that

$$K_{(p-1)}(x, y) = \omega^y_{(p-1)y}(x, y) + (-1)^p \omega^y_{(p-2)x}(x, y)$$

is analytic at all points x and y, except at $x = y$, where $r^{n-1}K_{(p-1)}(x, y)$ is analytic. In (5), let $M - \Delta$ be a geodesic sphere of radius ρ, which we make tend to zero, and centre x. Take $v(y) = \omega_{(p-1)}(x, y)$. By using normal coordinates we prove that

$$\lim_{\rho \to 0} \int_\Delta \omega^y_{(p-1)y}(x, y)\, u(y) = (-1)^{n(p-1)} \frac{(n-2)(n-p+1)}{n}\, a_n u(x) + \int_M u^y{}_y(y)\, \omega_{(p-1)}(y, x),$$

where
$$a_n = \int \sum_{i=1}^{n} (-1)^{i-1} x_i dx_1 \dots dx_{i-1} dx_{i+1} \dots dx_n$$

over
$$x_1^2 + x_2^2 + \dots + x_n^2 = 1$$

in Euclidean n-space. It is easily proved that the limit on the left-hand side exists. Now calculate

$$\left[\int_M \omega^y_{(p-2)}(x, y)\, u(y) \right]_x.$$

We find that this is equal to

$$\lim_{\rho \to 0} \int_\Delta \omega^y_{(p-2)x}(x, y)\, u(y) + (-1)^{(n-1)(p-1)} \frac{(n-2)(p-1)}{n}\, a_n u(x),$$

and, combining these results, we have

$$\int_M K_{(p-1)}(x, y)\, u(y) - (-1)^{n(p-1)}(n-2)\, a_n u(x)$$
$$= (-1)^p \left[\int_M \omega^y_{(p-2)} u(y) \right]_x + \int_M u^y{}_y(y)\, \omega_{(p-1)}(y, x)$$
$$= (-1)^n \int_M u^*{}_y(y)\, \omega_{(p-2)x}(y, x) + \int_M u^y{}_y(y)\, \omega_{(p-1)}(y, x). \quad (6)$$

† I have to thank Herr Kneser for suggesting this generalized form of $\omega_{p-1}(x, y)$.

We notice that, if $u(y) \sim 0$ (or $\to 0$),

$$\int_M K_{(p-1)}(x, y)\, u(y) \sim 0 \quad (\text{or} \to 0).$$

By a similar calculation we have

$$\Big[\int_M u^*(y)\, \omega_{(p-1)}(y, x)\Big]_x^x = \int_M u^*(y)\, K_{(p-1)}(y, x) - (-1)^{n(p-1)}(n-2)\, \alpha_n\, u^*(y) + (-1)^{p-1} \int_M u^*{}_y(y)\, \omega^x_{(p-2)}(y, x), \quad (7)$$

and $u^*(x) \sim 0$ (or $\to 0$) implies

$$\int_M u^*(y)\, K_{(p-1)}(y, x) \sim 0 \quad (\text{or} \to 0).$$

4. If u is a solution of (1), it must be a solution of

$$\int_M K(x, y)\, u(y) - (-1)^{n(p-1)}(n-2)\, \alpha_n\, u(x) = (-1)^n \int_M u^*{}_y(y)\, \omega_{(p-2)x}(y, x) + \int_M f(y)\, \omega_{(p-1)}(y, x).$$

We shall proceed conversely to use this equation to solve (1), and hence to establish the existence of R_p linearly independent harmonic forms of rank p. Since we have already established this result in the case $p=0$, we may assume that $p > 0$, and proceed by induction. Let us suppose that we know that there exist R_{p-1} independent harmonic $(p-1)$-forms

$$u^{(i)}_{(p-1)} \quad (i = 1, \ldots, R_{p-1}).$$

Since $$u^{(i)}_{(p-1)} \to 0, \quad u^{(i)*}_{(p-1)} \to 0,$$

we see from (6) that $u^{(i)}_{(p-1)}$ is a solution of the homogeneous equation

$$\int_M K_{(p-1)}(x, y)\, u(y) - (-1)^{n(p-1)}(n-2)\, \alpha_n\, u(x) = 0. \quad (8)$$

Suppose that (8) has M_{p-1} further solutions which are closed forms. Any closed form u is equal to

$$\Sigma \lambda_r u^{(r)}_{(p-1)} + \phi$$

for some set of constants λ_r, where

$$\int \phi$$

has no periods. (By a result due to de Rham, $\phi \sim 0$, but we do not have to use this fact.) Equation (8) has therefore M_{p-1} closed solutions $\phi^{(i)}$ $(i = 1, \ldots, M_{p-1})$, such that

$$\int \phi^{(i)}$$

has no periods. Finally, let (8) have N_{p-1} solutions no combination of which is closed, say

$$\psi^{(i)} \quad (i = 1, \ldots N_{p-1}).$$

Substitute $\phi^{(i)}$ and $\psi^{(i)}$ for u in (6) and we have

$$\int_M \phi^{(i)*}{}_y(y)\, \omega_{(p-2)}(y, x) \to 0,$$

$$\int_M \psi^{(i)y}{}_y(y)\, \omega_{(p-1)}(y, x) \sim 0.$$

From (7) we conclude that

$$\int_M \phi^{(i)*}(y)\, K_{(p-1)}(y, x) - (-1)^{n(p-1)}(n-2)\, a_n\, \phi^{(i)*}(x) = \Big[\int_M \phi^{(i)*}(y)\, \omega_{(p-1)}(y, x)\Big]^x_x, \quad (9)$$

$$\int_M \psi^{(i)y}{}_y(y)\, K_{(p-1)}(y, x) - (-1)^{n(p-1)}(n-2)\, a_n\, \psi^{(i)x}{}_x(x) = 0. \quad (10)$$

Equation (10) shows that the equation for the $(n-p+1)$-form v, namely

$$\int_M v(y)\, K_{(p-1)}(y, x) - (-1)^{n(p-1)}(n-2)\, a_n\, v(x) = 0, \quad (11)$$

has, among its $R_{p-1}+M_{p-1}+N_{p-1}$ solutions, the forms

$$\psi^{(i)x}{}_x \quad (i = 1, \ldots N_{p-1})$$

which are independent. For, if $\Sigma a_i \psi^{(i)x}{}_x = 0$, $\int \Sigma a_i \psi^{(i)}_x$ would be a harmonic integral without periods, and hence

$$\Sigma a_i \psi^{(i)} \to 0,$$

which is contrary to hypothesis.

By the lemma of § 2, the equation

$$\int_M v(y)\, K_{(p-1)}(y, x) - (-1)^{n(p-1)}(n-2)\,\alpha_n\, v(x) = \left[\int_M \phi^{*(i)}(y)\, \omega_{(p-1)}(y, x)\right]_x^x$$

has a solution $\xi^{(i)} \sim 0.$

Hence, by (9), $\phi^{(i)*} - \xi^{(i)}$

is a solution of (11). No linear combination of these solutions can be closed. For, if

$$\Sigma a_i(\phi^{(i)*} - \xi^{(i)}) \to 0,$$

then $\Sigma a_i \phi^{(i)*} \to 0,$

and hence $\int \Sigma a_i \phi^{(i)}$

would be a harmonic integral with all its periods zero. Again

$$u_{(p-1)}^{(i)*} \quad (i = 1, \ldots, R_{p-1} = R_{n-p+1})$$

are linearly independent closed (harmonic) forms and hence no combination can be null. But, from (7),

$$\int_M u_{(p-1)}^{(i)*}(y)\, K_{(p-1)}(y, x) - (-1)^{n(p-1)}(n-2)\,\alpha_n\, u_{(p-1)}^{(i)*}(x) = \left[\int_M u_{(p-1)}^{(i)*}(y)\, \omega_{(p-1)}(y, x)\right]_x^x,$$

and hence, by an argument similar to that given above, (11) has R_{p-1} solutions

$$u_{(p-1)}^{(i)*} - \eta^{(i)}, \quad \eta^{(i)} \sim 0,$$

no linear combination of which is a null form.

Equation (11) has, therefore, (i) N_{p-1} independent solutions of the form $\psi^{(i)x}{}_x$, (ii) M_{p-1} independent solutions $\phi^{(i)*} - \xi^{(i)}$, (iii) R_{p-1} independent solutions $u_{(p-1)}^{(i)*} - \eta^{(i)}$. These $N_{p-1} + M_{p-1} + R_{p-1}$ solutions are all independent, and only the first N_{p-1} are null. For, if

$$\Sigma a_i \psi^{(i)x}{}_x + \Sigma b_i(\phi^{(i)x} - \xi^{(i)}) + \Sigma c_i(u_{(p-1)}^{(i)*} - \eta^{(i)}) \sim 0,$$

then $b_i = 0$, since otherwise a linear combination of the M_{p-1} solutions (ii) would be closed, and this has been proved to be impossible. Similarly, $c_i = 0$, since otherwise a linear combination of the forms $u_{(p-1)}^{(i)*}$ would be null. Hence, *the only solutions of* (11) *which are null forms are*

$$\psi^{(i)x}{}_x \quad (i = 1, \ldots N_{p-1}).$$

Now let v be any solution of (11), and let

$$u = \int_M v(y)\, \omega_{(p-1)}(y, x).$$

From equation (7) and the fact that v is a solution of (11), we have

$$u^x{}_x = \left[(-1)^{p-1} \int_M u^*{}_y(y)\, \omega_{(p-2)}(y, x) \right]^x,$$

and hence $$u^{xx} \sim 0.$$

Therefore $$\int u^x{}_x$$

is a harmonic integral with all its periods zero, and so

$$u^x \to 0.$$

Therefore $$\int u_x$$

is a harmonic integral, and all its periods are zero; hence

$$u \to 0.$$

Therefore, for every solution v of (11),

$$\int_M v(y)\, \omega_{(p-1)}(y, x) \to 0.$$

Finally we show that, if the solutions of (8) or (11) are known, we can solve the simultaneous equations

$$\int_M v(y)\, \omega_{(p-1)}(y, x) \to 0,$$

$$v \sim 0.$$

From (7) it follows that v must be a null form satisfying (11), and hence

$$v = \sum_{i=1}^{N_{p-1}} a_i \psi^{(1)} x_x,$$

where $a_1, \ldots, a_{N_{p-1}}$ are arbitrary constants.

5. Let us now consider the equation

$$\int_M K_{(p-1)}(x, y)\, u(y) - (-1)^{n(p-1)}(n-2)\, \alpha_n\, u(x) = \int_M f(y)\, \omega_{(p-1)}(y, x). \quad (12)$$

This has a solution if

$$\int_M \int_M f(y)\,\omega_{(p-1)}(y,\,x)\,v(x) = 0$$

for all solutions v of (11). But we have proved above that, for all v in question,

$$\int_M \omega_{(p-1)}(y,\,x)\,v(x) \to 0;$$

hence these conditions are all satisfied if

$$f \sim 0.$$

Let u be a solution of (12). From (6) we see that

$$\int_M [u^y{}_y(y) - f(y)]\,\omega_{(p-1)}(y,\,x) \sim 0$$

and

$$u^x{}_x - f \sim 0.$$

Hence

$$u^x{}_x - f = \sum_{i=1}^{N_{p-1}} a_i \psi^{(i)x}{}_x,$$

where the a's are constants. Hence, if

$$U = u - \sum_{i=1}^{N_{p-1}} a_i \psi^{(i)},$$

$$U^x{}_x = f,$$

and hence we have found a solution of (1). In consequence we have established by induction that there exists a harmonic p-fold integral having arbitrarily assigned periods on R_p independent p-cycles of V.

6. One or two deductions from the foregoing theory may be mentioned. If

$$\int u$$

is any p-fold harmonic integral on V, u is a solution of

$$\int_M K_{(p)}(x,\,y)\,u(y) - (-1)^{np}(n-2)\,\alpha_n\,u(x) = 0.$$

It is a consequence of the theory of integral equations that the coefficients of u must be analytic functions of the local coordinates.

Secondly, if $$\int P \quad (P \to 0)$$

is a p-fold integral whose coefficients have continuous derivatives of the second order, and whose periods are all zero, we can find a $(p-1)$-form u such that

$$u^{x}{}_{x} = P^{*}{}_{x}.$$

Then $$\int (P - u_x)$$

is a harmonic integral with all its periods zero. Therefore

$$P = u_x \sim 0.$$

This result, due to de Rham, has been quoted, but not used, in the text.

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ON CERTAIN PROPERTIES OF NON-HOLONOMIC SUBMANIFOLDS OF A NON-RIEMANNIAN SPACE AND THE EFFECT ON THEM OF A GENERAL DEFORMATION

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1. *Introduction.*

The general theory of deformation, outlined by Schouten and van Kampen (**1**), leads one to inquire the nature of the deformation of such important entities as torsion and curvature in particular spaces. I have calculated the magnitudes of the deformations of certain of the fundamental quantities associated with a non-holonomic submanifold of a non-Riemannian space, basing the geometry of the submanifold on the important paper of Dienes (**2**). The determination of the deformation requires, in some cases, a rather more detailed geometrical interpretation of the formulae than that given in Dienes' paper, and in supplying these interpretations, particularly in the case of the curvature tensors, I found that more than one form of curvature could be calculated by transporting along different circuits. A brief note on circuits will be found in § 3 of this paper. The definitions of the contravariant and covariant frames at the beginning of § 2 are those given in a lecture by Dr. Dienes, for whose kind permission to quote them I am very grateful. The expressions $B_i{}^a \partial_j B_a{}^h$, which I call j_{ij}^h, recurred frequently in the working; several geometrical properties in which they occur as connection parameters of a local m-dimensional facet with zero torsion and curvature are given. This facet I have called the "pseudo-Euclidean tangent facet". The j_{ij}^h determine a local "absolute parallelism" which may be used independently for the facet or the span, giving an interesting property, that the alternated Ricci coefficient of rotation in the facet is the facet torsion tensor.

Throughout the paper, Greek suffixes are used for the original mathematical base, running from 1 to n, and Latin suffixes are used in the local geometrical base, the initial letters $a, b, c, \ldots$ running from 1 to n, the middle

letters $h, i, j, \ldots$ from 1 to m, and the later letters $p, q, r, \ldots$ from $m+1$ to n.

2. *The local base defined at a point of a non-Riemannian manifold.*

The points $P \equiv (x^1, x^2, \ldots, x^n)$ constitute the n-dimensional mathematical manifold X_n. A system of n quantities v^α, which transform like dx^α under a coordinate transformation, is called a *contravariant vector*, and a system of n quantities v_α, which transform like $\partial f / \partial x^\alpha$, is called a *covariant vector.*

The contravariant frame. When only the α-th coordinate of P varies, the locus of P is a one-dimensional range of points called an α-th mathematical parametric line. The range of points

$$(x^1, x^2, \ldots, x^{\alpha-1}, x^\alpha + t\, l^\alpha, x^{\alpha+1}, \ldots, x^n),$$

where only t varies from 0 to 1, can be conceived as an arrow $\underset{\alpha}{e}$, of the mathematical length l^α, issued from P in the direction of the α-th parametric line through P. The α-th component of any contravariant vector is some multiple of l^α, and this factor is called the α-th component, v^α, in the adopted mathematical frame $\underset{\alpha}{e}$. Thus we write $v^\alpha \underset{\alpha}{e}$ for any contravariant vector.

In a change of variables, we have

$$(2.1) \qquad dx^\alpha = \frac{\partial x^\alpha}{\partial y^a}\, dy^a = A_a{}^\alpha\, dy^a, \quad dy^b = \frac{\partial y^b}{\partial x^\alpha}\, dx^\alpha = A_\alpha{}^b\, dx^\alpha,$$

and thus, by definition, if $\bar{v}$ is the transform of v,

$$(2.2) \qquad v^\alpha = A_b{}^\alpha \bar{v}^b, \quad \bar{v}^b = A_\alpha{}^b v^\alpha.$$

Also the equations

$$(2.3) \qquad dy^1 = t\, A_\alpha{}^1 l^\alpha, \quad \ldots, \quad dy^n = t\, A_\alpha{}^n l^\alpha$$

define the new arrows when t varies from 0 to 1.

The above change of variables, as produced by the change of the arrows $\underset{\alpha}{e}$, may be described by putting

$$(2.4) \qquad \underset{\alpha}{e} = A_\alpha{}^b \underset{b}{e}, \quad \underset{b}{e} = A_b{}^\alpha \underset{\alpha}{e},$$

which gives

$$(2.5) \qquad v^\alpha \underset{\alpha}{e} = v^\alpha A_\alpha{}^b \underset{b}{e} = \bar{v}^b \underset{b}{e},$$

in conformity with (2.2).

This leads to the general idea of a frame where $A_a{}^\alpha$ and $A_\alpha{}^a$ are not necessarily gradients. In fact, in the above work, the only necessary restriction on $A_\alpha{}^a$, and thus on $A_a{}^\alpha$ also, is that their determinants are different from zero, so that the relations

$$A_a{}^\alpha A_\alpha{}^b = \delta_a{}^b, \quad A_a{}^\alpha A_\beta{}^a = \delta_\beta{}^\alpha \tag{2.6}$$

determine one set of transformation factors in terms of the other.

The covariant frame. If $\overset{\alpha}{e}$ are the components of any covariant vector, the α-th component of any other covariant vector is some multiple of $\overset{\alpha}{e}$. The factor v_α is called its α-th covariant component relative to the frame $\overset{\alpha}{e}$, and we write the covariant vector in the form $v_\alpha \overset{\alpha}{e}$.

By definition, in a change of variables, v_α are transformed like $\partial f/\partial x^\alpha$, that is

$$v_\alpha = A_\alpha{}^a \bar{v}_a, \quad \bar{v}_a = A_a{}^\alpha v_\alpha, \tag{2.7}$$

and the frame is, therefore, transformed by

$$\overset{\alpha}{e} = A_a{}^\alpha \overset{a}{e}, \quad \overset{a}{e} = A_\alpha{}^a \overset{\alpha}{e}. \tag{2.8}$$

Tensors are forms, that is homogeneous polynomial expressions in $\underset{\alpha}{e}$ and $\overset{\alpha}{e}$, like $v^\alpha_{\beta\gamma} \underset{\alpha}{e}\overset{\beta}{e}\overset{\gamma}{e}$. The transformations of the components are defined through the frame $(\underset{\alpha}{e}, \overset{\alpha}{e})$. By changing the variables and the frame, we obtain new systems of reference for the description of the *same* geometric objects, for example, tensors.

Connection and parallel transport in terms of the new base. The connection in the new base is defined to be such that two parallel vectors in the neighbourhood of the base point P, in the original mathematical base, retain this property in the local geometrical base.

Let P be the point (x^α) in the mathematical base, and the base point of the geometrical base, and let Q be the point $(x^\alpha + \xi^\alpha)$ in the mathematical base; then PQ has components ξ^a in the direction of the local base vectors. If the contravariant vector v, at Q, is parallel to the contravariant vector w, at P, then

$$w^\alpha(P \| Q) = w^\alpha - \Gamma^\alpha_{\beta\gamma} w^\beta \xi^\gamma = v^\alpha \tag{2.9}$$

in the original base, the $\Gamma^\alpha_{\beta\gamma}$ being calculated at P.

If γ_{bc}^{a} are the connection parameters in the new base, then, for parallelism, we have

$$w^a(P\,||\,Q) = w^a - \gamma_{bc}^{a}\, w^b\, \xi^c = v^a. \tag{2.10}$$

Since the v^a and v^a are components of the same contravariant vector at Q and the w^a and w^a at P, these two equations are equivalent, and from these, together with the relations

$$(A_a{}^{\alpha})_Q\,(v^a)_Q = (v^{\alpha})_Q \tag{2.11}$$

and

$$(A_a{}^{\alpha})_P\,(w^a)_P = (w^{\alpha})_P, \tag{2.12}$$

the γ_{bc}^{a} may be calculated in terms of the $\Gamma_{\beta\gamma}^{\alpha}$, leading to

$$A_{\alpha}{}^{a}\, A_b{}^{\beta}\, A_c{}^{\gamma}\, \Gamma_{\beta\gamma}^{\alpha} = \gamma_{bc}^{a} - A_{\alpha}{}^{a}\, \partial_c A_b{}^{\alpha}, \tag{2.13}$$

which is the equation for transformation of connection parameters under a change of base, in such a way that parallelism is preserved.

The expression $-A_{\alpha}{}^{a}\, \partial_c A_b{}^{\alpha}$ we write j_{bc}^{a}. Since, from (2.6), $A_a{}^{\alpha}\, A_{\alpha}{}^{b} = \delta_a{}^{b}$, $A_{\alpha}{}^{a}\, \partial_c A_b{}^{\alpha} + A_b{}^{\alpha}\, \partial_c A_{\alpha}{}^{a} = 0$, the j_{bc}^{a} may be written for either of the alternate forms

$$j_{bc}^{a} = A_b{}^{\alpha}\, \partial_c A_{\alpha}{}^{a} = -A_{\alpha}{}^{a}\, \partial_c A_b{}^{\alpha}, \tag{2.14}$$

the equation for transformation of connection parameters being then written

$$A_{\alpha}{}^{a}\, A_b{}^{\beta}\, A_c{}^{\gamma}\, \Gamma_{\beta\gamma}^{\alpha} = \gamma_{bc}^{a} + j_{bc}^{a}. \tag{2.15}$$

Change of local base. Another set of functions $\bar{A}_{\alpha}^{\alpha'}$ defines a new set of local base vectors, given by equations

$$\underset{b'}{\bar{e}} = \bar{A}_{b'}^{a}\, \underset{a}{e} \tag{2.16}$$

of the type of (2.4). On substitution in (2.16) for e from (2.4) we obtain

$$\underset{b'}{\bar{e}} = A_{b'}^{a}\, \underset{a}{e}, \tag{2.17}$$

where

$$A_{b'}^{a} = \bar{A}_{b'}^{\alpha}\, A_{\alpha}{}^{a}. \tag{2.18}$$

Similarly

$$A_b^{a'} = \bar{A}_a^{a'} A_b{}^a. \tag{2.19}$$

These last two equations, together with (2.13) and the analogous equation

$$\bar{A}_a^{a'} \bar{A}_{b'}^{\beta} \bar{A}_{c'}^{\gamma} \Gamma_{\beta\gamma}^{\alpha} = \bar{\gamma}_{b'c'}^{a'} - \bar{A}_a^{a'} \bar{\partial}_{c'} \bar{A}_{b'}^{a}$$

in the second local base, give the important relation

$$\partial_c A_b^{a'} = A_a^{a'} \gamma_{bc}^{a} - A_b^{b'} A_c^{c'} \bar{\gamma}_{b'c'}^{a'}, \tag{2.20}$$

relating the connections in different local bases which give the same parallelism†.

By differentiating (2.19) it is easily verified that

$$\partial_c A_b^{a'} = -A_a^{a'} j_{bc}^{a} + A_b^{b'} A_c^{c'} \bar{j}_{b'c'}^{a'}, \tag{2.21}$$

from which, by comparison with the previous equation, it is seen that the $-j_{bc}^{a}$ satisfy the equation for transformation of connection parameters. The j_{bc}^{a}, being simple functions of the $A_a{}^a$ and hence of the coordinates (x^a) of the base point P, may, therefore, be used as connection parameters to establish a local geometry in the neighbourhood of P.

3. *The non-holonomic submanifold at P.*

If P is the base point of the geometric base, its coordinates in the original mathematical base being (x^a), and Q is the point $(x^a + dx^a)$, then, in the given manifold X_n there are n degrees of freedom in the choice of Q.

If, however, the $n-m$ conditions

$$C_a{}^p \, dx^a = 0 \tag{3.1}$$

are imposed on the vectors dx^a, then Q lies in an m-space defined in the neighbourhood of P. Since n^2 functions $A_a{}^a$ are necessary to define a local base at P, in addition to the choice of the functions $C_a{}^p$, the remaining functions $B_a{}^i$ must also be defined in order to establish a local geometry.

† This is the non-holonomic equivalent of the usual equations for the change of base in non-Riemannian manifolds, namely

$$\frac{\partial^2 y^a}{\partial x^b \partial x^c} = \Gamma_{bc}^{a} \frac{\partial y^a}{\partial x^a} - \bar{\Gamma}_{\beta\gamma}^{a} \frac{\partial y^\beta}{\partial x^b} \frac{\partial y^\gamma}{\partial x^c}.$$

All vectors at P satisfying the equations

$$C_\alpha^{\ p} v^\alpha = 0 \tag{3.2}$$

will be said to lie in the "facet" at P of the non-holonomic sub-manifold defined at P. This facet may be likened to the tangent plane at a point to a two-dimensional surface. Facet components of vectors are obtained as projections on the local base of the vectors whose components in the X_n are v^α, by means of the equations

$$v^i = B_\alpha^{\ i} v^\alpha. \tag{3.3}$$

The pencil of vectors having no facet components lie in the "span" and satisfy the equations

$$B_\alpha^{\ i} v^\alpha = 0. \tag{3.4}$$

If v is any contravariant vector at P, then its projections in the local frame are given by the n equations

$$v^i = B_\alpha^{\ i} v^\alpha, \tag{3.3}$$

$$v^p = C_\alpha^{\ p} v^\alpha. \tag{3.5}$$

If these equations are solved for the v^α, they lead to

$$v^\alpha = A_a^{\ \alpha} v^a, \tag{3.6}$$

which we write in two groups separately for facet and span,

$$v^\alpha = B_i^{\ \alpha} v^i + C_p^{\ \alpha} v^p, \tag{3.7}$$

thus obtaining the reciprocal functions $B_i^{\ \alpha}$ and $C_p^{\ \alpha}$ which satisfy equations of the type of (2.6),

$$\begin{cases} B_i^{\ \alpha} B_\beta^{\ i} + C_p^{\ \alpha} C_\beta^{\ p} = \delta_\beta^{\ \alpha}, \\ B_i^{\ \alpha} B_\beta^{\ j} = \delta_i^{\ j}, \quad C_p^{\ \alpha} C_\alpha^{\ q} = \delta_p^{\ q}, \quad B_i^{\ \alpha} C_\alpha^{\ q} = 0, \quad B_\alpha^{\ i} C_p^{\ \alpha} = 0. \end{cases} \tag{3.8}$$

In defining a non-holonomic submanifold at a point P of a non-Riemannian space, equations (3.1) express the condition that a displacement from P should lie within the submanifold. These equations $C_\alpha^{\ p}\, dx^\alpha = 0$ are not, in general, integrable. If they are integrable, that is if the $C_\alpha^{\ p}$ are gradients, these equations are equivalent to differentials of equations of the type

$$f^p(x) = \text{constant}.$$

In this case the submanifold is holonomic.

For any change of geometrical base for which the facet and span bases are transformed independently, that is a change of base given by equations

$$(3.9)\qquad \begin{cases} \bar{B}^{i}_{i'} = B^{i}_{i'}\, B_{i}^{\alpha}, \quad \bar{B}^{i'}_{\alpha} = B^{i'}_{i}\, B_{\alpha}^{i}, \\ \bar{C}^{\alpha}_{p'} = C^{p}_{p'}\, C_{p}^{\ \alpha}, \quad \bar{C}^{p'}_{\alpha} = C^{p'}_{p}\, C_{\alpha}^{\ p}, \end{cases}$$

all quantities of the type $A^{i'}_{p}$, $A^{p'}_{i}$, or $A^{i}_{p'}$, $A^{p}_{i'}$ [compare equations (2.18) and (2.19)] are zero. The equations (2.20) for the transformation of connection parameters then split into eight groups:

(3.10.i) $\partial_j B^{h'}_{i} = B^{h'}_{h}\, \gamma^{h}_{ij} - B^{i'}_{i}\, B^{j'}_{j}\, \bar{\gamma}^{h'}_{i'j'}$ (arising in the calculation of the facet components after transport of a facet vector along a facet path).

(3.10.ii) $\partial_j A^{h'}_{q} = 0 = B^{h'}_{h}\, \gamma^{h}_{qj} - C^{q'}_{q}\, B^{j'}_{j}\, \bar{\gamma}^{h'}_{q'j'}$ (facet component after transport of span vector along facet path).

(3.10.iii) $\partial_j A^{p'}_{i} = 0 = C^{p'}_{p}\, \gamma^{p}_{ij} - B^{i'}_{i}\, B^{j'}_{j}\, \bar{\gamma}^{p'}_{i'j'}$ (span component after transport of facet vector along facet path).

(3.10.iv) $\partial_j C^{p'}_{q} = C^{p'}_{p}\, \gamma^{p}_{qj} - C^{q'}_{q}\, B^{j'}_{j}\, \bar{\gamma}^{p'}_{q'j'}$ (span component after transport of span vector along facet path).

(3.10.v) $\partial_r B^{h'}_{i} = B^{h'}_{h}\, \gamma^{h}_{ir} - B^{i'}_{i}\, C^{r'}_{r}\, \bar{\gamma}^{h'}_{i'r'}$ (facet component after transport of facet vector along span path).

(3.10.vi) $\partial_r A^{h'}_{q} = 0 = B^{h'}_{h}\, \gamma^{h}_{qr} - C^{q'}_{q}\, C^{r'}_{r}\, \bar{\gamma}^{h'}_{q'r'}$ (facet component after transport of span vector along span path).

(3.10.vii) $\partial_r A^{p'}_{i} = 0 = C^{p'}_{p}\, \gamma^{p}_{ir} - B^{i'}_{i}\, C^{r'}_{r}\, \bar{\gamma}^{p'}_{i'r'}$ (span component after transport of facet vector along span path).

(3.10.viii) $\partial_r C^{p'}_{q} = C^{p'}_{p}\, \gamma^{p}_{qr} - C^{q'}_{q}\, C^{r'}_{r}\, \bar{\gamma}^{p'}_{q'r'}$ (span component after transport of span vector along span path).

Similarly the equation (2.21) splits into eight groups

$$\partial_j B_i^{h'} = -B_h^{h'} j_{ij}^h + B_i^{i'} B_j^{j'} \bar{j}_{i'j'}^{h'}, \tag{3.11.i}$$

$$\partial_j A_q^{h'} = 0 = -B_h^{h'} j_{qj}^h + C_q^{q'} B_j^{j'} \bar{j}_{q'j'}^{h'}, \tag{3.11.ii}$$

$$\partial_j A_i^{p'} = 0 = -C_p^{p'} j_{ij}^p + B_i^{i'} B_j^{j'} \bar{j}_{i'j'}^{p'}, \tag{3.11.iii}$$

$$\partial_j C_q^{p'} = -C_p^{p'} j_{qj}^p + C_q^{q'} B_j^{j'} \bar{j}_{q'j'}^{p'}, \tag{3.11.iv}$$

$$\partial_r B_i^{h'} = -B_h^{h'} j_{ir}^h + B_i^{i'} C_r^{r'} \bar{j}_{i'r'}^{h'}, \tag{3.11.v}$$

$$\partial_r A_q^{h'} = 0 = -B_h^{h'} j_{qr}^h + C_q^{q'} C_r^{r'} \bar{j}_{q'r'}^{h'}, \tag{3.11.vi}$$

$$\partial_r A_i^{p'} = 0 = -C_p^{p'} j_{ir}^p + B_i^{i'} C_r^{r'} \bar{j}_{i'r'}^{p'}, \tag{3.11.vii}$$

$$\partial_r C_q^{p'} = -C_p^{p'} j_{qr}^p + C_q^{q'} C_r^{r'} \bar{j}_{q'r'}^{p'}. \tag{3.11.viii}$$

It is seen that in these transformations the $-j$ behave exactly like the connection parameters γ. Thus a facet exists at P which has $-j$ as connection parameters as well as the ordinary facet with "projected" connection parameters γ. The equations giving projected connection in each of the eight cases are of the same type, namely,

$$B_\alpha^h B_i^\beta B_j^\gamma \Gamma_{\beta\gamma}^\alpha = \gamma_{ij}^h + j_{ij}^h \quad \text{[compare (2.15)]}.$$

It is of interest to note that the connection parameters γ and $-j$ which arise in the calculation of facet (span) components after transporting span (facet) vectors along any path, numbers ii, iii, vi, vii of 3.10 and 3.11, transform as tensors under any transformation for which (3.9) are satisfied.

Infinitesimal circuits of the facet and in the submanifold. If $P(x^\alpha)$ is the base point, and if $Q(x^\alpha + \underset{1}{dx^\alpha})$, $R(x^\alpha + \underset{2}{dx^\alpha})$, $S(x^\alpha + \underset{3}{dx^\alpha})$ are points in the facet at P, that is, if

$$C_\alpha^p \underset{1}{dx^\alpha} = 0, \quad C_\alpha^p \underset{2}{dx^\alpha} = 0, \quad C_\alpha^p \underset{3}{dx^\alpha} = 0, \tag{3.12}$$

then the circuit $PQRSP$ is a circuit defined for the facet at P. If a contravariant vector λ at P is transported by parallelism to Q by means of the γ, then

$$\lambda^a(P \| Q) = \lambda^a - \gamma_{bh}^a \lambda^b \underset{1}{dx^h},$$

$$\lambda^a(Q \| R) = \lambda_{(Q)}^a - \gamma_{bh(Q)}^a \lambda_{(Q)}^b (\underset{2}{dx^h} - \underset{1}{dx^h}),$$

and, finally, the value at P after completing the circuit $PQRSP$ is

$$(\bar{\lambda}^a)_P = (\lambda^a)_P - (\partial_k \gamma^a_{bh} - \gamma^a_{eh}\gamma^e_{bk} + \gamma^a_{be} j^e_{hk})\,\lambda^b (\underset{1}{dx^h}\,\underset{2}{dx^k} - \underset{1}{dx^k}\,\underset{2}{dx^h} + \underset{2}{dx^h}\,\underset{3}{dx^k} - \underset{2}{dx^k}\,\underset{3}{dx^h}).$$

We may therefore write

$$(3.13)\quad \begin{aligned}(\bar{\lambda}^a)_P - (\lambda^a)_P &= (\partial_k \gamma^a_{bh} - \partial_h \gamma^a_{bk} + \gamma^a_{ek}\gamma^e_{bh} - \gamma^a_{eh}\gamma^e_{bk} + 2\gamma^a_{be} j^e_{[hk]})\,\lambda^b (\underset{1}{dx^h}\,\underset{2}{dx^k} + \underset{2}{dx^h}\,\underset{3}{dx^k}) \\ &= C^a_{bhk}\,\lambda^b (\underset{1}{dx^h}\,\underset{2}{dx^k} + \underset{2}{dx^h}\,\underset{3}{dx^k}),\end{aligned}$$

where C^a_{bhk} is a "facet curvature" tensor for transport of a general vector round a facet circuit at P.

If, however, ζ at P is a contravariant facet vector, that is

$$(3.14)\qquad (\zeta^p)_P = 0,$$

on transport along a facet vector from P to Q it loses its characteristic property of being a facet vector, since its span components do not, in general, remain zero. Hence, in transporting in any direction, account must be taken of the span components, and only when finally expressed in terms of quantities at P can use be made of the equation (3.14). The facet and span components at P of a facet vector after completing the circuit given at the beginning of this paragraph are, respectively,

$$(3.15)\qquad \begin{cases} (\bar{\zeta}^h)_P = \zeta^h + \zeta^i C^h_{ijk} (\underset{1}{dx^j}\,\underset{2}{dx^k} + \underset{2}{dx^j}\,\underset{3}{dx^k}), \\ (\bar{\zeta}^p)_P = \zeta^i C^p_{ijk} (\underset{1}{dx^j}\,\underset{2}{dx^k} + \underset{2}{dx^j}\,\underset{3}{dx^k}), \end{cases}$$

where the curvature tensor in the facet is

$$(3.16)\qquad C^h_{ijk} = \partial_k \gamma^h_{ij} - \partial_j \gamma^h_{ik} + \gamma^h_{ek}\gamma^e_{ij} - \gamma^h_{ej}\gamma^e_{ik} + \gamma^h_{ie} j^e_{jk} - \gamma^h_{ie} j^e_{kj}$$

and the span curvature tensor of a facet vector for a facet path is

$$(3.17)\qquad C^p_{ijk} = \partial_k \gamma^p_{ij} - \partial_j \gamma^p_{ik} + \gamma^p_{ek}\gamma^e_{ij} - \gamma^p_{ej}\gamma^e_{ik} + \gamma^p_{ie} j^e_{jk} - \gamma^p_{ie} j^e_{kj}.$$

These last two equations take into account the span components at the points Q, R, S, and at P after the circuit is completed, and in both equations a suffix runs from 1 to n in each of the last four terms. The equation is, however, intrinsic to the facet at P, in that, for any transformation of the span base which does not affect the facet, the value of C^h_{ijk} does not change. In the statement that span components exist at Q, R, S after transport, it is meant that "span components from the point of view of P" exist, where "if $\overline{PQ}$ is the vector dx at P and Q' is a point very near to Q, Q' lies in the facet at Q as viewed from P, if the vector $\overline{PQ'}$ has no span components at P".

A curvature tensor

$$(3.18) \qquad {}'C^h_{ijk} = \partial_k \gamma^h_{ij} - \partial_j \gamma^h_{ik} + \gamma^h_{lk}\gamma^l_{ij} - \gamma^h_{lj}\gamma^l_{ik} + \gamma^h_{ia} j^a_{jk} - \gamma^h_{ia} j^a_{kj}$$

is obtained by transporting a facet vector from P to Q, and then transporting the facet component of $\zeta(P \| Q)$ to R, and so on, suppressing the span components throughout the circuit, or, in other words, forming the expression for the submanifold as though the whole space X_n did not exist. This curvature tensor (3.18) is given by Dienes (2, 270). It is intrinsic to the facet, being independent of the span basé, and, like (3.16), transforms as a tensor for transformations of the facet base. Either may be taken as an expression for the curvature of the facet; I have, however, found (3.16) of greater service in practice, since its algebraic form conforms with the identity (3.23).

Both (3.16) and (3.18) are essentially "facet" curvatures as distinct from submanifold curvatures, since the path lies in the facet at P. If one could transport a vector from P to a point Q, in the facet at P, followed by a transport from Q to a point R lying in the "natural facet" at Q, defined by the $(A_a{}^\alpha)_Q$, and so complete a circuit, moving always in the natural facet at every point of the path, the tensor so obtained could be called the "submanifold curvature". Following a recent discussion of the subject with Dr. A. G. Walker, I received from him the following elegant proof of the impossibility of completing such a circuit in a non-holonomic submanifold. He has most kindly allowed me to quote his proof.

"Referring to the base $A_a{}^\alpha$, let P be the point (x) of X_n, P' be the point (x') of the path through P in direction λ^a, P'' be the point (x'') of the path through P' in direction μ^a; then there is a vector ν^a at P such that P'' is on the path defined by this direction, where the paths chosen are the 'autoparallels' for the connection giving the base vectors as absolutely parallel; that is, the

$$-\Lambda^\alpha_{\beta\gamma} = A_\beta{}^\alpha \partial_\gamma A_a{}^\alpha.$$

The components λ^a, μ^a, ν^a can be chosen so that the parametric distances PP', $P'P''$, PP'' are all 't' without restricting the circuit; then

$$x'^\alpha = x^\alpha + \lambda^\alpha t - \tfrac{1}{2}\Lambda^\alpha_{\beta\gamma}\lambda^\beta\lambda^\gamma t^2 + \ldots \qquad (\lambda^\alpha = A_a{}^\alpha \lambda^a),$$

$$x''^\alpha = x'^\alpha + \mu'^\alpha t - \tfrac{1}{2}\Lambda'^\alpha_{\beta\gamma}\mu'^\beta\mu'^\gamma t^2 + \ldots \qquad (\mu'^\alpha = A'^\alpha_a \mu^a),$$

$$x''^\alpha = x^\alpha + \nu^\alpha t - \tfrac{1}{2}\Lambda^\alpha_{\beta\gamma}\nu^\beta\nu^\gamma t^2 + \ldots \qquad (\nu^\alpha = A_a{}^\alpha \nu^a),$$

leading to

$$\nu^a = \lambda^a + \mu^a - \tfrac{1}{2}t(A_b{}^\alpha \partial_c A_\alpha{}^a - A_c{}^\alpha \partial_b A_\alpha{}^a)\lambda^b\mu^c + o(t^2).$$

If λ, μ are facet vectors, then circuits exist throughout the sub-space if ν is *always* a facet vector. Writing

$$\lambda^p = 0 = \mu^p,$$

we require $\nu^p = 0$ to be satisfied identically to order t, that is

$$(B_i^\alpha \partial_j C_\alpha^p - B_j^\alpha \partial_i C_\alpha^p)\lambda^i \mu^j = 0 \text{ for } all \text{ facet vectors } \lambda, \mu.$$

This condition can be written

$$B_i^\beta \partial_\beta B_j^\alpha - B_j^\beta \partial_\beta B_i^\alpha = K_{ij}^k B_k^\alpha, \tag{A}$$

where the K's are some functions of (x).

These conditions are necessary and sufficient for closed circuits to exist everywhere in the sub-space.

Consider now the set of linear differential equations

$$B_i^\alpha \frac{\partial f}{\partial x^\alpha} = 0,$$

that is

$$X_i f = 0,$$

where

$$X_i f \equiv B_i^\alpha \partial_\alpha f.$$

For integrability, consider the commutator operation

$$(X_i, X_j)f = X_i X_j f - X_j X_i f = (X_i A_j^\alpha - X_j A_i^\alpha)\partial_\alpha f.$$

The condition (A) above gives us

$$(X_i, X_j)f = K_{ij}^k A_k^\alpha \partial_\alpha f = K_{ij}^k X_k f;$$

that is, if

$$X_i f = 0,$$

then

$$(X_i, X_j)f = 0,$$

hence equations $X_i f = 0$ can be completely integrated to give $n-m$ functions $f^p(x)$.

Thus, if circuits exist, the vectors B_i^α are tangents to a family of m-dimensional holonomic sub-spaces and each circuit lies in a sub-space of the form

$$f^p(x) = \text{constant} \quad (p = m+1, \ldots, n)''.$$

Hence curvature of a *non-holonomic* submanifold, as distinct from its facet, cannot be defined.

The torsion of the submanifold.

Let $P(x^a)$ be the base point and let two vectors PQ and PR be chosen at P such that Q and R lie in the facet at P; then the point T may be found such that QT is parallel to PR with respect to the γ at P. If, also, RS is the vector at R parallel to PQ, then it may be shown that

$$(x^h)_T - (x^h)_R = (t^h_{ij} - t^h_{ji}) \underset{1}{u^i} \underset{2}{u^j},$$

where

$$t^h_{ij} = \gamma^h_{ij} + j^h_{ij}, \tag{3.19}$$

PQ being the vector $\underset{1}{u}$ and PR the vector $\underset{2}{u}$.

The functions t^h_{ij} are the torsion tensors of the facet, Dienes (**2**, 269). From the addition of (3.10.i) and (3.11.i) it is seen that the t^h_{ij} transform as tensors.

If ξ is any vector lying in the facet at P and we form the covariant derivatives of ξ^h with respect both to the γ^h_{ij} and the $-j^h_{ij}$, writing them, respectively, $\nabla_k \xi^h$ and $d_k \xi^h$, that is

$$\nabla_k \xi^h = \partial_k \xi^h + \gamma^h_{ik} \xi^i, \tag{3.20}$$

$$d_k \xi^h = \partial_k \xi^h - j^h_{ik} \xi^i, \tag{3.21}$$

then, on subtraction, we obtain

$$\nabla_k \xi^h - d_k \xi^h = t^h_{ik} \xi^i. \tag{3.22}$$

Thus the torsion tensors are also the coefficients in the difference between values of a vector transported by the projected connection parameters γ^h_{ik} and the $-j^h_{ik}$. If j^h_{ik} is expanded, it will be noticed that the right-hand side of (3.22),

$$t^h_{ik} \xi^i = (\gamma^h_{ik} - B_a{}^h \partial_k B_i{}^a) \xi^i,$$

is identical in form with the coefficients of rotation of Ricci [Levi-Civita (**6**)]*. This dual rôle of the t^h_{ik} as coefficients of rotation and, in the alternated form, as torsion tensors is of great interest. The idea of a coefficient of rotation suggests parallelism with respect to a congruence. From the following facts it will be seen that the introduction of the j as connection parameters is equivalent to superimposing a "pseudo-Euclidean tangent facet" on the local geometry. This behaves as a frame of reference in the same way as would a congruence defined in the neighbourhood. This

* See also Vranceanu (**8**, 124) for a discussion of the case where the γ's are zero and the j^a_{bc} reduce to coefficients of rotation.

explains the significance of the covariant derivative $d_k \xi^h$. In order to justify the description of the "psuedo-Euclidean facet", that is the facet obtained by replacing γ by $-j$, we notice that, from (3.19), the torsion is zero. By actual calculation it may readily be proved that

$$\partial_j j^h_{ik} - \partial_k j^h_{ij} = \partial_j (B_i{}^\alpha B_k{}^\beta \partial_\beta B_\alpha{}^h) - \partial_k (B_i{}^\alpha B_j{}^\beta \partial_\beta B_\alpha{}^h)$$

$$= -j^\alpha_{ij} j^h_{\alpha k} + j^\alpha_{ik} j^h_{\alpha j} - j^h_{i\alpha} j^\alpha_{kj} + j^h_{i\alpha} j^\alpha_{jk},$$

that is

$$-\partial_k j^h_{ij} + \partial_j j^h_{ik} + j^h_{\alpha k} j^\alpha_{ij} - j^h_{\alpha j} j^\alpha_{ik} - j^h_{i\alpha} j^\alpha_{jk} + j^h_{i\alpha} j^\alpha_{kj} = 0. \tag{3.23}$$

Since this is exactly the facet curvature (3.16) with the γ's replaced by $-j$'s, the facet formed with the $-j$'s has zero torsion and zero curvature. I have, therefore, called it pseudo-Euclidean, although no metric has been defined.

4. *Deformations of the submanifold.*

(a) *Local variation without displacement of base point.*

Without displacing the base from the point $P(x^\alpha)$ of the non-Riemannian space, the local split base may be modified. The equations defining such a small variation may be written

$$\begin{cases} \tilde{e}_k = e_k + \eta K_k{}^h e_h, \\ \tilde{e}_q = e_q + \eta K_q{}^p e_p, \end{cases} \tag{4.1}$$

the $K_k{}^h$ and $K_q{}^p$ being two independent sets of arbitrarily chosen functions of the (x^α) of the base point P, and η being any arbitrary small number.

Since the mathematical base vectors e^α are not altered,

$$\begin{cases} \tilde{e}_h = \tilde{B}_h{}^\alpha e_\alpha, \\ \tilde{e}_p = \tilde{C}_p{}^\alpha e_\alpha \end{cases} \tag{4.2}$$

give the functions $\tilde{B}_h{}^\alpha$ and $\tilde{C}_p{}^\alpha$ which define the split base. If

$$\tilde{B}_\alpha{}^h = B_\alpha{}^h + \eta b_\alpha{}^h \quad \text{and} \quad \tilde{C}_\alpha{}^p = C_\alpha{}^p + \eta c_\alpha{}^p, \tag{4.3}$$

then we have, at once,

$$\begin{cases} b_\alpha{}^h = K_k{}^h B_\alpha{}^k, \\ c_\alpha{}^p = K_q{}^p C_\alpha{}^q. \end{cases} \tag{4.4}$$

By making use of the equations

$$\tilde{B}_a{}^h \tilde{B}_k{}^a = \delta_k{}^h \quad \text{and} \quad \tilde{C}_a{}^p \tilde{C}_q{}^a = \delta_q{}^p, \tag{4.5}$$

we readily prove that the variation of the reciprocal functions is given by

$$\begin{cases} \tilde{B}_k{}^a = B_k{}^a - \eta K_k{}^h B_h{}^a = B_k{}^a - \eta b_k{}^a, \\ \tilde{C}_q{}^a = C_q{}^a - \eta K_q{}^p C_p{}^a = C_q{}^a - \eta c_q{}^a. \end{cases} \tag{4.6}$$

If this local variation is such that a vector v at some point Q of the facet, which is parallel to the vector w at P, transforms in this variation to a vector parallel to the transformed vector w at P, then the variation in the connection parameters must be such that

$$\tilde{\gamma}_{ij}^k = \gamma_{ij}^k - \eta(\nabla_j K_i{}^k + \gamma_{ih}^k K_j{}^h). \tag{4.7}$$

By actual differentiation of the B_i we obtain

$$\tilde{j}_{ij}^k = j_{ij}^k + \eta(d_j K_i{}^k - j_{ih}^k K_j{}^h), \tag{4.8}$$

so, by addition, the *torsion* of the submanifold after variation is

$$\tilde{t}_{ij}^k = t_{ij}^k - \eta(\nabla_j K_i{}^k - d_j K_i{}^k + t_{ih}^k K_j{}^h),$$

which is equivalent to

$$\tilde{t}_{ij}^k = t_{ij}^k - \eta(t_{hj}^k K_i{}^h - t_{ij}^h K_h{}^k + t_{ih}^k K_j{}^h). \tag{4.9}$$

This is a tensor.

The curvature of the submanifold after variation is given by

$$\tilde{C}_{ijk}^h = C_{ijk}^h + \eta \,|\, C_{ijk}^m K_m{}^h - C_{mjk}^h K_i{}^m - C_{imk}^h K_j{}^m - C_{ijm}^h K_k{}^m + (j_{kj}^b - j_{jk}^b)\,\partial_b K_i{}^h \,|,$$

which may be written

$$\tilde{C}_{ijk}^h = C_{ijk}^h + [R_{ijk}^m K_m{}^h - R_{mjk}^h K_i{}^m - C_{imk}^h K_j{}^m - C_{ijm}^h K_k{}^m + 2j_{[kj]}^b \nabla_b K_i{}^h], \tag{4.10}$$

where R_{ijk}^h is the Riemann-Christoffel tensor

$$\partial_k \gamma_{ij}^h - \partial_j \gamma_{ik}^h + \gamma_{lk}^h \gamma_{ij}^l - \gamma_{lj}^h \gamma_{ik}^l.$$

The presence of the last term of (4.10) destroys the tensor character of the expression, but, as pointed out by Dienes (**2**, 267 and 268), the C_{ijk}^h are relative structure tensors and not absolute tensors.

(*b*) *Local variation with displacement of the base point.*

The submanifold, its facet and span, may be deformed by a displacement of the split base within the non-Riemannian space. The geometrical objects associated with the split base are deformed by such a displacement. Their values at the base point of the displaced local base, $\bar{P}$ $(x^a+\epsilon\xi^a)$, may be calculated in three different ways, leading to three different variations.

(i) Any geometric object O is a function of the coordinates x^a of the point P, at which it is calculated. The "natural" value $O_{(\bar{P})}$ of the object at $\bar{P}$ may be then obtained, leading to the "natural variation".

(ii) The object O may be expressible in terms of the local base, and the facet, etc., may then be displaced point by point to a "displaced base" at $\bar{P}$. The object calculated at $\bar{P}$ will have a "displaced" (*mitgeschleppte*) value O'.

(iii) The base vectors may be transported by parallelism from P to $\bar{P}$. Any geometric object expressible in terms of the local base will then be said to have a "geodesic value" $O(P||\bar{P})$.

These lead to the three variation measures

$$(4.11)\qquad \begin{cases} \text{(i) The "natural" measure } O-O(\bar{P}), \\ \text{(ii) The "absolute" measure } O-O', \\ \text{(iii) The "geodesic" measure } O-O(P||\bar{P}). \end{cases}$$

These three variations are not, in general, invariant in themselves, but from them may be calculated three distinct differentials which are invariant.

They are, respectively:

$$(4.12)\qquad \begin{cases} \text{(i) The "Lie differential" } D'O=O(\bar{P})-O'. \\ \text{(ii) The "apparent" (*scheinbare*) differential} \\ \qquad\qquad \tilde{D}O=O(P||\bar{P})-O', \\ \text{(iii) The "covariant differential" } DO=O(\bar{P})-O(P||\bar{P}). \end{cases}$$

In determining these differentials we must first calculate the "natural", "displaced", and "transported" local bases. In each case the base vectors of the new facet and span $\overset{n}{v}{}^h, \overset{a}{v}{}^h, \overset{g}{v}{}^h$ are calculated, and from the condition

which we impose, that a facet vector when deformed shall be a vector of the new facet, we obtain the $\bar{B}_\alpha{}^h$, $\bar{B}_k{}^\alpha$; $\bar{C}_\alpha{}^p$, $\bar{C}_q{}^\beta$ from which the j^h_{ik}, etc., of the deformed space are directly obtained. The γ^h_{ik} of the deformed base are found in each case by giving them the values for which a vector w at Q is deformed to a vector $\overline{w}$ at $(\bar{Q})$ such that, if w is parallel to a vector v at P with respect to the γ^h_{ij}, then $\overline{w}$ is parallel to $\bar{v}$ at $\bar{P}$ with respect to the $\overline{\gamma}^h_{ij}$. The torsion and curvature of the deformed facet are obtained by making use of the $\overline{j}^h_{ij}$ and $\overline{\gamma}^h_{ij}$ in completing the circuits of the deformed submanifold. Considering $v^\alpha(Q||P)$ as a geometrical object defined at P, we can calculate the "transported connection parameters at $\bar{P}$", giving the three complete geometries at $\bar{P}$.

In the last paragraph $\bar{v}$, $\bar{j}$, etc., have been used to describe any of the three deforms $\overset{n}{v}, \overset{a}{v}, \overset{g}{v}$; $\overset{n}{j}, \overset{a}{j}, \overset{g}{j}$; etc.

(i) *The "natural" geometry at* $\bar{P}$ $(x^\alpha+\epsilon\xi^\alpha)$.

$$(4.13.\text{i}) \qquad \overset{(n)}{B}_\alpha{}^h = B_\alpha{}^h+\epsilon\,\partial_c B_\alpha{}^h\,\xi^c = B_\alpha{}^h+\epsilon B_\alpha{}^k j^h_{kc}\,\xi^c,$$

$$(4.13.\text{ii}) \qquad \overset{(n)}{C}_\alpha{}^p = C_\alpha{}^p+\epsilon\,\partial_c C_\alpha{}^p\,\xi^c = C_\alpha{}^p+\epsilon\, C_\alpha{}^q j^p_{qc}\,\xi^c.$$

$$(4.13.\text{iii}) \qquad \overset{(n)}{B}_h{}^\alpha = B_h{}^\alpha+\epsilon\,\partial_c B_h{}^\alpha\,\xi^c = B_h{}^\alpha-\epsilon\, B_k{}^\alpha j^k_{hc}\,\xi^c.$$

$$(4.13.\text{iv}) \qquad \overset{(n)}{C}_p{}^\alpha = C_p{}^\alpha+\epsilon\,\partial_c C_p{}^\alpha\,\xi^c = C_p{}^\alpha-\epsilon\, C_q{}^\alpha j^q_{pc}\,\xi^c.$$

$$(4.13.\text{v}) \qquad \overset{(n)}{v}{}^h = v^h+\epsilon\,\partial_c v^h\,\xi^c = v^h+\nabla_c v^h\,\xi^c-\epsilon\,\gamma^h_{kc} v^k\,\xi^c.$$

$$(4.13.\text{vi}) \qquad \overset{(n)}{v}_h = v_h+\epsilon\,\partial_c v_h\,\xi^c = v_h+\nabla_c v_h\,\xi^c+\epsilon\gamma^k_{hc} v_k\,\xi^c.$$

For parallelism to be preserved, the values of the connection parameters in the "natural" base are the "natural" values:

$$(4.13.\text{vii}) \qquad \overset{(n)}{\gamma}{}^h_{ij} = \gamma^h_{ij}+\epsilon\,\partial_c\gamma^h_{ij}\,\xi^c.$$

$$(4.13.\text{viii}) \qquad \overset{(n)}{j}{}^h_{ij} = j^h_{ij}+\epsilon\,\partial_c j^h_{ij}\,\xi^c$$

(this follows directly from equations i to iv above).

$$(4.13.\text{ix}) \qquad \overset{(n)}{t}{}^h_{ij} = t^h_{ij}+\epsilon(\nabla_c t^h_{ij}-\gamma^h_{bc} t^b_{ij}+\gamma^b_{ic} t^h_{bj}+\gamma^b_{jc} t^h_{ib})\,\xi^c.$$

$$(4.13.\text{x}) \qquad \overset{(n)}{C}{}^h_{ijk} = C^h_{ijk}+\epsilon(\nabla_c C^h_{ijk}-\gamma^h_{bc} C^b_{ijk}+\gamma^b_{ic} C^h_{bjk}+\gamma^b_{jc} C^h_{ibk}+\gamma^b_{kc} C^h_{ijb})\,\xi^c.$$

The last two of these equations are the natural values, at $\overline{P}$, of the curvature and torsion tensors, but they may also be obtained by actual completion of circuits in the "natural" base.

(ii) *The "displaced" (mitgeschleppte) geometry at* $\overline{P}$ $(x^a+\epsilon\xi^a)$.

If the *mitgeschleppten* values are calculated such that the displacement is relative to the "local geometric" base, leaving the original "mathematical" base not deformed, we obtain the following "displaced values":

(4.14.i) $$B'^h_a = B^h_a + \epsilon A^a_a \partial_a \xi^h.$$

(4.14.ii) $$B'^a_k = B^a_k - \epsilon A^a_b \partial_k \xi^b.$$

(4.14.iii) $$v'^h = v^h + \epsilon v^a \partial_a \xi^h.$$

(4.14.iv) $$v'_k = v_k - \epsilon v_b \partial_k \xi^b.$$

(4.14.v) $$\gamma'^h_{ij} = \gamma^h_{ij} - \epsilon(\partial_i\partial_j\xi^h + \gamma^h_{aj}\partial_i\xi^a + \gamma^h_{ia}\partial_j\xi^a - \gamma^a_{ij}\partial_a\xi^h)$$
$$= \gamma^h_{ij} - \epsilon\left(\overset{*}{\nabla}_j\overset{*}{\nabla}_i\xi^h + \gamma^h_{[aj]}\overset{*}{\nabla}_i\xi^a - \gamma^a_{[ij]}\overset{*}{\nabla}_a\xi^h + R^h_{ijb}\xi^b - \partial_b\gamma^h_{ij}\xi^b\right).$$

(4.14.vi) $$j'^h_{ij} = j^h_{ij} + \epsilon\left(\overset{*}{d}_j\overset{*}{d}_i\xi^h - j^h_{[aj]}d_i\xi^a + j^a_{[ij]}d_a\xi^h + j^h_{ia}j^a_{[jb]}\xi^b + \partial_b j^h_{ij}\xi^b\right).$$

(4.14.vii) $$t'^h_{ij} = t^h_{ij} - \epsilon(t^h_{aj}\partial_i\xi^a + t^h_{ia}\partial_j\xi^a - t^a_{ij}\partial_a\xi^h).$$

(4.14.viii) $$C'^h_{ijk} = C^h_{ijk} - \epsilon(C^h_{ajk}\partial_i\xi^a + C^h_{iak}\partial_j\xi^a + C^h_{ija}\partial_k\xi^a - C^a_{ijk}\partial_a\xi^h).$$

(iii) *The "transported geometry" at* $\overline{P}$ $(x^a+\epsilon\xi^a)$.

(4.15.i) $$\overset{(g)}{B}{}^h_a(\overline{P}) = B^h_a - \epsilon\gamma^h_{ac}A^a_a\xi^c.$$

(4.15.ii) $$\overset{(g)}{C}{}^p_a(\overline{P}) = C^p_a - \epsilon\gamma^p_{ac}A^a_a\xi^c.$$

(4.15.iii) $$\overset{(g)}{B}{}^a_k(\overline{P}) = B^a_k + \epsilon\gamma^a_{kc}A^a_a\xi^c.$$

(4.15.iv) $$\overset{(g)}{C}{}^a_p(\overline{P}) = C^a_p + \epsilon\gamma^a_{pc}A^a_a\xi^c.$$

(4.15.v) $$\overset{(g)}{v}{}^h(\overline{P}) = v^h - \epsilon\gamma^h_{ac}v^a\xi^c.$$

(4.15.vi) $$\overset{(g)}{v}{}^p(\overline{P}) = v^p - \epsilon\gamma^p_{ac}v^a\xi^c.$$

(4.15.vii) $$\overset{(g)}{v}_k(\overline{P}) = v_k + \epsilon\gamma^a_{kc}v_a\xi^c.$$

(4.15.viii) $$\overset{(g)}{v}_q(\overline{P}) = v_q + \epsilon\gamma^a_{qc}v_a\xi^c.$$

For parallelism to be preserved in the deformation, the following values must be given to the local connections:

$$(4.15.\text{ix}) \qquad \overset{(g)}{\gamma}{}^h_{ij}(\overline{P}) = \gamma^h_{ij} - \epsilon(\gamma^h_{ab}\gamma^a_{ij} - \gamma^a_{ib}\gamma^h_{aj} + \partial_j \gamma^h_{ib})\,\xi^b.$$

$$(4.15.\text{x}) \qquad \overset{(g)}{j}{}^h_{ij}(\overline{P}) = j^h_{ij} - \epsilon(\gamma^h_{ab} j^a_{ij} - \gamma^a_{ib} j^h_{aj} - \gamma^a_{jb} t^h_{ia} - \partial_j \gamma^h_{ib})\,\xi^b.$$

$$(4.15.\text{xi}) \qquad \overset{(g)}{t}{}^h_{ij}(\overline{P}) = t^h_{ij} - \epsilon(\gamma^h_{ab} t^a_{ij} - \gamma^a_{ib} t^h_{aj} - \gamma^a_{jb} t^h_{ia})\,\xi^b.$$

$$(4.15.\text{xii}) \qquad \overset{(g)}{C}{}^h_{ijk}(\overline{P}) = C^h_{ijk} - \epsilon(\gamma^h_{ab} C^a_{ijk} - \gamma^a_{ib} C^h_{ajk} - \gamma^a_{jb} C^h_{iak} - \gamma^a_{kb} C^h_{ija})\,\xi^b.$$

It should be noted that numbers (v)-(viii) above give the components in the transported facet and span of general vectors at P, which have both facet and span components in the original base. Components of facet vectors at P are obtained by replacing a by "i" in the second terms.

The "Lie differential" $D'O = \overset{(a)}{D}O - \overset{(n)}{D}O = O(\overline{P}) - O'$.

$$(4.16.\text{i}) \qquad D' B_a{}^h = -\epsilon A_a{}^a \overset{*}{d}_a \xi^h.$$

$$(4.16.\text{ii}) \qquad D' B_k{}^a = +\epsilon A_b{}^a d_k \xi^b.$$

$$(4.16.\text{iii}) \qquad D' v^h = \epsilon(\partial_a v^h \xi^a - v^a \partial_a \xi^h) = \epsilon(\nabla_a v^h \xi^a - v^a \overset{*}{\nabla}_a \xi^h)\dagger.$$

$$(4.16.\text{iv}) \qquad D' v_k = \epsilon(\partial_a v_k \xi^a + v_a \partial_k \xi^a) = \epsilon(\nabla_a v_k \xi^a + v_a \overset{*}{\nabla}_k \xi^a)\dagger.$$

$$(4.16.\text{v}) \qquad D' \gamma^h_{ij} = \epsilon(\overset{*}{\nabla}_j \overset{*}{\nabla}_i \xi^h + \gamma^h_{[aj]} \overset{*}{\nabla}_i \xi^a - \gamma^a_{[ij]} \overset{*}{\nabla}_a \xi^h + R^h_{ijb} \xi^b).$$

$$(4.16.\text{vi}) \qquad D' j^h_{ij} = -\epsilon(\overset{*}{d}_j \overset{*}{d}_i \xi^h - j^h_{[aj]} \overset{*}{d}_i \xi^a + j^a_{[ij]} d_a \xi^h + j^h_{ia} j^a_{[jb]} \xi^b).$$

$$(4.16.\text{vii}) \qquad D' t^h_{ij} = \epsilon(\nabla_a t^h_{ij} \xi^a - t^a_{ij} \overset{*}{\nabla}_a \xi^h + t^h_{aj} \overset{*}{\nabla}_i \xi^a + t^h_{ia} \overset{*}{\nabla}_j \xi^a).$$

$$(4.16.\text{viii}) \qquad D' C^h_{ijk} = \epsilon(\nabla_a C^h_{ijk} \xi^a - C^a_{ijk} \overset{*}{\nabla}_a \xi^h + C^h_{ajk} \overset{*}{\nabla}_i \xi^a + C^h_{iak} \overset{*}{\nabla}_j \xi^a + C^h_{ija} \overset{*}{\nabla}_k \xi^a).$$

† *Cf.* Dienes (3), equation 7, and Schouten and van Kampen (1), equation (1.6).

The apparent (scheinbare) differential $\tilde{D}O = \overset{(a)}{D}O - \overset{(g)}{D}O = O(P||\overline{P}) - O'$.

(4.17.i) $$\tilde{D}B_a^{\ h} = -\epsilon A_a^{\ a} \overset{*}{\nabla}_a \xi^h.$$

(4.17.ii) $$\tilde{D}B_k^{\ a} = +\epsilon A_b^{\ a} \overset{*}{\nabla}_k \xi^b.$$

(4.17.iii) $$\tilde{D}v^h = -\epsilon v^a \overset{*}{\nabla}_a \xi^h.$$

(4.17.iv) $$\tilde{D}v_k = +\epsilon v_b \overset{*}{\nabla}_k \xi^b.$$

(4.17.v) $$\tilde{D}\gamma_{ij}^h = \epsilon(\overset{*}{\nabla}_j \overset{*}{\nabla}_i \xi^h + \gamma_{[aj]}^h \overset{*}{\nabla}_i \xi^a - \gamma_{[ij]}^a \overset{*}{\nabla}_a \xi^h).$$

(4.17.vi) $$\tilde{D}j_{ij}^h = -\epsilon(\overset{*}{d}_j \overset{*}{d}_i \xi^h - j_{[aj]}^h \overset{*}{d}_i \xi^a + j_{[ij]}^a d_a \xi^h + t_{ia}^h j_{[jc]}^a \xi^c - C_{ijc}^h \xi^c + \nabla_c t_{ij}^h \xi^c).$$

(4.17.vii) $$\tilde{D}t_{ij}^h = \epsilon(t_{aj}^h \overset{*}{\nabla}_i \xi^a + t_{ia}^h \overset{*}{\nabla}_j \xi^a - t_{ij}^a \overset{*}{\nabla}_a \xi^h).$$

(4.17.viii) $$\tilde{D}C_{ijk}^h = \epsilon(C_{ajk}^h \overset{*}{\nabla}_i \xi^a + C_{iak}^h \overset{*}{\nabla}_j \xi^a + C_{ija}^h \overset{*}{\nabla}_k \xi^a - C_{ijk}^a \overset{*}{\nabla}_a \xi^h).$$

The covariant differential $DO = \overset{(g)}{D}O - \overset{(n)}{D}O = O(\overline{P}) - O(P||\overline{P})$.

(4.18.i) $$DB_a^{\ h} = \epsilon \nabla_c B_{(a)}^h \xi^c.$$

(4.18.ii) $$DB_k^{\ a} = \epsilon \nabla_c B_k^{(a)} \xi^c.$$

(4.18.iii) $$Dv^h = \epsilon \nabla_c v^h \xi^c.$$

(4.18.iv) $$Dv_k = \epsilon \nabla_c v_k \xi^c.$$

(4.18.v) $$D\gamma_{ij}^h = \epsilon R_{ijc}^h \xi^c.$$

(4.18.vi) $$Dj_{ij}^h = \epsilon(\nabla_c t_{ij}^h - R_{ijc}^h) \xi^c.$$

(4.18.vii) $$Dt_{ij}^h = \epsilon \nabla_c t_{ij}^h \xi^c.$$

(4.18.viii) $$DC_{ijk}^h = \epsilon \nabla_c C_{ijk}^h \xi^c.$$

In numbers (v) and (vi), R_{ijc}^h is the Riemann-Christoffel tensor

$$\partial_c \gamma_{ij}^h - \partial_j \gamma_{ic}^h + \gamma_{bc}^h \gamma_{ij}^b - \gamma_{bj}^h \gamma_{ic}^b.$$

5. In paragraph 3 stress was laid on the geometry of the submanifold and its facet, which is dependent on the local connection parameters. A simple deformation of the base, as given by Schouten and van Kampen (1),

gives the magnitude and nature of vectors at the point $\bar{P}$, and from this it is possible to calculate a form for the deformation of tensors. In actual problems one requires, however, to deform the submanifold and not merely the base, and to know the effect of a particular deformation on certain of its essential properties, such as torsion and curvature, which are dependent on the connection. By regarding a vector at Q, transported by parallelism to the base point P, as a geometrical object defined at P and thus applying it to the deformation, the values (4.13.vii), (4.14.v), and (4.15.ix) for γ_{ij}^{h} at $\bar{P}$ may be obtained†. The corresponding values for j_{ij}^{h} are obtained by differentiating the $B_h{}^{\alpha}$'s at $\bar{P}$ and thus the whole geometry may be built up, and the Lie and apparent (*scheinbare*) differentials of the torsion and curvature tensors follow logically. These are typical tensors and the forms of their deformations serve also as an indication of the deformations of other tensors.

The forms (4.18.v and vi) are not covariant derivatives themselves, since the γ's and j's are not tensors, but they cannot logically be excluded if the torsion and curvature at $\bar{P}$ are calculated. I have, therefore, included them under the same heading as the covariant derivatives of other quantities.

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† Compare the treatment of the X_n by Dienes (4), 514.

THE ABSOLUTE CESÀRO SUMMABILITY OF A FOURIER SERIES

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1. *Introduction.*

The object of this paper is to prove some general theorems concerning the absolute Cesàro summability of a Fourier series analogous to known results concerning its Cesàro summability. The summability theorems were introduced by Hardy and Littlewood*, and extended by other writers†. They set up a connection between the summability of the Fourier series at a point and the limit of a generalised mean of the function. The absolute summability theorems set up a similar connection between the absolute summability of the Fourier series and the total variation of the generalised mean of the function. The simplest cases of these theorems have been given elsewhere‡, and the theorems of this paper have also been discussed in a preliminary account§.

We suppose throughout that $f(t)$ is a periodic function, with period 2π, integrable in the Lebesgue sense. We write

$$\phi(t) = \tfrac{1}{2}\{f(x+t)+f(x-t)\},$$

and, for $t > 0$,

$$\Phi_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \phi(u)\, du \quad (\alpha > 0),$$

$$\Phi_0(t) = \phi(t),$$

$$\phi_\alpha^!(t) = \Gamma(\alpha+1)\, t^{-\alpha} \Phi_\alpha(t) \qquad (\alpha \geqslant 0).$$

Then $\phi_\alpha(t)$ is the mean value, in a certain sense, of order α of $\phi(t)$.

* Hardy and Littlewood, **7**, **9**.

† For references see Bosanquet, **4**.

‡ Bosanquet, **5**.

§ Bosanquet, **2**.

The Fourier series of $f(t)$ at $t=x$ is the same as that of $\phi(t)$ at $t=0$. Accordingly we write

$$\phi(t) \sim \sum_0^\infty a_n \cos nt,$$

and we suppose, as we may without loss of generality, that $a_0 = 0$.

We also write s_n^α and τ_n^α for the n-th Cesàro means of order α of $s_n = a_0 + \ldots + a_n$ and $\tau_n = na_n$ respectively*.

If the series $\Sigma |s_n^\alpha - s_{n-1}^\alpha|$ is convergent, the series Σa_n is said to be *absolutely summable* (C, α), or *summable* $|C, \alpha|$.

If $F(x) = \Sigma a_n x^n$ converges for $|x|<1$ and $F(x)$ is of bounded variation in $(0, 1)$, the series Σa_n is said to be *absolutely summable* (A) or *summable* $|A|$.

We have the following lemmas of consistency.

LEMMA 1†. *If $\alpha \geqslant 0$ and $\phi_\alpha(t)$ is of bounded variation in an interval $(0, \eta)$, then the same is true if α is replaced by $\beta > \alpha$.*

More precisely, for $\beta > \alpha \geqslant 0$,

$$\int_0^\eta |d\phi_\alpha(u)| \geqslant \int_0^\eta |d\phi_\beta(u)| .$$

LEMMA 2‡. *If $\alpha \geqslant 0$ and Σa_n is summable $|C, \alpha|$, then it is summable $|C, \beta|$, for $\beta > \alpha$, and summable $|A|$.*

More precisely§, for $\beta > \alpha \geqslant 0$,

$$\Sigma |s_n^\alpha - s_{n-1}^\alpha| \geqslant \Sigma |s_n^\beta - s_{n-1}^\beta| \geqslant \int_0^1 |F'(x)|\, dx.$$

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* Thus, for $\alpha > -1$,
$$s_n^\alpha = \frac{S_n^\alpha}{A_n^\alpha} = \frac{1}{A_n^\alpha} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} s_\nu \quad (s_{-1}^\alpha = 0),$$
where
$$\sum_0^\infty A_n^\alpha x^n = (1-x)^{-\alpha-1} \quad \text{for} \quad |x|<1.$$

† Bosanquet, 3.

‡ Kogbetliantz, 11, Fekete, 6.

§ The first inequality was given by Kogbetliantz. The second was given by Fekete in the case when β is an integer. It may be proved as follows:

$$\int_0^1 |F'(x)|\, dx = \int_0^1 |\Sigma \tau_n x^{n-1}|\, dx = \int_0^1 |(1-x)^\beta \Sigma T_n^\beta x^{n-1}|\, dx$$
$$\leqslant \int_0^1 (1-x)^\beta \Sigma |T_n^\beta| x^{n-1} = \Sigma |T_n^\beta| \int_0^1 (1-x)^\beta x^{n-1}\, dx = \Sigma n^{-1} |\tau_n^\beta|$$
$$= \Sigma |s_n^\beta - s_{n-1}^\beta|.$$

The summability $|A|$ of a Fourier series was first investigated by J. M. Whittaker*, and his results have been extended by B. N. Prasad† and the present writer‡.

In this paper we prove the following theorems.

THEOREM 1. *If $\phi_\alpha(t)$ is of bounded variation in $(0, \pi)$, then the Fourier series of $f(t)$ is summable $|C, \beta|$ at the point $t = x$, where $\beta > \alpha \geqslant 0$.*

THEOREM 2. *If the Fourier series of $f(t)$ is summable $|C, \alpha|$ at the point $t = x$, then $\phi_\beta(t)$ is of bounded variation in $(0, \pi)$, where $\beta > \alpha + 1$ $(\alpha \geqslant 0)$.*

These theorems have been proved elsewhere in the case $\alpha = 0$, and it has also been shown in that case that the results become false when $\beta = \alpha$ in the first theorem and when $\beta = \alpha + 1$ in the second§.

Since a Lebesgue integral is absolutely continuous, it is plain that, for $\alpha \geqslant 1$, $\phi_\alpha(t)$ is of bounded variation in any range (η, π), $\eta > 0$. Therefore, the summability $|C, \beta|$ of a Fourier series, for $\beta > 1$, depends only on the behaviour of the function in the neighbourhood of the point under consideration. On the other hand, it is known that the absolute convergence, or summability $|C, 0|$, of a Fourier series is not a local property, and in fact depends on the behaviour of the function throughout the whole interval of definition. The most interesting problem that arises out of this paper is the question whether summability $|C, \beta|$, for $0 < \beta \leqslant 1$, is or is not a local property of the function. This problem is not yet solved.

2. In order to simplify the proofs of the theorems, it is convenient first to give some inequalities satisfied by certain special functions.

We write $g^\alpha(n, t)$ for the n-th Cesàro mean of order α of $2\pi^{-1} \sin nt$. Then we have, for $\sigma > 0$, $0 < t < \pi$, $n > 0$,

$$\left| \left(\frac{d}{dt}\right)^\lambda g^\sigma(n, t) \right| \begin{cases} \leqslant An^\lambda & (\lambda \geqslant 0), \\ \leqslant An^{-1}t^{-\lambda-1} & (0 \leqslant \lambda \leqslant \sigma - 1), \\ \leqslant An^{\lambda-\sigma}t^{-\sigma} & (\lambda > \sigma - 1). \end{cases} \quad (1)$$

These results have been given by Obrechkoff||, and may also be obtained by arguments given by Zygmund¶.

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* Whittaker, 15.
† Prasad, 13, 14.
‡ Bosanquet, 3.
§ Bosanquet, 5.
|| Obrechkoff, 12.
¶ Zygmund, 16.

We also write, for $\alpha > 0$,

$$\gamma_\alpha(x) = \int_0^1 (1-u)^{\alpha-1} \cos xu \, du.$$

Then, for $\sigma > 0$, $x > 0$,

$$|\gamma_\sigma^{(\lambda)}(x)| \begin{cases} \leqslant A & (\lambda \geqslant 0), \\ \leqslant Ax^{-\lambda-2} & (0 \leqslant \lambda \leqslant \sigma-2), \\ \leqslant Ax^{-\sigma} & (\lambda > \sigma-2). \end{cases} \tag{2}$$

These results may be obtained from an asymptotic formula due to Hardy*, and may also be obtained by arguments given by Hobson†.

Writing $\Delta f(n) = f(n) - f(n+1)$, we obtain‡, from the identity

$$\Delta^p f(n) = \int_{n+1}^{n} dx_1 \int_{x_1+1}^{x_1} dx_2 \ldots \int_{x_{p-1}+1}^{x_{p-1}} \left(\frac{d}{dx_p}\right)^p f(x_p)\, dx_p,$$

for $\sigma > 0$, $t > 0$, $n > 0$,

$$\left| \Delta^p \left(\frac{d}{dt}\right)^\lambda \gamma_\sigma(nt) \right| \begin{cases} \leqslant An^\lambda t^p & (\lambda \geqslant 0,\ p \geqslant 0), \\ \leqslant An^{-p-2} t^{-\lambda-2} & (p+\lambda \leqslant \sigma-2), \\ \leqslant An^{\lambda-\sigma} t^{p-\sigma} & (p+\lambda > \sigma-2). \end{cases} \tag{3}$$

3. *Proof of Theorem* 1. There is no loss of generality in supposing that $0 < \alpha < \beta < h+1$, where h is the greatest integer not greater than α. Since $s_n{}^\beta - s_{n-1}^\beta = n^{-1}\tau_n{}^\beta$§, we have to show that $\Sigma n^{-1}|\tau_n{}^\beta|$ converges, and, from

$$na_n = \frac{2}{\pi} \int_0^\pi \phi(t) \frac{d}{dt} \sin nt \, dt,$$

we have

$$\tau_n{}^\beta = \int_0^\pi \phi(t) \frac{d}{dt} g^\beta(n, t)\, dt.$$

Now, on integration by parts,

$$\int_0^\pi \phi(t) \frac{d}{dt} g^\beta(n, t)\, dt = \left[\sum_{\rho=1}^{h} (-1)^{\rho-1} \Phi_\rho(t) \left(\frac{d}{dt}\right)^\rho g^\beta(n, t) \right]_0^\pi + (-1)^h \int_0^\pi \Phi_h(t) \left(\frac{d}{dt}\right)^{h+1} g^\beta(n, t)\, dt.$$

* *Cf.* Hardy and Littlewood, 8.

† Hobson, 10, 564–567.

‡ *Cf.* Bosanquet, 1.

§ Kogbetliantz, 11.

Also

$$\int_0^\pi \Phi_h(t)\left(\frac{d}{dt}\right)^{h+1} g^\beta(n,t)\,dt$$

$$= \frac{1}{\Gamma(1+h-\alpha)}\int_0^\pi \left(\frac{d}{dt}\right)^{h+1} g^\beta(n,t)\int_0^t (t-u)^{h-\alpha}\,d\Phi_\alpha(u)$$

$$= \frac{1}{\Gamma(1+h-\alpha)}\int_0^\pi d\Phi_\alpha(u)\int_u^\pi (t-u)^{h-\alpha}\left(\frac{d}{dt}\right)^{h+1} g^\beta(n,t)\,dt$$

$$= \int_0^\pi J(n,u)\,d\Phi_\alpha(u)$$

$$= \Big[\Phi_\alpha(u)\,J(n,u)\Big]_0^\pi - \int_0^\pi \Phi_\alpha(u)\,\frac{d}{du}\,J(n,u)\,du,$$

where

$$J(n,u) = \frac{1}{\Gamma(1+h-\alpha)}\int_u^\pi (t-u)^{h-\alpha}\left(\frac{d}{dt}\right)^{h+1} g^\beta(n,t)\,dt.$$

Further, since $\phi_\alpha(+0)$ is finite,

$$\int_0^\pi \Phi_\alpha(u)\,\frac{d}{du}\,J(n,u)\,du$$

$$= \frac{1}{\Gamma(\alpha+1)}\int_0^\pi \phi_\alpha(u)\,u^\alpha\,\frac{d}{du}\,J(n,u)\,du$$

$$= \frac{1}{\Gamma(\alpha+1)}\Big[\phi_\alpha(u)\int_0^u v^\alpha\,\frac{d}{dv}\,J(n,v)\,dv\Big]_0^\pi$$

$$- \frac{1}{\Gamma(\alpha+1)}\int_0^\pi d\phi_\alpha(u)\int_0^u v^\alpha\,\frac{d}{dv}\,J(n,v)\,dv$$

$$= \phi_\alpha(\pi)\,V(n,\pi) - \int_0^\pi V(n,u)\,d\phi_\alpha(u),$$

where

$$V(n,u) = \frac{1}{\Gamma(\alpha+1)}\int_0^u v^\alpha\,\frac{d}{dv}\,J(n,v)\,dv.$$

Collecting our results, we have

$$\tau_n^\beta = \Big[\sum_{\rho=1}^{h}(-1)^{\rho-1}\Phi_\rho(t)\left(\frac{d}{dt}\right)^\rho g^\beta(n,t)\Big]_0^\pi + (-1)^h\Big[\Phi_\alpha(u)\,J(n,u)\Big]_0^\pi$$

$$+ (-1)^{h+1}\phi_\alpha(\pi)\,V(n,\pi) + (-1)^h\int_0^\pi V(n,u)\,d\phi_\alpha(u).$$

Assuming, for the moment, that, for $0 < u < \pi$,

$$|J(n, u)| \begin{cases} \leqslant An^{\alpha}, \\ \leqslant A\, n^{\alpha-\beta} u^{-\beta}, \end{cases} \tag{4}$$

we have, from (1) and (4),

$$\tau_n{}^{\beta} = \sum_{\rho=1}^{h-1} O(n^{-1}) + O(n^{h-\beta}) + O(n^{\alpha-\beta}) + (-1)^{h+1} \phi_{\alpha}(\pi)\, V(n, \pi) + (-1)^{h} \int_0^{\pi} V(n, u)\, d\phi_{\alpha}(u). \tag{5}$$

If, in particular, we suppose that $\phi(t) = 1$ for all t, in which case $\phi_{\alpha}(t) = 1$ for all t and $\tau_n{}^{\beta} = 0$ for every n, we obtain

$$0 = O(n^{\alpha-\beta}) + (-1)^{h+1} V(n, \pi). \tag{6}$$

It follows from (5) and (6) that

$$\tau_n{}^{\beta} = O(n^{\alpha-\beta}) + (-1)^{h} \int_0^{\pi} V(n, u)\, d\phi_{\alpha}(u). \tag{7}$$

We now have

$$\Sigma n^{-1} |\tau_n{}^{\beta}| \leqslant A \Sigma n^{-1+\alpha-\beta} + A \Sigma n^{-1} \left| \int_0^{\pi} V(n, u)\, d\phi_{\alpha}(u) \right|$$

$$\leqslant A + A \Sigma n^{-1} \int_0^{\pi} |V(n, u)|\, |d\phi_{\alpha}(u)|$$

$$= A + A \int_0^{\pi} |d\phi_{\alpha}(u)|\, \Sigma n^{-1} |V(n, u)|.$$

If then we prove that, for $0 < u < \pi$,

$$|V(n, u)| \begin{cases} \leqslant An^{\alpha} u^{\alpha}, \\ \leqslant An^{\alpha-\beta} u^{\alpha-\beta}, \end{cases} \tag{8}$$

we shall have

$$\Sigma n^{-1} |V(n, u)| = \sum_{n \leqslant u^{-1}} n^{-1} O(n^{\alpha} u^{\alpha}) + \sum_{n > u^{-1}} n^{-1} O(n^{\alpha-\beta} u^{\alpha-\beta})$$

$$= O(1) + O(1),$$

uniformly in u, since $\beta > \alpha > 0$. Finally, since $\int_0^{\pi} |d\phi_{\alpha}(u)|$ is finite, it will follow that $\Sigma n^{-1} |\tau_n{}^{\beta}|$ converges.

We now establish the inequalities (4). To prove these we write, for $u+n^{-1}<\pi$*,

$$J(n, u)=\int_u^{u+n^{-1}}+\int_{u+n^{-1}}^{\pi}=K_1+K_2.$$

Then, since, for $\sigma<\lambda+1$,

$$\left(\frac{d}{dt}\right)^{\lambda} g^{\sigma}(n, t)=O(n^{\lambda})\min\{1, (nt)^{-\sigma}\},$$

we have

$$\begin{aligned}|K_1| &\leqslant An^{h+1}\int_u^{u+n^{-1}}(t-u)^{h-\alpha}\min\{1, (nt)^{-\beta}\}dt\\ &\leqslant An^{h+1}\min\{1, (nu)^{-\beta}\}\int_n^{u+n^{-1}}(t-u)^{h-\alpha}dt\\ &\leqslant An^{h+1}\min\{1, (nu)^{-\beta}\}O(n^{-1-h+\alpha})\\ &=O(n^{\alpha})\min\{1, (nu)^{-\beta}\},\end{aligned}$$

while

$$\begin{aligned}K_2 &=\left(\frac{1}{n}\right)^{h-\alpha}\int_{u+n^{-1}}^{\zeta}\left(\frac{d}{dt}\right)^{h+1}g^{\beta}(n, t)\,dt\dagger\\ &=O(n^{\alpha-h})\,O(n^{h})\min\{1, (nu)^{-\beta}\}\\ &=O(n^{\alpha})\min\{1, (nu)^{-\beta}\}.\end{aligned}$$

Finally, in order to prove (8), we have, on the one hand,

$$\begin{aligned}\Gamma(\alpha+1)\,V(n, u) &=u^{\alpha}J(n, u)-\alpha\int_0^u v^{\alpha-1}J(n, v)\,dv\\ &=O(n^{\alpha}u^{\alpha})+\int_0^u v^{\alpha-1}O(n^{\alpha})\,dv\\ &=O(n^{\alpha}u^{\alpha}).\end{aligned}$$

On the other hand,

$$\Gamma(\alpha+1)\{V(n, \pi)-V(n, u)\}=\Big[v^{\alpha}J(n, v)\Big]_u^{\pi}-\alpha\int_u^{\pi}v^{\alpha-1}J(n, v)\,dv,$$

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* When $u+n^{-1}\geqslant\pi$, the integral need not be split up, and the argument is simpler.

† Where $u+n^{-1}<\zeta<\pi$.

and hence, using (6), we have

$$O(n^{\alpha-\beta})-V(n,u)=O(n^{\alpha-\beta})+O(n^{\alpha-\beta}u^{\alpha-\beta})+\int_u^\pi v^{\alpha-1}O(n^{\alpha-\beta}v^{-\beta})\,dv$$
$$=O(n^{\alpha-\beta}u^{\alpha-\beta}),$$

since $\beta>\alpha$, from which the second inequality in (8) follows. This completes the proof of the theorem.

4. *Proof of Theorem* 2. There will be no loss of generality in supposing that $0<\alpha<\beta-1<h+1$, where h is the greatest integer not greater than α. Since, for $\gamma\geqslant 1$, $\phi_\gamma(t)$ is absolutely continuous in every range (η,π), $\eta>0$, it will be enough to show that $\int_0^\pi|\phi_\beta'(t)|\,dt$ is finite.

Since $a_n=o(1)$, we have, on partial summation, by (3)*,

$$\beta^{-1}\phi_\beta(t)=\sum_{n=0}^{\infty}a_n\gamma_\beta(nt)$$
$$=\sum_{n=0}^{\infty}S_n{}^h\Delta^{h+1}\gamma_\beta(nt)$$
$$=\sum_{n=0}^{\infty}\Delta^{h+1}\gamma_\beta(nt)\sum_{m=0}^{n}A_{n-m}^{h-\alpha}S_m^{\alpha-1}$$
$$=\sum_{m=0}^{\infty}S_m^{\alpha-1}\sum_{n=m}^{\infty}A_{n-m}^{h-\alpha}\Delta^{h+1}\gamma_\beta(nt)$$
$$=\sum_{m=0}^{\infty}S_m^{\alpha-1}J_m(t)$$
$$=\sum_{m=0}^{\infty}S_m{}^\alpha\Delta J_m(t)$$
$$=\sum_{m=0}^{\infty}s_m{}^\alpha A_m{}^\alpha\Delta J_m(t)$$
$$=\sum_{m=0}^{\infty}s_m{}^\alpha\Delta V_m(t)$$
$$=\sum_{m=0}^{\infty}(s_m{}^\alpha-s_{m-1}^\alpha)V_m(t),$$

* *Cf.* Bosanquet, 1. If we used the fact that $S_v{}^\alpha=O(v^\alpha)$, which is included in the hypotheses, it would be possible to perform an extra partial summation before the inversion of the repeated sum, instead of after it. The method chosen is, however, more convenient for several reasons. In particular, it brings out the analogy between the proofs of Theorems 1 and 2.

where $$J_m(t) = \sum_{n=m}^{\infty} A_{n-m}^{h-\alpha} \Delta^{h+1} \gamma_\beta(nt)$$

and $$V_m(t) = \sum_{\nu=m}^{\infty} A_\nu{}^\alpha \Delta J_\nu(t),$$

provided that we show that $J_m(t) = O(m^{-\beta})$ and $V_m(t) = O(m^{\alpha-\beta})$. The inversion of the repeated integral is justified by an argument used elsewhere*.

We next have

$$\beta^{-1}\phi_\beta'(t) = \sum_{m=0}^{\infty} (s_m{}^\alpha - s_{m-1}^\alpha) V_m'(t), \tag{9}$$

provided that differentiation under the integral sign is permissible, and hence

$$\beta^{-1}\int_0^\pi |\phi_\beta'(t)|\,dt = \int_0^\pi \left| \sum_{m=0}^{\infty} (s_m{}^\alpha - s_{m-1}^\alpha) V_m'(t) \right| dt$$

$$\leqslant \int_0^\pi \sum_{m=0}^{\infty} |s_m{}^\alpha - s_{m-1}^\alpha|\,|V_m'(t)|\,dt$$

$$= \sum_{m=0}^{\infty} |s_m{}^\alpha - s_{m-1}^\alpha| \int_0^\pi |V_m'(t)|\,dt.$$

If now we prove that

$$|V_m'(t)| \begin{cases} \leqslant A m^\alpha t^{\alpha-1}, \\ \leqslant A m^{1+\alpha-\beta} t^{\alpha-\beta}, \end{cases} \tag{10}$$

it will follow, since $\Sigma|s_m{}^\alpha - s_{m-1}^\alpha|$ converges, that (9) converges uniformly for $t \geqslant \epsilon$, $\epsilon > 0$, which will justify the differentiation. Also we shall have

$$\int_0^\pi |V_m'(t)|\,dt = \int_0^{m^{-1}} O(m^\alpha t^{\alpha-1})\,dt + \int_{m^{-1}}^\pi O(m^{1+\alpha-\beta} t^{\alpha-\beta})\,dt$$

$$= O(1) + O(1),$$

uniformly in m, since $\beta > \alpha+1$. Since $\Sigma|s_m{}^\alpha - s_{m-1}^\alpha|$ converges, it will follow that $\int_0^\pi |\phi_\beta'(t)|\,dt$ is finite.

* *Cf.* Bosanquet, **1**.

Before establishing (10), we prove that

$$|J_m'(t)| \begin{cases} \leqslant At^{\alpha-1}, \\ \leqslant Am^{1-\beta}t^{\alpha-\beta}. \end{cases} \tag{11}$$

We first observe that

$$J_m'(t) = \sum_{n=m}^{\infty} A_{n-m}^{h-\alpha}\Delta^{h+1}\frac{d}{dt}\gamma_\beta(nt),$$

for, on partial summation,

$$\sum_{n=N}^{\infty} A_{n-m}^{h-\alpha}\Delta^{h+1}\frac{d}{dt}\gamma_\beta(nt) = \sum_{n=N}^{\infty} A_{n-m}^{h-\alpha-1}\Delta^h\frac{d}{dt}\gamma_\beta(nt) + A_{N-m-1}^{h-\alpha}\Delta^h\frac{d}{dt}\gamma_\beta(Nt),$$

from which it follows that the differentiated series is uniformly convergent for $t \geqslant \epsilon$, $\epsilon > 0$, since $\beta > 1$. Now, writing

$$J_m'(t) = \sum_{n=m}^{m+p} + \sum_{n=m+p+1}^{\infty} = \Sigma_1 + \Sigma_2,$$

where p is the greatest integer not exceeding $1/t$, we have, by (3),

$$\begin{aligned} |\Sigma_1| &\leqslant At^h \sum_{n=m}^{m+p}(n-m+1)^{h-\alpha}\min\{1,\ (nt)^{1-\beta}\} \\ &\leqslant At^h \min\{1,\ (mt)^{1-\beta}\}\sum_{n=m}^{m+p}(n-m+1)^{h-\alpha} \\ &= t^h \min\{1,\ (mt)^{1-\beta}\}\, O\{(1/t)^{1+h-\alpha}\} \\ &= O(t^{\alpha-1})\min\{1,\ (mt)^{1-\beta}\}, \end{aligned}$$

while

$$\begin{aligned} |\Sigma_2| &\leqslant A_{p+1}^{h-\alpha}\max_{p'>p+1}\left|\sum_{n=m+p+1}^{m+p'}\Delta^{h+1}\frac{d}{dt}\gamma_\beta(nt)\right| \\ &= O\{(1/t)^{h-\alpha}\}\, O(t^{h-1})\min\{1,\ (\overline{m+p}\,t)^{1-\beta}\} \\ &= O(t^{\alpha-1})\min\{1,\ (mt)^{1-\beta}\}. \end{aligned}$$

This proves (11), and in a similar way it may be shown that

$$J_m(t) = O(m^{-\beta}).$$

We now establish (10). We have, on partial summation,

$$\sum_{\nu=m}^{\infty} A_\nu^{\alpha} \Delta J_\nu'(t) = \sum_{\nu=m}^{\infty} A_\nu^{\alpha-1} J_\nu'(t) + A_{m-1}^{\alpha} J_m'(t)$$

$$= \sum_{\nu=m}^{\infty} O(\nu^{\alpha-1}) O(\nu^{1-\beta} t^{\alpha-\beta}) + O(m^{\alpha}) O(m^{1-\beta} t^{\alpha-\beta})$$

$$= O(m^{1+\alpha-\beta} t^{\alpha-\beta}).$$

Observing that these series converge uniformly for $t \geqslant \epsilon$, $\epsilon > 0$, we see that

$$V_m'(t) = \sum_{\nu=m}^{\infty} A_\nu^{\alpha} \Delta J_\nu'(t),$$

and the second inequality in (10) is obtained. Before establishing the first inequality in (10), we observe that, if we take the particular case when $\phi(t) = 1$ for all t, so that $\phi_\beta(t) = 1$ for all t and $s_n^{\beta} = 1$ for every n, we are led to the identity

$$\beta^{-1} = \sum_{m=0}^{\infty} A_m^{\alpha} \Delta J_m(t).$$

From this we obtain

$$0 = \sum_{m=0}^{\infty} A_m^{\alpha} \Delta J_m'(t),$$

the resulting series being uniformly convergent for $t \geqslant \epsilon$, $\epsilon > 0$, as may be shown by partial summation. We now have

$$V_m'(t) = -\sum_{\nu=0}^{m-1} A_\nu^{\alpha} \Delta J_\nu'(t)$$

$$= -\sum_{\nu=0}^{m} A_\nu^{\alpha-1} J_\nu'(t) + A_m^{\alpha} J_m'(t)$$

$$= \sum_{\nu=0}^{m} O\{(\nu+1)^{\alpha-1}\} O(l^{\alpha-1}) + O(m^{\alpha}) O(l^{\alpha-1})$$

$$= O(m^{\alpha} l^{\alpha-1}),$$

since $\alpha > 0$, which establishes the first inequality in (10). This completes the proof of the theorem.

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KINEMATICAL RELATIVITY (II): EQUIVALENT OBSERVERS IN UNIFORM RELATIVE MOTION IN SKEW TRAJECTORIES

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1. *Introduction and summary.*

The present note is concerned with the development of further consequences of the notion of kinematical equivalence. The simplest case of observers with non-intersecting world-lines has been considered in the previous note. We now pass to the case of equivalent non-intersecting observers, who agree that they are in uniform relative motion when each maps his observations in a Euclidean space. In his analysis of the space-time relations of two equivalent observers Milne, in order to afford an easy method for synchronizing the zeros of their clocks, imposed the restriction that they once parted company*. If two equivalent observers are moving in skew rectilinear trajectories, we impose the arbitrary restriction that they synchronize their time zeros at the epoch of nearest approach. *It will be found that two such observers cannot correlate their space-time frames uniquely merely from observations on each other.*

We shall find, however, that if we impose certain additional restrictions this ambiguity may be avoided. With this object we introduce a third observer, who is to be permanently located at the perihelion with respect to one of the two original observers of the path of the other. If each observer adopts a Euclidean space for his map of events and assumes that all "cross country" light paths may be mapped rectilinearly, it will be found that there exists one and only one set of transformations connecting the space-time frames of the three observers, which is compatible with their equivalence in pairs.

* E. A. Milne, *Relativity, gravitation, and world-structure* (Oxford, 1935), § 14. In future references, this work will be referred to as WS.

After these investigations had been completed there appeared an important paper by Leigh Page*, in which he criticized the definition of equivalence adopted by Milne. Milne's definition is that two observers A, B are said to be equivalent if the totality of observations which A can make on B can be described by A in the same way as the totality of observations which B can make on A can be described by B. Leigh Page maintains that Milne's definition implies the synchronization of the time zeros of the clocks kept by A, B respectively and that this is unnecessary. In the case of two observers only, as Leigh Page himself points out, there is no content to his criticism, and in the particular case, which he discusses at length, of three relatively accelerated observers he explicitly assumes that their time zeros are synchronized. Consequently he does not really demonstrate that his own definition is an improvement on Milne's. In so far as the subject of the present note is concerned, his discussion is not rigorous and he succeeds in deriving only one of the possible transformations correlating the observations of two equivalent observers in uniform relative motion in skew trajectories†.

2. *Construction of an equation for the signal function.*

As in previous problems, we begin by constructing an equation for the function correlating the times of emission and reception of signals passing between the two observers A, B. Let A emit at any instant t_1 a signal, which is instantaneously reflected by B at t', returning to A at t_2. Since A, B are equivalent, it follows that, if $t' = \Omega(t_1)$, then $t_2 = \Omega(t')$ and hence‡

$$t_2 = \Omega\Omega(t_1). \tag{1}$$

A can tabulate the values of $\frac{1}{2}c(t_2-t_1)$ against the corresponding values of $\frac{1}{2}(t_2+t_1)$, c being any arbitrarily chosen number, of the dimensions of velocity§. If A finds that $\frac{1}{2}c(t_2-t_1)$ has a unique minimum which is strictly positive, he will regard this as the perihelion distance of B's trajectory. A can then adjust his clock so that the corresponding epoch $\frac{1}{2}(t_2+t_1)$ is his time zero. We now postulate that for all time A finds that B moves in

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* Leigh Page, *Phys. Rev.* (2), 49 (1936), 254–68.

† Leigh Page, *loc. cit.*, 262–3.

‡ Whitrow, *Quart. J. of Math.* (Oxford), 6 (1935), 250; H. P. Robertson, *Astrophys. J.*, 82 (1935), 289–290; Leigh Page, *loc. cit.*, 256.

§ Milne, WS, § 16.

such a way that t_1, t_2 are related by an equation of the form

$$\tfrac{1}{2}c(t_2-t_1) = \{y_0^2+\tfrac{1}{4}V^2(t_2+t_1)^2\}^{\frac{1}{2}}, \tag{2}$$

where V is a positive constant of the dimensions of velocity, its value being less than c, and y_0 is a constant of the dimensions of length. Moreover, we postulate that A observes B to move in a plane in which A himself lies*, the angle χ between the light ray leaving B at t' and the perihelion radius vector of B being given by

$$\tan\chi = \frac{V(t_2+t_1)}{2y_0}. \tag{3}$$

It is then readily seen that, if A plots his observations in a Euclidean space, he will regard B as moving with constant velocity V in a rectilinear trajectory, the perihelion distance being y_0. Since A, B are equivalent, it follows that B constructs exactly the same picture of A's motion as A does of B's.

Substituting in equation (2) for t_2 in terms of t_1 by equation (1), we find that

$$\tfrac{1}{2}c[\Omega\Omega(t_1)-t_1] = [y_0^2+\tfrac{1}{4}V^2\{\Omega\Omega(t_1)+t_1\}^2]^{\frac{1}{2}}. \tag{4}$$

This relation holds for all values of t_1 and so must be a functional equation for Ω, the operator correlating the times of emission and reception of a signal passing between the two observers. Replacing t_1 by x and squaring (4), we find that $\Omega\Omega(x)$ must be a root of the quadratic equation

$$\{\Omega\Omega(x)\}^2 - 2\,\frac{1+V^2/c^2}{1-V^2/c^2}\,x\{\Omega\Omega(x)\}+x^2-4\,\frac{y_0^2/c^2}{1-V^2/c^2} = 0. \tag{5}$$

Since this equation is symmetrical in $\Omega\Omega(x)$ and x, it follows that, if $\Omega\Omega(x)$ is one root, then $\Omega^{-1}\Omega^{-1}(x)$ must be the other.

At this stage it is convenient to introduce numbers κ, γ_1, γ_2 defined thus:

$$\kappa^2 = \frac{4y_0^2/c^2}{1-V^2/c^2}, \quad \gamma_1 = \tfrac{1}{2}\left(\alpha^2+\frac{1}{\alpha^2}\right), \quad \gamma_2 = \tfrac{1}{2}\left(\alpha^2-\frac{1}{\alpha^2}\right), \tag{6}$$

where

$$\alpha = \left(\frac{1+V/c}{1-V/c}\right)^{\frac{1}{2}}. \tag{7}$$

* "Plane" and "angle" will, of course, have no observable significance so long as A, B observe only themselves. These postulates are necessary only when additional observers are introduced.

We observe that $$\gamma_1^2-\gamma_2^2=1. \tag{8}$$

With this notation, we find that equation (5) may be written in the form

$$\{\Omega\Omega(x)\}^2-2\gamma_1 x\{\Omega\Omega(x)\}+x^2-\kappa^2=0, \tag{9}$$

the solutions of which are given by

$$\Omega\Omega(x)=\gamma_1 x\pm(\gamma_2^2 x^2+\kappa^2)^{\frac{1}{2}}. \tag{10}$$

The reason for the ambiguity of sign arises from the fact that we have squared (4). Since $t_2>t_1$, it follows that

$$\Omega\Omega(x)>x, \tag{11}$$

and hence we must take the positive sign for the radical. Consequently

$$\Omega\Omega(x)=\gamma_1 x+(\gamma_2^2 x^2+\kappa^2)^{\frac{1}{2}}. \tag{12}$$

As we have already remarked, the other solution must be $\Omega^{-1}\Omega^{-1}(x)$ and so

$$\Omega^{-1}\Omega^{-1}(x)=\gamma_1 x-(\gamma_2^2 x^2+\kappa^2)^{\frac{1}{2}}. \tag{13}$$

It may be immediately verified that these two functions are actual algebraic inverses.*

3. *A particular solution of the equation for the signal function.*

To solve equations (12) and (13) for Ω we introduce an auxiliary function θ such that

$$\Omega\theta(x)\equiv\theta(x+1). \tag{14}$$

Hence $$\Omega\Omega\theta(x)\equiv\theta(x+2), \quad \Omega^{-1}\Omega^{-1}\theta(x)\equiv\theta(x-2).$$

Substituting $\theta(x)$ for x in equations (12) and (13), we find that

$$\left.\begin{aligned}\theta(x+2)&=\gamma_1\theta(x)+[\gamma_2^2\{\theta(x)\}^2+\kappa^2]^{\frac{1}{2}},\\ \theta(x-2)&\equiv\gamma_1\theta(x)-[\gamma_2^2\{\theta(x)\}^2+\kappa^2]^{\frac{1}{2}}.\end{aligned}\right\} \tag{15}$$

Consequently

$$\left.\begin{aligned}\theta(x+2)-2\gamma_1\theta(x)+\theta(x-2)&\equiv 0,\\ \{\theta(x+2)\}^2-2\gamma_1\theta(x+2)\,\theta(x)+\{\theta(x)\}^2&\equiv\kappa^2.\end{aligned}\right\} \tag{16}$$

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* Equations (12) and (13) were originally obtained by Milne, who used a somewhat different method. I should like to express my thanks to Professor Milne for drawing my attention to this problem.

Substituting for γ_1 in terms of α, we see that the former equation of (16) may be written as

$$\theta(x+2)-\alpha^2\theta(x) \equiv \alpha^{-2}\{\theta(x)-\alpha^2\theta(x-2)\},$$

and hence
$$\theta(x+2)-\alpha^2\theta(x) \equiv C(x)\,\alpha^{-x}, \tag{17}$$

where $C(x)$ is an arbitrary function of period 2. The complete solution of (17) is readily seen to be

$$\theta(x) \equiv A(x)\,\alpha^x + B(x)\,\alpha^{-x}, \tag{18}$$

where $A(x)$, $B(x)$ are arbitrary functions of period 2.

Substituting this solution for $\theta(x)$ in the latter equation of (16), we obtain the condition

$$A(x)\,B(x) = -(\kappa/2\gamma_2)^2. \tag{19}$$

Consequently, if $\kappa \neq 0$,

$$A(x) \equiv \frac{\kappa}{2\gamma_2}\,\lambda(x), \qquad B(x) \equiv -\frac{\kappa}{2\gamma_2}\,\frac{1}{\lambda(x)}, \tag{20}$$

where $\lambda(x)$ is an arbitrary function of period 2. Hence the general form of $\theta(x)$ is

$$\theta(x) \equiv \frac{\kappa}{2\gamma_2}\left\{\lambda(x)\,\alpha^x - \frac{1}{\lambda(x)}\,\alpha^{-x}\right\}. \tag{21}$$

A particular solution is therefore given by

$$\theta(x) \equiv \frac{\kappa}{2\gamma_2}\left\{\lambda\alpha^x - \frac{1}{\lambda}\,\alpha^{-x}\right\}, \tag{22}$$

where λ is an arbitrary constant.

Defining new numbers w, z by the relations

$$2w \equiv \lambda\alpha^x + \frac{1}{\lambda}\,\alpha^{-x}, \qquad 2z \equiv \lambda\alpha^x - \frac{1}{\lambda}\,\alpha^{-x},$$

we find that
$$w^2 = 1+z^2$$

and that
$$x \equiv \log_\alpha(z+w) - \log_\alpha\lambda.$$

Consequently, if $\theta(x)$ is given by (22), then $\theta^{-1}(x)$ is given by

$$\theta^{-1}(x) \equiv \log_\alpha\left[\gamma_2 x + \{\gamma_2^2 x^2 + \kappa^2\}^{\frac{1}{2}}\right] - \log_\alpha(\lambda\kappa). \tag{23}$$

Since
$$\theta^{-1}\,\Omega(x) \equiv \theta^{-1}(x) + 1, \tag{24}$$

we see that

$$\log_a\left[\gamma_2\,\Omega(x)+\left\{\gamma_2^2\left(\Omega(x)\right)^2+\kappa^2\right\}^{\frac{1}{2}}\right]\equiv\log_a[\gamma_2 x+(\gamma_2^2x^2+\kappa^2)^{\frac{1}{2}}]+1,$$

and hence

$$\gamma_2\,\Omega(x)+[\gamma_2^2\{\Omega(x)\}^2+\kappa^2]^{\frac{1}{2}}\equiv a[\gamma_2 x+(\gamma_2^2x^2+\kappa^2)^{\frac{1}{2}}]. \tag{25}$$

This is an implicit equation for $\Omega(x)$, from which we may immediately reproduce the original form of $\Omega\Omega(x)$. For, if we substitute $\Omega(x)$ for x, we obtain the equation

$$\gamma_2\,\Omega\Omega(x)+[\gamma_2^2\{\Omega\Omega(x)\}^2+\kappa^2]^{\frac{1}{2}}\equiv a^2[\gamma_2 x+(\gamma_2^2x^2+\kappa^2)^{\frac{1}{2}}] \tag{26}$$

the solution of which is

$$\Omega\Omega(x)\equiv\gamma_1 x+(\gamma_2^2x^2+\kappa^2)^{\frac{1}{2}}.$$

Defining a new function $\phi(x)$ and its inverse $\phi^{-1}(x)$ by the equations

$$\left.\begin{aligned}\phi(x)&\equiv\gamma_2 x+(\gamma_2^2x^2+\kappa^2)^{\frac{1}{2}},\\ \phi^{-1}(x)&\equiv\frac{1}{2\gamma_2}\left(x-\frac{\kappa^2}{x}\right),\end{aligned}\right\} \tag{27}$$

we see that (25) gives

$$\Omega\equiv\phi^{-1}a\phi, \tag{28}$$

and it is readily found by direct substitution that the algebraic form of $\Omega(x)$ is

$$\Omega(x)\equiv\frac{x+\dfrac{V/c}{\gamma_2}(\gamma_2^2x^2+\kappa^2)^{\frac{1}{2}}}{(1-V^2/c^2)^{\frac{1}{2}}}. \tag{29}$$

The corresponding solution for $\Omega^{-1}(x)$ is given by changing the sign of the radical and hence is

$$\Omega^{-1}(x)\equiv\frac{x-\dfrac{V/c}{\gamma_2}(\gamma_2^2x^2+\kappa^2)^{\frac{1}{2}}}{(1-V^2/c^2)^{\frac{1}{2}}}, \tag{30}$$

and it can easily be verified that these two functions are actual algebraic inverses. Equations (29) and (30) therefore yield a particular solution for the signal function correlating two observers in uniform relative motion in skew trajectories.

4. *The general solution of the equation for the signal function.*

We shall now look for the general Ω in the form

$$\Omega \equiv \phi^{-1}\psi\phi, \tag{31}$$

where ϕ, ϕ^{-1} are still given by (27). From this it follows that

$$\Omega\Omega \equiv \phi^{-1}\psi\psi\phi. \tag{32}$$

But from (26) we see that

$$\Omega\Omega \equiv \phi^{-1}\alpha^2\phi. \tag{33}$$

Consequently, since ϕ and its inverse are both single-valued, we find, by identifying (32) and (33), that $\psi(x)$ must be a solution of the functional equation

$$\psi\psi(x) \equiv \alpha^2 x. \tag{34}$$

It is obvious that $\phi(x) > 0$ for all finite values of x and that it converges to zero only as x diverges negatively. Hence its inverse is defined only for positive values of the argument. Consequently, if Ω is given by (31), then $\psi\phi$ must be positive. Since ϕ is always positive, we see that $\psi(x)$ must be any solution of (31) which is positive when x is positive. Moreover, we are interested only in the behaviour of such a solution for $x > 0$. Finally, since Ω is a signal function, it must be monotonic increasing, and consequently, since ϕ, ϕ^{-1} are monotonic increasing, ψ must be monotonic increasing also. Using these facts, we may construct graphically the most general admissible solution.

We first show that any such solution $z = \psi(x)$ intersects the line

$$z = \alpha x$$

infinitely often. Let x_1 be a particular value of x for which

$$\psi(x_1) < \alpha x_1,$$

then, since ψ is monotonic increasing,

$$\psi\psi(x_1) < \psi(\alpha x_1).$$

Hence, writing $\alpha^2 x_1$ for $\psi\psi(x_1)$ and x_2 for αx_1, we find that

$$\psi(x_2) > \alpha x_2.$$

Similarly

$$\psi\psi(x_2) > \psi(\alpha x_2),$$

and hence, writing x_3 for αx_2, we find that

$$\psi(x_3) < \alpha x_3,$$

and so on indefinitely. Since $z = \psi(x)$ is continuous, it must therefore meet the line $z = ax$ infinitely often. In particular, if x_0 is the abscissa of a particular point of intersection, then ax_0 is the abscissa of another such point, and so on. Moreover, it follows, from the fact that every solution of (34) is also a solution of

$$\psi(a^2 x) \equiv a^2 \psi(x),$$

that the derivative of $\psi(x)$ must satisfy the functional equation

$$\psi'(a^2 x) \equiv \psi'(x).$$

Consequently the graph of $z = \psi(x)$ must be as shown below. In general,

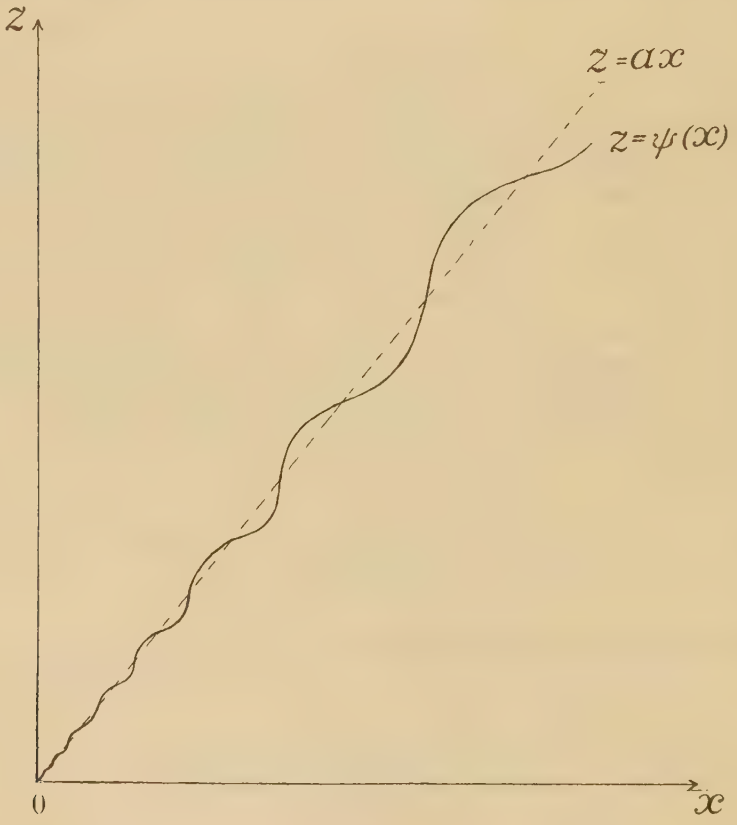

Fig. 1.

of course, there will be other points of intersection of the line $z = ax$ and the curve $z = \psi(x)$ between the consecutive associated points (x_0, ax_0) and $(ax_0, a^2 x_0)$.

As $x \to 0$ the curve fluctuates infinitely often about the line $z = ax$. However, since we are concerned with $\psi\phi(x)$ and $\phi(x) \to 0$ only as $x \to -\infty$, it follows that there is no singularity in the behaviour of the derivative of $\psi\phi(x)$ as $\phi(x) \to 0$. Hence the argument by which it was possible to eliminate all solutions of (34) other than $\psi(x) \quad ax$, in the case of intersecting observers in uniform relative motion, does not apply in the case of non-intersecting observers*. Moreover, it is not possible to rule out the alternative solutions by a dimensional argument, since we have a constant κ, of the dimensions of time, at our disposal. *We therefore conclude that the fact that two observers describe each other as equivalent and moving uniformly in skew trajectories is not sufficient to determine uniquely the function expressing the time of arrival of a signal at either observer in terms of its time of emission from the other.* Since this ambiguity has been found to exist for the simplest case of non-intersecting observers, viz. relatively stationary observers, it is not surprising that it arises in other cases.

5. *The synchronization of perihelion-epochs.*

We have assumed that A assigns the conventional epoch $t = 0$ to the instant at which B is at perihelion with respect to A. Similarly, since B is equivalent to A, B must assign the conventional epoch $t' = 0$ to the instant at which A is at perihelion with respect to B. We now inquire whether B can also assign the epoch $t' = 0$ to the event at himself to which A has assigned the conventional epoch $t = 0$, and correlatively whether A can assign the epoch $t = 0$ to the event at himself to which B has assigned the conventional epoch $t' = 0$.† Since A, B are equivalent, it is clearly sufficient to consider the former question only.

If t_0' is the time recorded by B at the epoch of B's nearest approach to A, according to A, then, since $t = \frac{1}{2}(t_2 + t_1)$, where $t_2 = \Omega(t')$, $t_1 = \Omega^{-1}(t')$, it follows that

$$0 = \tfrac{1}{2}\{\Omega(t_0') + \Omega^{-1}(t_0')\}.$$

Hence t_0' is zero if, and only if,

$$\Omega(0) + \Omega^{-1}(0) = 0. \tag{35}$$

Since the general form of $\Omega(t)$ is given by (31), where ψ is a solution of (34), it is easily seen that

$$\Omega(0) = \frac{1}{2\gamma_2}\{\psi(\kappa) - \kappa^2/\psi(\kappa)\}, \quad \Omega^{-1}(0) = \frac{1}{2\gamma_2}\{\psi^{-1}(\kappa) - \kappa^2/\psi^{-1}(\kappa)\}.$$

* Whitrow, *Quart. J. of Math.* (Oxford), 4 (1933), 161–72, § 10.

† This point was first raised by Milne.

Hence, if the relation (34) is true, then we must have

$$\psi(\kappa)+\psi^{-1}(\kappa)=\kappa^2\{1/\psi(\kappa)+1/\psi^{-1}(\kappa)\}. \tag{36}$$

Since $$\psi\psi(x)\equiv a^2 x,$$

it follows that $$\psi^{-1}(x)\equiv\psi(x)/a^2,$$

and hence (36) may be rewritten in the form

$$\{\psi(\kappa)\}^2=a^2\kappa^2.$$

Since $\psi(x)>0$ when $x>0$, it follows that

$$\psi(\kappa)=a\kappa.$$

Consequently, either* $\psi(x)\equiv ax$, or else κ must be the abscissa of one of the infinitely many points in which the curve $z=\psi(x)$ meets the line $z=ax$.

We therefore find that, in general, B will not assign the epoch $t'=0$ to the event at himself, to which A has assigned the conventional epoch $t=0$. Moreover, even if this additional condition is imposed, the signal function correlating A, B is still not unique, although the curve $z=\psi(x)$ is now restricted to pass through the point $(\kappa, a\kappa)$. This result is of interest in connection with the construction of cosmological models possessing a unique epoch of minimum configuration.

6. *The Lorentz formulae.*

We have seen that, if two observers A, B can describe each other as equivalent and moving with uniform relative motion in skew trajectories, this does not suffice to determine uniquely the running of the clock kept by either in terms of that kept by the other. However, if for the moment we assume that they are correlated by the particular signal function given by (29), that is

$$\Omega_{12}(t')\equiv\frac{t'+\dfrac{V/c}{\gamma_2}(\gamma_2^2 t'^2+\kappa^2)^{\frac{1}{2}}}{(1-V^2/c^2)^{\frac{1}{2}}}, \tag{37}$$

then, since

$$\Omega_{12}^{-1}(t')\equiv\frac{t'-\dfrac{V/c}{\gamma_2}(\gamma_2^2 t'^2+\kappa^2)^{\frac{1}{2}}}{(1-V^2/c^2)^{\frac{1}{2}}}, \tag{38}$$

* This case gives rise to the modified Lorentz formulae. See § 6.

it follows that, if t is the conventional time assigned by A to the instant t' at B,

$$t = \tfrac{1}{2}\{\Omega_{12}(t') + \Omega_{12}^{-1}(t')\} = t'(1-V^2/c^2)^{\frac{1}{2}}. \quad (39)$$

From this relation it can be shown that the space-time frames of A, B are related by the modified Lorentz formulae*

$$x' = (x-Vt)(1-V^2/c^2)^{-\frac{1}{2}}, \quad y' = y-y_0, \quad t' = (t-Vx/c^2)(1-V^2/c^2)^{\frac{1}{2}}, \quad (40)$$

provided that we assume that each observer maps all "cross country" light paths rectilinearly. This result may be established by an exactly similar method to that employed by Milne in deducing the usual Lorentz formulae†.

We now consider a third observer C equivalent to A and moving with the same relative velocity as B from a position of coincidence with A at time zero. Then, if any light track emanating from either observer is mapped rectilinearly by the other, the space time frames of A, C are related by the formulae‡

$$x'' = (x-Vt)(1-V^2/c^2)^{-\frac{1}{2}}, \quad y'' = y, \quad t'' = (t-Vx/c^2)(1-V^2/c^2)^{-\frac{1}{2}}. \quad (41)$$

Combining (40) and (41), we readily see that B, C describe each other as equivalent and relatively stationary. Hence, if the signal function correlating A, B is given by (37), then, subject to the light-path postulate, we find that A, B, C are all equivalent in pairs. We now prove that for this to be the case it is not merely sufficient for the signal function correlating A, B to be of the form (37), but it is also necessary.

Since the space-time frames of A, C are related by the Lorentz formulae (41), it follows that, if B is moving in a skew trajectory with respect to A from a position of perihelion at time zero, and if B, C have the same velocity relative to A, then C must regard B as relatively stationary. Hence, if A, B, C are equivalent in pairs, then B must regard C as relatively stationary. Moreover, B must describe A as moving uniformly so that A and C are coincident at the epoch of A's perihelion with respect to B. From this it follows that B regards C as permanently located at the perihelion point of A's path.

The signal function correlating A, C is unique, being

$$\Omega_{13}(t) \equiv at, \quad (42)$$

* Unprimed symbols refer to A's measures; primed symbols refer to B's.

† Milne, WS, §§ 32–36.

‡ Double-primed symbols refer to C's measures.

where

$$\alpha = \left(\frac{1+V/c}{1-V/c}\right)^{\frac{1}{2}}, \tag{43}$$

if V is the relative velocity*. The signal function correlating B, C is $\Omega_{23}(t)$, where

$$\Omega_{23}\,\Omega_{23}(t) \equiv t + 2y_0/c, \tag{44}$$

if y_0 is the fixed distance which B, C assign to each other. Consequently, if a signal is emitted by A in the direction of C at local time t_0, it will arrive at local time αt_0. If C reflects it instantaneously in the direction of B, it will arrive at local time $\Omega_{23}(\alpha t_0)$.

It is also possible for A, B to signal to each other directly and to determine the function $f_{12}(T)$ expressing the local time of any event at B in terms of the conventional epoch T assigned to it by A. Hence the epoch T assigned by A to the arrival at B of the signal considered above must be such that

$$f_{12}(T) = \Omega_{23}(\alpha t_0). \tag{45}$$

But if it is possible for A to map all "cross country" light paths rectilinearly, he can calculate the epoch T in the following manner. Since C is receding from A with velocity V, it follows that the conventional epoch t assigned by A to the instant of the arrival of the signal at C is given by

$$t = \frac{t_0}{1-V/c}. \tag{46}$$

In travelling from C to B the signal will be regarded by A as having a velocity component V in the direction AC. Hence A will assign to its arrival at B the epoch

$$T = t + \frac{y_0/c}{(1-V^2/c^2)^{\frac{1}{2}}}. \tag{47}$$

Consequently, if the assumptions made by the observers are consistent, it follows automatically that

$$\Omega_{23}(\alpha t_0) = f_{12}\left\{\frac{t_0}{1-V/c} + \frac{y_0/c}{(1-V^2/c^2)^{\frac{1}{2}}}\right\}.$$

* It is readily proved that if A, C are in uniform relative motion, then $\Omega_{13}\,\Omega_{13}(t) \equiv \alpha^2 t$, and the only monotonic increasing solution of this functional equation which has a continuous derivative as $t \to 0$ positively is $\Omega_{13}(t) \equiv \alpha t$. *Vide* Whitrow, *loc. cit.*, § 10.

This must be true for all values of t_0 and hence, on replacing αt_0 by any variable t, we find that

$$\Omega_{23}(t) \equiv f_{12}\left\{\frac{t+y_0/c}{(1-V^2/c^2)^{\frac{1}{2}}}\right\}. \tag{48}$$

We now consider a signal emitted by B in the direction of C at any local time τ_0. It will arrive at C at local time $\Omega_{23}(\tau_0)$. Let it be reflected by C in the direction of A. Then it will arrive at A at local time $\alpha\Omega_{23}(\tau_0)$. Since A, B are equivalent, it follows that the epoch T' assigned by B to the arrival of the signal at A must be such that

$$f_{12}(T') = \alpha\Omega_{23}(\tau_0). \tag{49}$$

But if it is possible for B to map all "cross country" light paths rectilinearly, he can calculate the epoch T' as

$$T' = \frac{\tau_0+y_0/c}{1-V/c}. \tag{50}$$

Hence, for consistency, we must have

$$\alpha\Omega_{23}(t) \equiv f_{12}\left\{\frac{t+y_0/c}{1-V/c}\right\} \tag{51}$$

if we replace τ_0 by the variable t.

Comparing equations (48) and (51), we see that, if we introduce a new variable z, defined so that

$$z \equiv (t+y_0/c)(1-V^2/c^2)^{-\frac{1}{2}}, \tag{52}$$

then we must have $$f_{12}(\alpha z) \equiv \alpha f_{12}(z). \tag{53}$$

Since the function $\Omega_{23}(t)$ must have a continuous derivative, it follows that $f_{12}(z)$ must have one also. The only solution of (53) which has a continuous derivative at $z = 0$ is

$$f_{12}(z) \equiv kz, \tag{54}$$

where k is an arbitrary constant*. Hence

$$\Omega_{23}(t) \equiv k\,\frac{t+y_0/c}{(1-V^2/c^2)^{\frac{1}{2}}},$$

and so $$\Omega_{23}\,\Omega_{23}(t) \equiv t+2y_0/c,$$

* Whitrow, *loc. cit.*, § 10.

if and only if $$k = (1-V^2/c^2)^{\frac{1}{2}}. \tag{55}$$

From this it follows immediately that

$$f_{12}(t) \equiv t(1-V^2/c^2)^{\frac{1}{2}}, \tag{56}$$

and so A, B are related by the modified Lorentz formulae as before. *Hence we have proved that, if A, B are equivalent observers in uniform relative motion in skew trajectories, there is one and only one transformation, correlating their space-time frames, which is compatible with the equivalence of a third observer permanently located at the perihelion of the path of either with respect to the other, and with the assumption that all "cross country" light tracks are to be mapped rectilinearly by each observer in his Euclidean space*.*

7. *Einstein's formula for the Doppler effect.*

It is of interest to verify directly that the particular signal function (37) actually leads to the correct formula for the Doppler shift of a light source moving obliquely with uniform velocity relative to the observer. To prove this we must assume that the atomic vibrators keep the same time as the macroscopic clocks concerned. Let B emit signals at local times t', $t'+\delta t'$ respectively, and let these be received by A at local times t_2, $t_2+\delta t_2$ respectively. Since

$$t_2 = \Omega_{12}(t'),$$

it follows that the Doppler shift ratio observed by A is given by

$$s = \frac{\nu(\textit{proper frequency})}{\nu'(\textit{apparent frequency})} = \frac{\delta t_2}{\delta t'} = \Omega'_{12}(t'). \tag{57}$$

Since $$t' = t(1-V^2/c^2)^{\frac{1}{2}},$$

where t is the conventional time assigned by A to the epoch t' at B, it follows that the Doppler shift of an atomic signal leaving B at epoch t, according to A, is

$$s(t) = \Omega'_{12}\{(1-V^2/c^2)^{\frac{1}{2}}t\}.$$

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* It can easily be proved *a posteriori* that if the frames of A, B are related by the modified Lorentz formulae (40) then the signal function correlating them must be of the form (37).

Substituting for the derivative of Ω_{12}, we get

$$s(t) = \frac{1+\frac{V}{c}\left\{\frac{Vt}{(V^2t^2+y_0^2)^{\frac{1}{2}}}\right\}}{(1-V^2/c^2)^{\frac{1}{2}}}. \tag{58}$$

Since the angle ξ between B's trajectory relative to A and the radius vector AB is given by

$$\cos\xi = Vt/(V^2t^2+y_0^2)^{\frac{1}{2}}, \tag{59}$$

we see that

$$s(t) = \frac{1+V_r/c}{(1-V^2/c^2)^{\frac{1}{2}}}, \tag{60}$$

where V_r is the radial velocity of B at the instant at which the signal is emitted. As is well known, this formula was originally due to Einstein*.

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* Einstein, *Ann. der Phys.*, 17 (1905), 911.

AN INEQUALITY FOR THE RIEMANN ZETA-FUNCTION

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1. The inequality in question is an Ω-result for the critical strip. Two such inequalities are known*: Titchmarsh's, that $|\zeta(\sigma+it)|$, for fixed σ ($\frac{1}{2} \leqslant \sigma < 1$), sometimes takes values greater than any power of $\log t$; and Valiron's, that the minimum of $|\zeta(\sigma+it)|$ on the line $-1 \leqslant \sigma \leqslant 2$ is generally greater than t^{-A}. We consider the line $\frac{1}{2}+\delta \leqslant \sigma \leqslant 2$, and show that the minimum of $|\zeta(s)|$ on it is sometimes greater than a number $L(\delta)$ independent of t.

2. The proof depends on an inequality established in a recent publication†, namely that, if

(*a*) $$f(z) \text{ is regular in } |z| < 1;$$

(*b*) $$|f(z)| \leqslant M \quad (|z| < 1);$$

(*c*) $$|f(a_k)| \leqslant L \quad (k = 0, 1, \ldots, N),$$

$a_0, a_1, \ldots, a_N$ being any distinct points in $|z| \leqslant r < 1$; then

$$|f(z)| \leqslant M\left(\frac{2r}{1+r^2}\right)^{N+1} + \frac{6}{1-r^4} L\Delta(2r)^N \quad (|z| \leqslant r), \tag{1}$$

where

$$\Delta = \sum_{k=0}^{N} \prod_{i}{}' \frac{1}{|a_i - a_k|}, \tag{2}$$

* E. C. Titchmarsh, *The zeta-function of Riemann* (Cambridge Tract, 1930), 26.

† J. M. Whittaker, *Interpolatory function theory* (Cambridge Tract, 1935), 57. The reference to Pólya given on this page should be Pólya (5, Part II, Satz V).

and $\prod_i'$ means that the factor for which $i = k$ is to be omitted. If $L = 0$ the inequality reduces to

$$|f(z)| \leqslant M\left(\frac{2r}{1+r^2}\right)^{N+1} \quad (|z| \leqslant r), \tag{3}$$

a result which may be regarded as a generalisation of Schwarz's lemma and has been extensively used in one form or another. Pólya* has formulated the underlying principle in a very general manner by applying conformal representation to (3).

If $L > 0$ we cannot expect anything so simple as (3). For, if we are told that $f(z)$ is zero at certain points of $|z| \leqslant r$, it is enough to know the *number* of these points in order to obtain an upper bound for $|f(z)|$. If, however, we are merely told that $f(z)$ is small at certain points, it is evident that $f(z)$ will also be small at all neighbouring points; so the hypothesis "$f(z)$ is small at N points" asserts no more than "$f(z)$ is small at one point" unless some information is given about the *distribution* of the N points. An inequality such as (1) must therefore contain a "distribution factor" Δ, and it is sometimes a little difficult to evaluate it. The particular Δ which we use below has, however, been discussed already in a different connection by Pólya†.

3. The best results concerning $\zeta(s)$ naturally depend on the deeper properties of the function, but a certain amount can be proved merely by making use of the fact that $\zeta(s)$ is regular and of finite order in any strip $(a < \sigma < b,\ t > 1)$.

Let $f(s)$ be any function which is regular in the strip. As usual we define‡ the order $\mu(\sigma)$ to be the lower bound of numbers ξ such that

$$|f(\sigma+it)| = O(t^{\xi}). \tag{4}$$

Define also the function $\lambda(\alpha, \beta)$ to be the lower bound of numbers ξ such that

$$\min_{\alpha \leqslant \sigma \leqslant \beta} |f(\sigma+it)| = O(t^{\xi}) \quad (a < \alpha < \beta < b). \tag{5}$$

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* G. Pólya, "Untersuchungen über Lücken und Singularitäten von Potenzreihen, II", *Annals of Math.*, 34 (1933), 731–777 (774).

† G. Pólya, "Uber die Irreducibilität gewisser ganzzahliger Polynome", *Jahresbericht der deutschen Math. Vereinigung*, 28 (1919), 31–36.

‡ *Cf.* Titchmarsh, *op. cit.*, 19.

Evidently $\lambda(c-h, c+h)$ increases (in the wide sense) as h decreases, and never exceeds $\mu(c)$. Hence

$$\lambda(c-h, c+h) \to \lambda(c) \leqslant \mu(c) \quad \text{as} \quad h \to 0. \tag{6}$$

THEOREM 1. *Suppose that*

(i) $f(s)$ *is regular in* $S(a < \sigma < b, t > t_0)$;

(ii) $|f(s)| < Kt^A$ *in* S;

then

$$\lambda(\sigma) = \mu(\sigma) \quad (a < \sigma < b). \tag{7}$$

In view of (6) we have to show that, corresponding to given c, ϵ, there is a number h such that

$$\lambda(c-h, c+h) > \mu(c) - \epsilon. \tag{8}$$

By making a linear transformation we can suppose that $c = 0$ and that (ii) is satisfied in $-1 < \sigma < 1$. If (8) is false, then, for each h,

$$\lambda(-h, h) \leqslant \mu(0) - \epsilon = \mu - \epsilon$$

(say), and so

$$\min_{-h \leqslant \sigma \leqslant h} |f(\sigma+it)| < t^{\mu - \frac{1}{2}\epsilon} \quad (t \geqslant t_h). \tag{9}$$

With a fixed value of h, which will be chosen later, choose r so that $h < r < 1$ and consider the circles C_1, C_2 in the s-plane, of centre $(0, t)$ and radii r, 1. C_1 has, in common with the strip $-h \leqslant \sigma \leqslant h$, a rectangle of sides $2h$, $2\sqrt{(r^2-h^2)} = 2r\theta$ (say). Divide this rectangle into N (which will be chosen later) equal strips by drawing lines parallel to the σ-axis.

In view of (9) there are points $a_0, a_1, \ldots, a_N$, one on each of the lines so drawn, such that

$$|f(a_k)| < (Ct)^{\mu - \frac{1}{2}\epsilon}.$$

Again, if d is the width of a strip,

$$|a_i - a_k| \geqslant (k-i)d \quad (i = 0, 1, \ldots, k-1),$$

$$\geqslant (i-k)d \quad (i = k+1, k+2, \ldots, N),$$

so that

$$\Delta \leqslant \frac{1}{d^N} \sum_{k=0}^{N} \frac{1}{k(k-1)\ldots 2 \,.\, 1 \,.\, 1 \,.\, 2 \ldots (N-k)}$$

$$= \frac{2^N}{d^N N!} < \left(\frac{2e}{dN}\right)^N = \left(\frac{2e}{2r\theta}\right)^N.$$

Apply (1) to $f(s)$ in C_2. Then, for s in C_1,

$$|f(s)| \leqslant Kt^A \left(\frac{2r}{1+r^2}\right)^{N+1} + \frac{6}{1-r^4}(Ct)^{\mu-\frac{1}{2}\epsilon}\left(\frac{e}{r\theta}\right)^N (2r)^N. \tag{10}$$

Now choose N to be a function of t such that

$$\left(\frac{1+r^2}{2r}\right)^N \sim t^{A-\mu+\epsilon}, \tag{11}$$

so that

$$|f(it)| \leqslant Kt^{\mu-\epsilon} + \frac{6}{1-r^4}(Ct)^{\mu-\frac{1}{2}\epsilon}\left(\frac{2e}{\theta}\right)^N. \tag{12}$$

If now we take $r = 2h$, so that $\theta = \sqrt{3}/2$, and choose h so that

$$(A-\mu+\epsilon)\log\frac{4e}{\sqrt{3}} \Big/ \log\frac{1+r^2}{2r} < \frac{\epsilon}{4}, \tag{13}$$

it will follow from (11), (12) that

$$|f(it)| \leqslant Kt^{\mu-\epsilon} + K' t^{\mu-\frac{1}{2}\epsilon} \quad (t \geqslant t_h),$$

a contradiction.

That (i) is not in itself sufficient to ensure the truth of (7) is shown by the example

$$f(s) = e^{-is^2} = e^{2\sigma t - i(\sigma^2 - t^2)}$$

for which

$$\lambda(0) = -\infty, \quad \mu(0) = 0. \tag{14}$$

In the case of $\zeta(s)$ it is known that

$$\mu(\sigma) \geqslant 0 \quad (\text{all } \sigma).$$

Corresponding to given ϵ and σ there is therefore a number h such that

$$\lambda(\sigma-h, \sigma+h) = \Omega(t^{-\epsilon}),$$

and any interval (a, b) of the σ-axis can be covered by a finite number of the intervals $(\sigma-h, \sigma+h)$. Hence any interval (a, b) can be divided into a finite number of sub-intervals (α, β), in each of which

$$\min_{\alpha \leqslant \sigma \leqslant \beta} |\zeta(\sigma+it)| = \Omega(t^{-\epsilon}). \tag{15}$$

4. So far the hypotheses have been of a very general nature. In the case of the zeta-function it is possible to obtain much more exact information by making use of a mean-value theorem. The precise result required, contained in one due to Littlewood*, is

$$\int_T^{T+\sqrt{T}} |\zeta(\sigma+it)|^2\,dt = \sqrt{T}\,\zeta(2\sigma)+O(T^{\frac{1}{2}-\frac{1}{2}\delta}) \quad (\tfrac{1}{2}+\delta \leqslant \sigma \leqslant 3), \tag{16}$$

the constant implied in the O being absolute.

Let $f(s)$ satisfy condition (i) of Theorem 1 and let k be a positive integer. Write

$$I(T,\sigma) = \int_{t_0}^{T} |f(\sigma+it)|^k\,dt, \tag{17}$$

$$M(t;\alpha,\beta,d) = \text{maximum of } |f(s)| \text{ in the rectangle } R \text{ with corners } \alpha+it,\ \beta+it,\ \beta+it+id,\ \alpha+it+id, \text{ where } a<\alpha<\beta<b, \tag{18}$$

$$M(t) = M(t+\delta;\ \alpha+\delta,\ \beta-\delta,\ d-2\delta), \tag{19}$$

so that $M(t)$ is the maximum of $|f(s)|$ in a rectangle R_δ inside R and distant δ from it. If s is in or on R_δ,

$$\{f(s)\}^k = \frac{1}{2\pi i}\int_R \frac{\{f(w)\}^k}{w-s}\,dw \quad (w=u+iv),$$

and so

$$\{M(t)\}^k \leqslant \frac{1}{2\pi\delta}\int_R |f(w)|^k\,|dw|.$$

Hence

$$\begin{aligned}\int_{T_1}^{T_2}\{M(t)\}^k\,dt &\leqslant \frac{1}{2\pi\delta}\int_{T_1}^{T_2}\left\{\int_R |f(w)|^k\,|dw|\right\}dt\\ &= \frac{1}{2\pi\delta}\int_{T_1}^{T_2}\left\{\int_\alpha^\beta [|f(u+it)|^k+|f(u+it+id)|^k]\,du\right.\\ &\qquad \left.+\int_0^d [|f(\alpha+it+iv)|^k+|f(\beta+it+iv)|^k]\,dv\right\}dt\\ &= \frac{1}{2\pi\delta}\int_\alpha^\beta\left\{\int_{T_1}^{T_2}|f(u+it)|^k\,dt+\int_{T_1}^{T_2}|f(u+it+id)|^k\,dt\right\}du\\ &\qquad +\frac{1}{2\pi\delta}\int_{T_1}^{T_2}\left\{\int_0^d F(t+v)\,dv\right\}dt \quad \text{(say)}.\end{aligned}$$

* *Cf.* A. E. Ingham, "Mean-value theorems in the theory of the Riemann zeta-function", *Proc. London Math. Soc.* (2), 27 (1928), 273–300, for a proof and refinements.

Now, if $F(t)$ is any positive continuous function,

$$\int_{T_1}^{T_2}\left\{\int_0^d F(t+v)\,dv\right\}dt = \int_{T_1}^{T_2}\left\{\int_t^{t+d} F(x)\,dx\right\}dt$$

$$= \int_{T_1}^{T_2}\{F_1(t+d)-F_1(t)\}\,dt, \quad \text{where} \quad F_1(t)=\int_{t_0}^{t} F(x)\,dx,$$

$$= \int_{T_1+d}^{T_2+d} F_1(t)\,dt - \int_{T_1}^{T_2} F_1(t)\,dt$$

$$= \int_{T_2}^{T_2+d} F_1(t)\,dt - \int_{T_1}^{T_1+d} F_1(t)\,dt$$

$$= dF_1(T_2+\theta_1 d) - dF_1(T_1+\theta_2 d) \quad (0<\theta_1,\theta_2<1).$$

Hence

$$(20)\quad \int_{T_1}^{T_2}\{M(t)\}^k\,dt$$

$$\leqslant \frac{1}{2\pi\delta}\int_\alpha^\beta \{I(T_2,u)-I(T_1,u)+I(T_2+d,u)-I(T_1+d,u)\}\,du$$

$$+\frac{d}{2\pi\delta}\{I(T_2+\theta_1 d,\alpha)-I(T_1+\theta_2 d,\alpha)+I(T_2+\theta_1 d,\beta)-I(T_1+\theta_2 d,\beta)\},$$

where $0<\theta_1,\ \theta_2<1$.

Now take $f(s)=\zeta(s)$, $k=2$, and write

$$U(t)=M(t+\delta;\ \tfrac{1}{2}+2\delta,\ 2+2\delta,\ 2-2\delta),$$

$$J(T,\sigma)=\int_1^T |\zeta(\sigma+it)|^2\,dt.$$

It follows from (16), (20) that

$$\int_T^{T+\sqrt{T}}\{U(t)\}^2\,dt$$

$$\leqslant \frac{1}{2\pi\delta}\int_{\frac{1}{2}+\delta}^{2+3\delta}\{J(T+\sqrt{T},u)-J(T,u)+J(T+\sqrt{T}+2,u)-J(T+2,u)\}\,du$$

$$+\frac{1}{\pi\delta}\{J(T+\sqrt{T}+\phi_1,\ \tfrac{1}{2}+\delta)-J(T+\phi_2,\ \tfrac{1}{2}+\delta)+J(T+\sqrt{T}+\phi_1,\ 2+3\delta)$$

$$-J(T+\phi_2,\ 2+3\delta)\} \qquad (0<\phi_1,\ \phi_2<2)$$

$$=\frac{1}{2\pi\delta}\int_{\frac{1}{2}+\delta}^{2+3\delta}\{2\sqrt{T}\,\zeta(2u)+O(T^{\frac{1}{2}-\frac{1}{2}\delta})\}\,du$$

$$+\frac{1}{\pi\delta}\{\sqrt{T}\,\zeta(1+2\delta)+\sqrt{T}\zeta(4+6\delta)+O(T^{\frac{1}{2}-\frac{1}{2}\delta})\}$$

$$<\frac{\sqrt{T}}{\delta}\,\zeta(1+2\delta)<\frac{\sqrt{T}}{\delta^2} \qquad (0<\delta<\delta_1,\ T\geqslant 1).$$

We are now ready to make use of (1). Let C_1, C_2 be the circles of centre $\frac{5}{4}+2\delta+i(t+\frac{3}{4}+\delta)$, radii $\frac{3}{4}-\delta$, $\frac{3}{4}$, where $T \leqslant t \leqslant T+\sqrt{T}$. C_2 is therefore contained in the rectangle in which the maximum modulus of $\zeta(s)$ is $U(t)$, and the strip $\frac{1}{2}+4\delta \leqslant \sigma \leqslant 2$ has in common with C_1 a rectangle Z of sides $\frac{3}{2}-4\delta$, $(\frac{3}{2}-2\delta)\theta$, where

$$\theta = \{1-(\tfrac{3}{2}-4\delta)^2(\tfrac{3}{2}-2\delta)^{-2}\}^{\frac{1}{2}} \sim (\tfrac{8}{3}\delta)^{\frac{1}{2}}.$$

Suppose now that every horizontal line across Z contains a point at which

$$|\zeta(\sigma+it)| \leqslant L(\delta).$$

Divide Z into N strips, as before, leaving N to be chosen later except that we suppose that N is the same for all t in $(T, T+\sqrt{T})$. (1) gives

$$|\zeta(s)|^2 \leqslant \{U(t)\}^2\left(\frac{2r}{1+r^2}\right)^{N+1} + \frac{6L^2}{1-r^4}\left(\frac{2e}{\theta}\right)^N \quad (s \text{ in } C_1),$$

where $r = 1-\frac{4}{3}\delta$. Hence, since the point $\frac{1}{2}+3\delta+i(t+\frac{3}{4}+\delta)$ is on C_1,

$$\int_T^{T+\sqrt{T}} |\zeta\{\tfrac{1}{2}+3\delta+i(t+\tfrac{3}{4}+\delta)\}|^2\,dt \leqslant \left(\frac{2r}{1+r^2}\right)^{N+1}\int_T^{T+\sqrt{T}} \{U(t)\}^2\,dt + \frac{6L^2}{1-r^4}\left(\frac{2e}{\theta}\right)^N\sqrt{T}.$$

Now the expression on the left is equal to

$$\sqrt{T}\,\zeta(1+6\delta)+O(T^{\frac{1}{2}-\delta}) > \frac{\sqrt{T}}{6\delta} - AT^{\frac{1}{2}-\delta}$$

$$> \frac{\sqrt{T}}{12\delta} + \sqrt{T}\left(\frac{1}{12\delta}-A\right) > \frac{\sqrt{T}}{12\delta} \quad (0<\delta<\delta_2,\ T \geqslant 1).$$

Moreover, for sufficiently small δ,

$$\frac{2r}{1+r^2} = (1-\tfrac{4}{3}\delta)(1-\tfrac{4}{3}\delta+\tfrac{8}{9}\delta^2)^{-1} < 1-\tfrac{1}{2}\delta^2,$$

$$1-r^4 = 1-(1-\tfrac{16}{3}\delta+\ldots) > 3\delta,$$

$$\theta > \sqrt{\delta}.$$

A combination of these inequalities gives

$$\frac{\sqrt{T}}{12\delta} < (1-\tfrac{1}{2}\delta^2)^N\frac{\sqrt{T}}{\delta^2} + \frac{6L^2}{3\delta}\left(\frac{2e}{\sqrt{\delta}}\right)^N\sqrt{T},$$

that is,

$$1 < \frac{12}{\delta}(1-\tfrac{1}{2}\delta^2)^N + 24L^2\left(\frac{2e}{\sqrt{\delta}}\right)^N.$$

If we now choose N so that

$$(1-\tfrac{1}{2}\delta^2)^N < \tfrac{1}{24}\delta,$$

a contradiction will result provided that

$$L^2 \leqslant \tfrac{1}{48}\left(\frac{\sqrt{\delta}}{2e}\right)^N,$$

and a little calculation shows that it is sufficient to take

$$L = e^{-2\{(1/\delta)\log(1/\delta)\}^2}.$$

If we write δ in place of 4δ, the following result has therefore been established:

THEOREM 2. *Given δ $(0<\delta<\delta_0)$, the inequality*

$$|\zeta(\sigma+it)| > e^{-40\{(1/\delta)\log(1/\delta)\}^2} \quad (\tfrac{1}{2}+\delta \leqslant \sigma \leqslant 2) \tag{21}$$

is satisfied for at least one value of t in every interval $(T, T+\sqrt{T})$.

No significance attaches to the number 40 which occurs in (21), nor to the precise value of the numerical constant δ_0.

5. Theorem 2 can be used to study the function

$$S_1(t) = \int_0^t S(u)\,du$$

connected with the zeros of $\zeta(s)$. It is known* that

$$S_1(t) = \frac{1}{\pi}\int_{\frac{1}{2}}^{2} \log|\zeta(\sigma+it)|\,d\sigma + O(1), \tag{22}$$

and it follows from this that

$$S_1(t) = O(\log t). \tag{23}$$

This can be improved to $o(\log t)$ by assuming Lindelöf's hypothesis, and much more follows from Riemann's hypothesis. What we prove falls far short of what can be deduced from the latter, but any result which can be proved without hypothesis has a certain interest. We have, by (22),

$$S_1(t) = \frac{1}{\pi}\int_{\frac{1}{2}}^{\frac{1}{2}+\delta} \log|\zeta(\sigma+it)|\,d\sigma + \int_{\frac{1}{2}+\delta}^{2} \log|\zeta(\sigma+it)|\,d\sigma + O(1).$$

Suppose that $\delta = (\log\log T/\log T)^{\frac{1}{3}}$ and that t $(T \leqslant t \leqslant T+\sqrt{T})$ has one of the values for which (21) is satisfied, so that we can obtain a lower

* Titchmarsh, *op. cit.*, 59.

bound for the second integral. The method used to deduce (23) from (22) can be applied to the first integral and it is found that

$$S_1(t) > -K\delta \log \frac{1}{\delta} \log t - 60\left(\frac{1}{\delta} \log \frac{1}{\delta}\right)^2.$$

Finally, inserting the value of δ, we have the following result:

THEOREM 3. *Every interval* $(T, T+\sqrt{T})$ *contains values of* t *for which*

$$S_1(t) > -A(\log t)^{\frac{3}{5}} (\log\log t)^{\frac{4}{5}}, \tag{24}$$

where A *is an absolute constant.*

[*Added* 14 *December*, 1935. It follows from the inequality

$$\int_T^{T+\sqrt{T}} \{U(t)\}^2 dt < \frac{\sqrt{T}}{\delta^2}$$

that

$$U(t) < \frac{1}{\delta}$$

for values of t in $(T, T+\sqrt{T})$, and so that

$$\log|\zeta(\sigma+it)| < \log\frac{1}{\delta} \quad (\tfrac{1}{2}+2\delta \leqslant \sigma \leqslant 2-2\delta)$$

for such values of t. Hence, taking $\delta = 1/\log T$, there are values of t in $(T, T+\sqrt{T})$ such that

$$\begin{aligned} S_1(t) &< \frac{1}{\pi}\int_{\frac{1}{2}}^{\frac{1}{2}+2\delta} \log|\zeta(\sigma+it)|\,d\sigma + \frac{1}{\pi}\int_{\frac{1}{2}+2\delta}^{2-2\delta} \log|\zeta(\sigma+it)|\,d\sigma + O(1) \\ &< \delta\log t + \frac{2}{\pi}\log\frac{1}{\delta} + O(1) \\ &< \frac{2}{\pi}\log\log t + O(1). \end{aligned}$$

Since $S_1(t)$ is a continuous function, there must be values of t for which this inequality and (24) are both satisfied, *i.e.*

$$-A(\log t)^{\frac{3}{5}} (\log\log t)^{\frac{4}{5}} < S_1(t) < \log\log t \tag{25}$$

for values of t in every interval $(T, T+\sqrt{T})$, where $T \geqslant T_0$.]

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